2.8 Integrators and Differentiators

Reading Assignment: 105-113

Op-amp circuits can also (and often do) implement reactive elements such as inductors and capacitors.

**HO: OP-AMP CIRCUITS WITH REACTIVE ELEMENTS**

One important op-amp circuit is the inverting differentiator.

**HO: THE INVERTING DIFFERENTIATOR**

Likewise the inverting integrator.

**HO: THE INVERTING INTEGRATOR**

**HO: AN APPLICATION OF THE INVERTING INTEGRATOR**

Let's do some examples of op-amp circuit analysis with reactive elements.

**EXAMPLE: A NON-INVERTING NETWORK**

**EXAMPLE: AN INVERTING NETWORK**
**Example:** Another Inverting Network

**Example:** A Complex Processing Circuit
Op-Amp circuits with reactive elements

Now let's consider the case where the op-amp circuit includes reactive elements:

Q: Yikes! How do we analyze this?

A: Don't panic! Remember, the relationship between $v_{out}$ and $v_{in}$ is linear, so we can express the output as a convolution:

$$v_{out}(t) = \mathcal{L}[v_{in}(t)] = \int_{-\infty}^{t} g(t-t') v_{in}(t') dt'$$
Just find the Eigen value

Q: I'm still panicking—how do we determine the impulse response $g(t)$ of this circuit?

A: Say the input voltage $v_{in}(t)$ is an Eigen function of linear, time-invariant systems:

$$v_{in}(t) = e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t}e^{j\omega t}$$

Then, the output voltage is just a scaled version of this input:

$$v_{out}(t) = L[e^{-st}] = \int_{-\infty}^{t} g(t-t') e^{-st'} dt' = G(s)e^{-st}$$

where the “scaling factor” $G(s)$ is the complex Eigen value of the linear operator $L$. 
Express the input as a superposition of eigen values (i.e., the Laplace transform)

Q: First of all, how could the input (and output) be this complex function $e^{st}$? Voltages are real-valued!

A: True, but the real-valued input and output functions can be expressed as a weighted superposition of these complex Eigen functions!

\[
\begin{align*}
v_{in}(s) &= \int_{0}^{+\infty} v_{in}(t) e^{-st} \, dt \\
\text{The Laplace transform} \rightarrow \\
v_{out}(s) &= \int_{0}^{+\infty} v_{out}(t) e^{-st} \, dt
\end{align*}
\]

Such that:

\[
v_{out}(s) = G(s)v_{in}(s)
\]
Find the eigenvalue from circuit theory and impedance

Q: Still, I don't know how to find the eigen value $G(s)$!

A: Remember, we can find $G(s)$ by analyzing the circuit using the Eigen value of each linear circuit element—a value we know as complex impedance!

$$\frac{v(s)}{i(s)} = Z(s)$$

$+$ $\frac{v(s)}{i(s)}$ $-$

$Z(s)$
For example, consider this amplifier in with the inverting configuration, where the resistors have been replaced with complex impedances:

What is the open-circuit voltage gain $A_{vo}(s) = \frac{v_{oc}(s)}{v_{in}(s)}$?
The eigen value of this linear operator

From KCL:

\[ i_1(s) = i_2(s) \]

Since \( v_+(s) = 0 \), we find from Ohm's Law:

\[ i_1(s) = \frac{v_{in}(s)}{Z_1(s)} \]

And also from Ohm's Law:

\[ i_2(s) = \frac{-v^{oc}_{out}(s)}{Z_2(s)} \]

Equating the last two expressions:

\[ \frac{v_{in}(s)}{Z_1(s)} = \frac{v^{oc}_{out}(s)}{Z_2(s)} \]

Rearranging, we find the open-circuit voltage gain:

\[ A_v^{oc}(s) = \frac{v^{oc}_{out}(s)}{v_{in}(s)} = -\frac{Z_2(s)}{Z_1(s)} \]
The result passes the sanity check

Note that this complex voltage gain \( A_v(s) \) is the Eigen value \( G(s) \) of the linear operator relating \( v_{in}(t) \) and \( v_{out}(t) \):

\[
 v_{out}(t) = \mathcal{L} \left[ v_{in}(t) \right]
\]

Note also that if the impedances \( Z_1(s) \) and \( Z_2(s) \) are real valued (i.e., they're resistors!):

\[
 Z_1(s) = R_1 + j0 \quad \text{and} \quad Z_2(s) = R_2 + j0
\]

Then the voltage gain simplifies to the familiar:

\[
 A_v(s) = \frac{v_{oc}(s)}{v_{in}(s)} = -\frac{R_2}{R_1}
\]
Or, we can use the Fourier transform

Now, recall that the variable \( s \) is a complex frequency:

\[
s = \sigma + j \omega.
\]

If we set \( \sigma = 0 \), then \( s = j \omega \), and the functions \( Z(s) \) and \( A_{vo}(s) \) in the Laplace domain can be written in the frequency (i.e., Fourier) domain!

\[
A_{vo}(\omega) = A_{vo}(s)\big|_{\sigma=0}
\]

And therefore, for the inverting configuration:

\[
A_{vo}(\omega) = \frac{v_{out}^{oc}(\omega)}{v_{in}(\omega)} = -\frac{Z_2(\omega)}{Z_1(\omega)}
\]
For the non-inverting

Likewise, for the non-inverting configuration, we find:

\[
A_vo(\omega) = \frac{v_{oc}^{oc}(\omega)}{v_{in}(\omega)} = 1 + \frac{Z_2(\omega)}{Z_1(\omega)}
\]

\[
A_vo(s) = \frac{v_{oc}^{oc}(s)}{v_{in}(s)} = 1 + \frac{Z_2(s)}{Z_1(s)}
\]
The Inverting Differentiator

The circuit shown below is the inverting differentiator.

Since the circuit uses the inverting configuration, we can conclude that the circuit transfer function is:

$$G(s) = \frac{v_{oc}(s)}{v_{in}(s)} = -\frac{Z_2(s)}{Z_1(s)}$$
Know the impedance; know the answer

For the capacitor, we know that its complex impedance is:

\[ Z_1(s) = \frac{1}{sC} \]

And the complex impedance of the resistor is simply the real value:

\[ Z_2(s) = R \]

Thus, the eigen value of the linear operator relating \( v_{in}(t) \) to \( v_{out}^{oc}(t) \) is:

\[ G(s) = -\frac{Z_2(s)}{Z_1(s)} = -\frac{R}{\frac{1}{sC}} = -sRC \]

In other words, the (Laplace transformed) output signal is related to the (Laplace transformed) input signal as:

\[ v_{out}^{oc}(s) = -s(RC) \cdot v_{in}(s) \]

From our knowledge of Laplace Transforms, we know this means that the output signal is proportional to the derivative of the input signal!
Converting back to time domain

Taking the inverse Laplace Transform, we find:

\[ v_{oc}^{out}(t) = -RC \frac{d v_{in}(t)}{dt} \]

For example, if the input is:

\[ v_{in}(t) = \sin \omega t \]

then the output is:

\[ v_{oc}^{out}(t) = -RC \frac{d v_{in}(t)}{dt} = -RC \frac{d \sin \omega t}{dt} = -\omega RC \cos \omega t \]
Or, with Fourier analysis

We likewise could have determined this result using Fourier analysis (i.e., frequency domain):

\[
G(\omega) = \frac{v_{out}^{oc}(\omega)}{v_{in}(\omega)} = \frac{Z_2(\omega)}{Z_1(\omega)} = -\frac{R}{(1/j\omega C)} = -j\omega RC
\]

Thus, the magnitude of the transfer function is:

\[
|G(\omega)| = |-j\omega RC| = \omega RC
\]

And since:

\[-j = e^{-j(\pi/2)} = \cos(-\pi/2) + j\sin(-\pi/2)\]

the phase of the transfer function is:

\[
\angle G(\omega) = -\pi/2 \text{ radians} = -90^\circ
\]
Look at the magnitude and phase

So given that:

$$|v_{out}^{oc}(\omega)| = |G(\omega)| |v_{in}(\omega)|$$

and:

$$\angle v_{out}^{oc}(\omega) = \angle G(\omega) + \angle v_{in}(\omega)$$

we find for the input:

$$v_{in}(t) = \sin \omega t$$

where:

$$|v_{in}(\omega)| = 1 \quad \text{and} \quad \angle v_{in}(\omega) = 0$$

that the output of the inverting differentiator is:

$$|v_{out}^{oc}(\omega)| = |G(\omega)| |v_{in}(\omega)| = \omega RC$$

and:

$$\angle v_{out}^{oc}(\omega) = \angle G(\omega) + \angle v_{in}(\omega) = -90^\circ + 0 = -90^\circ$$
The result is the same!

Therefore, the output is:

\[
v_{oc}^{out}(t) = \omega RC \sin(\omega t - 90^\circ) = -\omega RC \cos(\omega t)
\]

Exactly the same result as before (using Laplace transforms)!

If you are still unconvinced that this circuit is a differentiator, consider this time-domain analysis.
Let’s do a time-domain analysis

From our elementary circuits knowledge, we know that the current through a capacitor \( i_1(t) \) is:

\[
i_1(t) = C \frac{d v_c(t)}{dt}
\]

and from the circuit we see from KVL that:

\[
v_c(t) = v_{in}(t) - v_o(t) = v_{in}(t)
\]
	herefore the input current is:

\[
i_1(t) = C \frac{d v_{in}(t)}{dt}
\]
Laplace, Fourier, time-domain: the result is the same!

From KCL, we likewise know that:

\[ i_1(t) = i_2(t) \]

and from Ohm's Law:

\[ i_2(t) = \frac{v_1(t) - v_{oc}^{out}(t)}{R} = -\frac{v_{oc}^{out}(t)}{R} \]

Combining the two previous equations:

\[ v_{oc}^{out}(t) = -i_1(t)R \]

and thus:

\[ v_{oc}^{out}(t) = -i(t)R = -\left(C \frac{dv_{in}(t)}{dt}\right)R = -RC \frac{dv_{in}(t)}{dt} \]

The same result as before!
The Inverting Integrator

The circuit shown below is the inverting integrator.
It’s the inverting configuration!

Since the circuit uses the **inverting** configuration, we can conclude that the circuit transfer function is:

\[
G(s) = \frac{v_{oc}^c(s)}{v_{in}(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{(1/sC)}{R} = \frac{-1}{sRC}
\]

In other words, the output signal is related to the input as:

\[
v_{out}(s) = \frac{-1}{RC} \frac{v_{in}(s)}{s}
\]

From our knowledge of **Laplace Transforms**, we know this means that the output signal is proportional to the **integral** of the input signal!
The circuit integrates the input

Taking the inverse Laplace Transform, we find:

\[ v_{oc}(t) = \frac{-1}{RC} \int_{0}^{t} v_{in}(t') dt' \]

For example, if the input is:

\[ v_{in}(t) = \sin \omega t \]

then the output is:

\[ v_{oc}^{out}(t) = \frac{-1}{RC} \int_{0}^{t} \sin \omega t' dt' = \frac{-1}{RC} \frac{-1}{\omega} \cos \omega t = \frac{1}{\omega RC} \cos \omega t \]
**Or, in the Fourier domain**

We likewise could have determined this result using **Fourier Analysis** (i.e., frequency domain):

\[
G(\omega) = \frac{v_{out}(\omega)}{v_{in}(\omega)} = -\frac{Z_2(\omega)}{Z_1(\omega)} = -\frac{1/j\omega C}{R} = \frac{j}{\omega RC}
\]

Thus, the **magnitude** of the transfer function is:

\[
|G(\omega)| = \left| \frac{j}{\omega RC} \right| = \frac{1}{\omega RC}
\]

And since:

\[
j = e^{j(\pi/2)} = \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right)
\]

the **phase** of the transfer function is:

\[
\angle G(\omega) = \frac{\pi}{2} \text{ radians} = 90^\circ
\]
Magnitude and phase

Given that:

\[ |v_{out}^{\text{oc}}(\omega)| = |G(\omega)| |v_{in}(\omega)| \]

and:

\[ \angle v_{out}^{\text{oc}}(\omega) = \angle G(\omega) + \angle v_{in}(\omega) \]

we find for the input:

\[ v_{in}(t) = \sin \omega t \]

where:

\[ |v_{in}(\omega)| = 1 \quad \text{and} \quad \angle v_{in}(\omega) = 0 \]

that the output of the inverting integrator is:

\[ |v_{out}^{\text{oc}}(\omega)| = |G(\omega)| |v_{in}(\omega)| = \frac{1}{\omega RC} \]

and:

\[ \angle v_{out}^{\text{oc}}(\omega) = \angle G(\omega) + \angle v_{in}(\omega) = 90^\circ + 0 = 90^\circ \]
See, it’s an integrator

Therefore:

\[ v_{out}^{oc}(t) = \frac{1}{wRC} \sin(wt + 90^\circ) \]

\[ = \frac{1}{wRC} \cos(wt) \]

Exactly the same result as before!

If you are still unconvinced that this circuit is an integrator, consider this time-domain analysis.
The time-domain solution

From our elementary circuits knowledge, we know that the voltage across a capacitor is:

\[ v_c(t) = \frac{1}{C} \int_0^t i_2(t') dt' \]

and from the circuit we see that:

\[ v_c(t) = v_-(t) - v_{out}^{oc}(t) = -v_{out}^{oc}(t) \]

due to the output voltage is:

\[ v_{out}^{oc}(t) = -\frac{1}{C} \int_0^t i_2(t') dt' \]
The same result no matter how we do it!

From KCL, we likewise know that:

\[ i_1(t) = i_2(t) \]

and from Ohm's Law:

\[ i_1(t) = \frac{v_{in}(t)-v_c(t)}{R_1} = \frac{v_{in}(t)}{R_1} \]

Therefore:

\[ i_2(t) = \frac{v_{in}(t)}{R_1} \]

and thus:

\[ v_{out}^{oc}(t) = \frac{-1}{C} \int_0^t i_2(t') \, dt' \]

\[ = \frac{-1}{RC} \int_0^t v_{in}(t') \, dt' \]

The same result as before!
An Application of the Inverting Integrator

Note the time average of a signal $v(t)$ over some arbitrary time $T$ is mathematically stated as:

$$\text{average of } v(t) = \bar{v}(t) = \frac{1}{T} \int_{0}^{T} v(t) \, dt$$

Note that this is exactly the form of the output of an op-amp integrator!

We can use the inverting integrator to determine the time-averaged value of some input signal $v(t)$ over some arbitrary time $T$. 

Make sure you see this!

For example, say we wish to determine the time-averaged value of the input signal:

\[ v_{in}(t) = \begin{cases} 
5 & 0 < t < 2 \\
-5 & 2 < t < 3 \\
0 & t > 3 
\end{cases} \]

The time average of this function over a period from \(0 < t < T=3\) is therefore:

\[ \overline{v_{in}(t)} = \frac{1}{3} \int_{0}^{3} v_{in}(t) \, dt = \frac{5}{3} \]
This better make sense to you!

We could likewise determine this average using an inverting integrator. We select a resistor $R$ and a capacitor $C$ such that the product $RC = 3$ seconds.

The output of this integrator would be:

$$v_{out}(t) = \frac{-1}{3} \int_{0}^{t} v_{in}(t') \, dt' = \begin{cases} \frac{-5t}{3} & 0 < t < 2 \\ \frac{5t - 20}{3} & 2 < t < 3 \\ \frac{-5}{3} & t > 3 \end{cases}$$

![Graph of $v_{out}(t)$](image)
We must sample a the correct time!

Note that the value of the output voltage at \( t = 3 \) is:

\[
\nu_{out}(t = 3) = \frac{-1}{3} \int_{0}^{3} \nu_{in}(t') \, dt' = -\frac{5}{3}
\]

The time-averaged value (times -1)!

Thus, we can use the inverting integrator, along with a voltage sampler (e.g., A to D converter) to determine the time-averaged value of a function over some time period \( T \).
Example: An Inverting Network

Now let’s determine the complex transfer function of this circuit:

\[ \text{Complex Transfer Function} \]
It's the inverting configuration!

Note this circuit uses the inverting configuration, so that:

\[ G(\omega) = -\frac{Z_2(\omega)}{Z_1(\omega)} \]

where \( Z_1 = R_1 \), and:

\[ Z_2 = R_2 \left[ \frac{1}{j\omega C} = \frac{R_2}{1 + j\omega R_2C} \right] \]

Therefore, the transfer function of this circuit is:

\[ G(\omega) = \frac{v_{out}^{oc}(\omega)}{v_{in}(\omega)} = -\frac{R_2}{R_1} \frac{1}{1 + j\omega R_2C} \]
Another low-pass filter

Thus, the transfer function magnitude is:

$$|G(\omega)|^2 = \left( -\frac{R_2}{R_1} \right)^2 \frac{1}{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

where:

$$\omega_0 = \frac{1}{R_2 C}$$

Thus, just as with the previous example, this circuit is a low-pass filter, with cutoff frequency $\omega_0$ and pass-band gain $(R_2/R_1)^2$. 
Example: A Non-Inverting Network

Let's determine the transfer function \( G(\omega) = \frac{v_{oc}^\omega}{v_{in}(\omega)} \) for the following circuit:
**Some enjoyable circuit analysis**

From KCL, we know:

\[ i_3(\omega) = i_c(\omega) + i_+(\omega) = i_c(\omega) + 0 = i_c(\omega) \]

where:

\[ i_3(\omega) = \frac{v_{in}(\omega) - v_+(\omega)}{R_3} \quad \text{and} \quad i_c(\omega) = \frac{v_+(\omega) - 0}{\frac{1}{j\omega C}} = j\omega C v_+(\omega) \]

Equating, we find an expression involving \( v_{in}(\omega) \) and \( v_2(\omega) \) only:

\[ \frac{v_{in}(\omega) - v_+(\omega)}{R_3} = j\omega C v_+(\omega) \]

and performing a little algebra, we find:

\[ v_2(\omega) = \frac{v_{in}(\omega)}{1 + j\omega R_3 C} \]
No need to go further: we have a template!

The remainder of the circuit is simply the non-inverting amplifier that we studied earlier.

We know that:

\[ v_{out}^{oc}(\omega) = \left(1 + \frac{R_2}{R_1}\right)v_+(\omega) \]

Combining these two relationships, we can determine the complex transfer function for this circuit:

\[ G(\omega) = \frac{v_{out}(\omega)}{v_{in}(\omega)} = \left(1 + \frac{R_2}{R_1}\right)\left(\frac{1}{1 + j\omega R_3 C}\right) \]
It's a low-pass filter!!!

The magnitude of this transfer function is therefore:

\[ |G(\omega)|^2 = \left(1 + \frac{R_2}{R_1}\right)^2 \frac{1}{1 + \left(\frac{\omega}{\omega_0}\right)^2} \]

where:

\[ \omega_0 = \frac{1}{R_3C} \]

This is a low-pass filter—one with pass-band gain!
Example: Another Inverting Network

Consider now the transfer function of this circuit:
Some more enjoyable circuit analysis

To accomplish this analysis, we must first...

Wait! You don’t need to explain this to me.

It is obvious that we can divide this is circuit into two pieces—the first being a complex voltage divider and the second a non-inverting amplifier.
Can we analyze the circuit this way?

The transfer function of the complex voltage divider is:

\[
\frac{v_3(\omega)}{v_{in}(\omega)} = \frac{1}{j\omega C} = \frac{1}{R_3 + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega R_3 C}
\]

and that of the inverting amplifier:

\[
\frac{v_{oc}^{out}(\omega)}{v_3(\omega)} = -\frac{R_2}{R_1}
\]

And so of course I have correctly determined that the transfer function of this circuit is:

\[
\frac{v_{oc}^{out}(\omega)}{v_{in}(\omega)} = \frac{v_{oc}^{out}(\omega)}{v_3(\omega)} \frac{v_3(\omega)}{v_{in}(\omega)} = -\frac{R_2}{R_1} \frac{1}{1 + j\omega R_3 C}
\]
No, we cannot

NO! This is not correct:

\[
\frac{v_o(\omega)}{v_r(\omega)} = -\frac{R_2}{R_1} \frac{1}{1 + j\omega R_C}
\]

The problem with the above “analysis” is that we cannot apply this complex voltage divider equation to determine \(v_3(\omega)\):

\[
v_3(\omega) \neq \frac{1}{j\omega C} \frac{1}{R_3 + \frac{1}{j\omega R_3}} v_{in}(\omega)
\]

The reason of course is that the output of this voltage divider is not open-circuited, and thus current \(i_3(\omega) \neq i_C(\omega)\).
My computer suspiciously crashed while writing this (really, it did!)

We cannot divide this circuit into two independent pieces, we must analyze it as one circuit.

Of course what I meant to say was that we should determine the impedance $Z_1$ of input network, and then use the inverting configuration equation $T(\omega) = -Z_2/Z_1$. 
An even worse idea than Vista

NO! This idea is as bad as the last one!

We cannot specify an impedance for the input network:

\[ Z_1 = \frac{V_{in} - V_3}{I_3} \quad \text{or} \quad Z_1 = \frac{V_{in} - V_1}{I_1} \]

After all, would we define this impedance as:
Don’t look for templates: trust what you know

So, there is no easy or direct way to solve this circuit, we must consult Mr. Kirchoff and his laws!

We know that $i_1 = i_2$, where:

$$i_1 = \frac{v_3 - v}{R_1} = \frac{v_3}{R_1} \quad \text{and} \quad i_2 = \frac{v_+ - v_{out}}{R_2} = \frac{-v_{out}}{R_2}$$

Combining these equations, we get the expected result:

$$v_{out} = -\frac{R_2}{R_1} v_3$$
Don’t forget virtual ground!

We must therefore determine \( v_3 \) in terms of \( v_i \):

\[
\begin{align*}
R_1 & \parallel \frac{1}{j \omega C} \\
\frac{v_3}{v_{in}} &= \frac{R_1}{R_3 + \left( R_1 \parallel \frac{1}{j \omega C} \right)}
\end{align*}
\]

where:

\[
R_1 \parallel \frac{1}{j \omega C} = \frac{R_1 \left( \frac{1}{j \omega C} \right)}{R_1 + \frac{1}{j \omega C}} = \frac{R_1}{1 + j \omega R_1 C}
\]

Note \( R_f \) and \( C \) are connected in parallel!

Thus, from voltage division, we find:
The Eigen value at last!

Performing some algebra, we find:

\[ v_3 = \left( \frac{R_1}{(R_1 + R_3) + j \omega R_1 R_3 C} \right) v_{in} \]

and since:

\[ v_{out} = \frac{-R_2}{R_1} v_3 \]

we finally discover that:

\[ G(\omega) = \frac{v_{out}(\omega)}{v_{in}(\omega)} = \left( \frac{-R_2}{(R_1 + R_3) + j \omega R_1 R_3 C} \right) \]
This again is a low-pass filter

We can rearrange this transfer function to find that this circuit is likewise a low-pass filter with pass-band gain:

\[
G(\omega) = \frac{v_{out}(\omega)}{v_{in}(\omega)} = \frac{-R_2}{R_1 + R_3} \left( \frac{1}{1 + j\left(\frac{\omega}{\omega_0}\right)} \right)
\]

where the cutoff frequency \( \omega_0 \) is:

\[
\omega_0 = \frac{1}{\left( \frac{R_1 R_3}{R_1 + R_3} \right) C} = \frac{1}{(R_1 \parallel R_3) C}
\]

I wish I had a nickel for every time my software has crashed—oh wait, I do!
Example: A Complex Processing Circuit using the Inverting Configuration

Note that we can combine inverting amplifiers to form a more complex processing system.

For example, say we wish to take three input signals \( v_1(t) \), \( v_2(t) \), and \( v_3(t) \), and process them such that the open-circuit output voltage is:

\[
\nu_{out}(t) = 5\nu_1(t) + \int_{-\infty}^{t} \nu_2(t') \, dt' + \frac{d}{dt} \nu_3(t)
\]

Assuming that we use ideal (or near ideal) op-amps, with an output resistance equal to zero (or at least very small), we can realize the above signal processor with the following circuit:
This circuit performs this operation!

\[ v_{out}(t) = 5v_1(t) + \int_{-\infty}^{t} v_2(t') \, dt' + \frac{d v_3(t)}{dt} \]