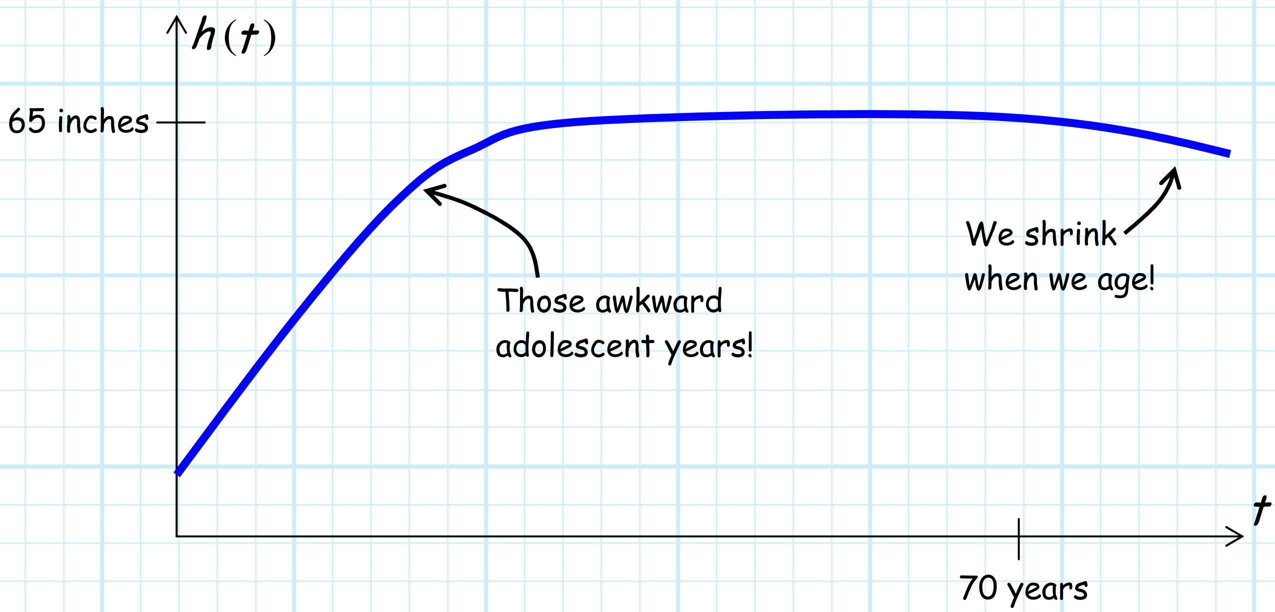
A "Small-Signal Analysis" of Human Growth

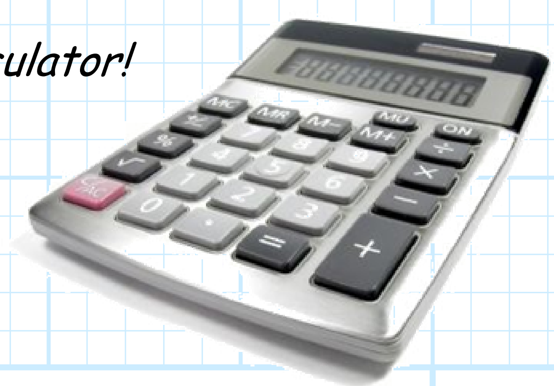
Say the **average height** h of a human (in **inches**) is related to his/her age t in **months** by this equation:

$$h(t) = 65 - 3.66\text{x}10^{-10}(45 - t/12)^{6.75} \text{ inches}$$



Say that we want to **calculate** the average height of a human at an age of t =58, 59, 59.5, 60, 60.5, 61, and 62 months.

Whew! Let me get out my calculator!



$$h(t = 58.0) = 40.48 \text{ inches}$$

$$h(t = 59.0) = 40.82 \text{ inches}$$

$$h(t = 59.5) = 40.99 \text{ inches}$$

$$h(t = 60.0) = 41.16 \text{ inches}$$

$$h(t = 60.5) = 41.32 \text{ inches}$$

$$h(t = 61.0) = 41.49 \text{ inches}$$

$$h(t = 62.0) = 41.82 \text{ inches}$$

Q: *Wow, this was hard. Isn't there an **easier** way to calculate these values?*

A: Yes, there is! We can make a "**small-signal**" approximation.

For a small-signal approximation, we simply need to calculate **two** values. First:

$$h(t)\Big|_{t=60} = h(t = 60) = 41.16 \text{ inches}$$

In other words, the average height of a human at **60 months** (i.e., 5 years) is **41.16 inches**.

Likewise, we calculate the **time derivative** of $h(t)$, and **then evaluate** the result at 60 months:

$$\begin{aligned} \left.\frac{d\,h(t)}{dt}\right|_{t=60} &= \left(2.059\times10^{-10}(45 - t/12)^{5.75}\right)\Big|_{t=60} \\ &= 2.059\times10^{-10}(45 - 60/12)^{5.75} \\ &= 0.34 \text{ inches/month} \end{aligned}$$

In other words, the average 5 year old **grows** at a rate of **0.34 inches/month**!

Now let's again consider the earlier problem.

If we know that an average 5-year old is 41.16 inches tall, and grows at a rate of 0.34 inches/month, then at 5 years **and one month** (i.e., 61 months), the little bugger will approximately be:

$$41.16 + (0.34)(1) = 41.50 \text{ inches}$$

Compare this to the exact value of 41.49 inches—a **very accurate approximation**.

We can likewise **approximate** the average height of a **59-month** old (i.e., 5 years **minus one** month):

$$41.16 + (0.34)(-1) = 40.83 \text{ inches}$$

or of a **62-month** old (i.e., 5 years **plus two** months):

$$41.16 + (0.34)(2) = 41.83 \text{ inches}$$

Note again the **accuracy** of these approximations!

For this approximation, let us define time t=60 as the **evaluation point**, or bias point T:

$$T \doteq \text{evaluation point}$$

We can then define:

$$\Delta t = t - T$$

In this example then, T = 60 months, and the values of Δt range from –2 to +2 months.

For example, t = 59 months can be expressed as $t = T + \Delta t$, where $T = 60$ months and $\Delta t = -1$ month.

We can therefore write our approximation as:

$$h(t) \approx h(t)\Big|_{t=T} + \left.\frac{d\,h(t)}{dt}\right|_{t=T} \Delta t$$

For the example where T=60 months we find:

$$\begin{aligned} h(t) &\approx h(t)\Big|_{t=60} + \left.\frac{d\,h(t)}{dt}\right|_{t=60} \Delta t \\ &= 41.16 + 0.34\ \Delta t \end{aligned}$$

This approximation is **not accurate**, however, if $|\Delta t|$ is **large**.

For example, we can determine from the **exact** equation that the average height of a **forty-year old** human is:

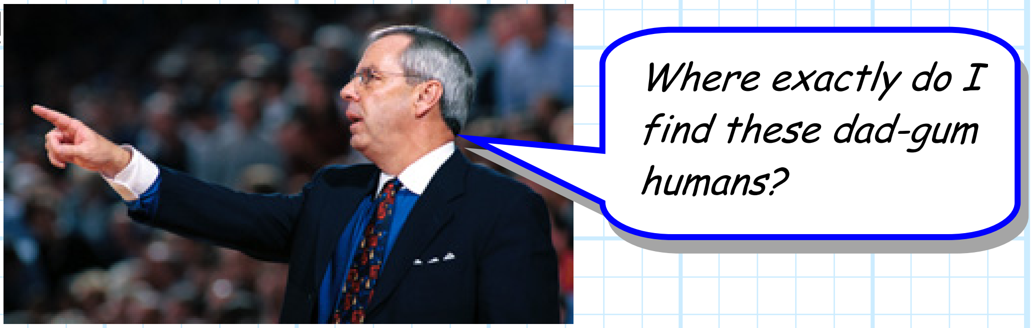
$$h(t = 480) = 65 \text{ inches}$$

or about **5 feet 5 inches**.

However, if we were to use our **approximation** to determine the average height of a 40-year old ($\Delta t = t - T = 480 - 60 = 420$), we would find:

$$\begin{aligned} h(t) &\approx 41.16 + 0.34\,(420) \\ &= 181.86 \text{ inches} \end{aligned}$$

The approximation says that the average 40-year old human is **over 15 feet tall!**



The reason that the above approximation provides an **inaccurate** answer is because it is based on the assumption that humans grow at a rate of 0.34 inches/month.

This **is** true for 5-year olds, but **not** for 40-year olds (unless, of course, you are referring to their **waistlines**)!

We thus refer to the approximation function as a "**small-signal**" approximation, as it is valid only for times that are **slightly different** from the nominal (evaluation) time T (i.e., Δt is small).

If we wish to have an **approximate** function for the growth of humans who are near the age of forty, we would need to **construct a new approximation**:

$$h(t) \approx h(t)\Big|_{t=480} + \left.\frac{d\,h(t)}{dt}\right|_{t=480} \Delta t$$

$$= 65.0 + 2.2 \times 10^{-6}\ \Delta t$$

Note that forty-year old humans have **stopped growing**!

The mathematically astute will recognize the small-signal model as a first-order **Taylor Series** approximation!