5.9 Frequency Response of the Common-Emitter Amp

Reading Assignment: 491-503

Amplifiers made with BJTs are similar to amplifiers made with opamps—the both exhibit finite bandwidth.

**HO: AMPLIFIER BANDWIDTH**

The gain within the bandwidth is usually constant with respect to frequency—we call this value the mid-band gain.

**HO: MID-BAND GAIN**

Large capacitors (e.g., COUS) determine the low-frequency limit of amplifier bandwidth. We can explicitly determine this value be analyzing the low-frequency small-signal circuit.

**HO: THE LOW-FREQUENCY RESPONSE**

Parasitic capacitors (e.g., Cπ) determine the high-frequency limit of amplifier bandwidth. We can explicitly determine this value be analyzing the high-frequency small-signal circuit.

**HO: THE HIGH-FREQUENCY RESPONSE**
Amplifier Bandwidth

BJT amplifiers are band-limited devices—in other words, they exhibit a finite bandwidth.

Q: ???

A: Say the input to a BJT small-signal amplifier is the eigen function of linear, time-invariant system:

\[ V_{in} \cos \omega t = V_{in} \text{Re} \left\{ e^{-j\omega t} \right\} \]

Since the small-signal BJT amp is (approximately) a linear system, the output will likewise be the eigen function—an undistorted sinusoidal function of precisely the same frequency \( \omega \) as the input!

Q: Of course that’s true! We know that:
\[ v_{out}(t) = A_v v_{in}(t) \]

Therefore the magnitudes of the input and output sinusoids are related as:

\[ V_{out} = A_v V_{in} \]

Right?

\[ A: \] Not necessarily!

The small-signal, open-circuit voltage gain of a BJT amplifier depends on the frequency \( \omega \) of the input signal!

\[ Q: \] Huh!?! We determined earlier that the small-signal voltage gain of this amplifier:
was:

\[ A_{io} = \frac{V_o}{V_i} = -200 \]

So then if the small-signal input is:

\[ v_i(t) = V_{in} \cos \omega t \]

isn't the small-signal output simply:

\[ v_o(t) = -200 \ V_{in} \cos \omega t \]

\[ ?????????? \]

A: Maybe—or maybe not!

Again, the gain of the amplifier is frequency dependent. We find that if \( \omega \) is too high (i.e., large) or too low (i.e., small), then the output might be much less than the 200 times larger than the input (e.g., only 127.63 times larger than the input—Doh!).

Now, the signal frequencies \( \omega \) for which

\[ v_o(t) = -200 \ V_{in} \cos \omega t \]

is an accurate statement, are frequencies that are said to lie within the bandwidth of this amplifier (\( \omega \) is just right!).

Conversely, frequencies \( \omega \) for which:

\[ v_o(t) \neq -200 \ V_{in} \cos \omega t \]
are frequencies $\omega$ that lie outside this amplifier’s bandwidth.

Fortunately, the frequencies that compose an amplifier’s bandwidth typically form a **continuum**, such that the frequencies outside this bandwidth are either **higher** or **lower** than all frequencies within the bandwidth.

Perhaps a **plot** would help.

![Amplifier Bandwidth Diagram](image)

The frequencies **between** $\omega_L$ and $\omega_H$ thus lie within the **bandwidth** of the amplifier. The gain within the bandwidth is sometimes referred to as the **midband gain**.

For signals with frequencies less than $\omega_L(f_L)$, the amplifier gain will be **less** than the midband gain—likewise for frequencies greater than $\omega_H(f_H)$. 
Q: So what then is the value:

\[ A_v = \frac{v_o}{v_i} = -200 \]

determined for the example amplifier? It doesn't seem to be a function of frequency!

A: The value -200 calculated for this amplifier is the midband gain—it's the gain exhibited for all signals that lie within the amplifier bandwidth. Your book at times uses the variable \( A_M \) to denote this value:

| \( A_v(\omega) \) |
| \( A_M \) |

Q: So it's actually the midband gain that we've been determining from our small-signal circuit analysis (e.g. \( A_M = -200 \))?

A: That's exactly correct!
Q: So how do we determine the frequency dependent gain $A_v(\omega)$? More specifically, how do we determine midband gain $A_M$, along with $f_L$ and $f_H$?

A: The function $A_v(\omega)$ is simply the eigen value of the linear operator relating the small-signal input and the small signal output:

$$v_o(t) = \mathcal{L}\{v_i(t)\} \implies V_o(\omega) = A_v(\omega) V_i(\omega)$$

Q: Yikes! How do we determine the eigen value of this linear operator?

A: We simply analyze the small-signal circuit, determining $V_o(\omega)$ in terms of $V_i(\omega)$.

Specifically, we must explicitly consider the capacitance in the small-signal amplifier—no longer can we make approximations!

So, instead of vaguely labeling large capacitors as Capacitors Of Unusual Size, let’s explicitly consider the exact values of these large capacitors:
Likewise, we must consider the **parasitic capacitances** of the BJT—specifically $C_\mu$ and $C_\pi$.

The **small-signal circuit**—when we explicitly consider these capacitances—is thus:
We analyze this circuit to determine \( V_\omega \), and then the eigen value—the small-signal gain—is determined as:

\[
A_\omega = \frac{V_\omega}{V_i}
\]

Q: So what again is the meaning of \( V_i \) and \( V_\omega \)?

A: It’s the Fourier transform of \( v_i(t) \) and \( v_\omega(t) \)!

\[
V_i(\omega) = \int_{-\infty}^{\infty} v_i(t) e^{j\omega t} dt \quad \text{and} \quad V_\omega(\omega) = \int_{-\infty}^{\infty} v_\omega(t) e^{j\omega t} dt
\]

Q: So—I can’t recall—what’s the relationship between \( v_i(t) \) and \( v_\omega(t) \)?
**A:** If:

\[ V_o(\omega) = A_v(\omega) V_i(\omega) \]

Then in the time domain, we find that the input and output are related by the always enjoyable convolution integral!!!

\[ v_o(t) = \int_{-\infty}^{\infty} g(t - t') v_i(t') dt' \]

where the **impulse response** of the amplifier is of course:

\[ g(t) = \int_{-\infty}^{\infty} A_v(\omega) e^{-j\omega t} d\omega \]

**Q:** What the heck? What happened to solutions like:

\[ v_o(t) = -200 v_i(t) \]

**A:** This result implies that the **impulse response** of the amplifier is:

\[ g(t) = -200 \delta(t) \]

Such that:

\[ v_o(t) = \int_{-\infty}^{\infty} g(t - t') v_i(t') dt' \]

\[ = -200 \int_{-\infty}^{\infty} \delta(t - t') v_i(t') dt' \]

\[ = -200 v_i(t) \]
Q: You say that the result:

\[ v_o(t) = -200 v_i(t) \]

"implies" that the impulse response of the amplifier is:

\[ g(t) = -200 \delta(t) \]

Are you saying the impulse response of the common-emitter example is not this function?

A: It is definitely not that function. The impulse response

\[ g(t) = -200 \delta(t) \]

is ideal—the impulse response of an amplifier with an infinite bandwidth!

Q: So all our small-signal analysis up to this point has been incorrect and useless???

A: Not at all! The small-signal gain we have been evaluating up to this point (e.g., -200) is the amplifier midband gain \( A_M \).

As long as the small-signal input \( v_i(t) \) resides completely within the amplifier bandwidth, then the output will be:

\[ v_o(t) = -200 v_i(t) \]
The problem occurs when the input signal lies—at least partially—outside the amplifiers bandwidth.

In that case, we find that:

\[ v_o(t) \neq -200v_i(t) \]

And instead:

\[ v_o(t) = \int_{-\infty}^{\infty} g(t-t')v_i(t')dt' \]

where:

\[ g(t) = \int_{-\infty}^{\infty} A_{vo}(\omega) e^{-j\omega t} d\omega \]

and the eigen value \[ A_{vo}(\omega) = \frac{V_o(\omega)}{V_i(\omega)} \] is determined by evaluating this small-signal circuit:
Q: What do you mean when you say that a signal lies "within the amplifier bandwidth" or "outside the amplifier bandwidth? How can we tell?

A: Use the Fourier Transform!

If we plot the magnitude of the Fourier Transform $V_i(\omega)$ of the input signal $v_i(t)$, we can see the spectrum of the input signal:

For example, if you are attempting to amplify a signal representing the audio of symphonic music, the spectrum $|V_i(\omega)|$ will include low-frequency signals (e.g., from the tubas), mid-range frequency signals (e.g., from the trumpets), and high-frequency signals (e.g., from the flutes).
Now, if it is your desire to reproduce exactly this music at the output of your amplifier, then the amplifier bandwidth must be wide enough to include all these spectral components!

For the case above, the input signal resides completely within the bandwidth of the amplifier (i.e., between $f_L$ and $f_H$), and so we find (for $A_M = -200$) that:

$$|V_o(\omega)| = |A_o(\omega)||V_i(\omega)|$$

and:

$$V_o(t) = -200 V_i(t)$$

However, if the input spectrum resides (at least partially) outside the amplifier bandwidth, e.g.:
then we find that:

\[ |V_o(\omega)| \neq (200) |V_i(\omega)| \]

and:

\[ v_o(t) \neq -200 v_i(t) \]

Instead, we find the more general (and more difficult!) expressions:

\[ V_o(\omega) = A_\omega(\omega) V_i(\omega) \]

and:

\[ v_o(t) = \int_{-\infty}^{\infty} g(t-t') v_i(t') dt' \]
where the **impulse response** of the amplifier is:

\[ g(t) = \int_{-\infty}^{\infty} A_v(\omega) e^{-j\omega t} d\omega \]

**Q:** So just what *causes* this amplifier to have a finite bandwith?

**A:** For mid-band frequencies \( f_m \) (i.e., between \( f_L < f_m < f_H \)), we will find that the Capacitors Of Unusual Size exhibit an impedance that is pretty small—approximately an **AC short** circuit:

\[ |Z_{C_{OUS}}(\omega_m)| = \frac{1}{\omega_m C_{ous}} \approx 0 \]

Likewise, the tiny parasitic capacitances \( C_\mu \) and \( C_\pi \) exhibit an impedance that is very large for mid-band frequencies—approximately an **open** circuit:

\[ |Z_{C_\mu}(\omega_m)| = \frac{1}{\omega_m C_\mu} \approx \infty \]

However, when the signal frequency \( \omega \) drops too low, the COUS will **no longer** be a small-signal short.

The result is that the amplifier **gain is reduced**—the values of the COUS determine the **low-end amplifier bandwidth** \( f_L \).
Likewise, when the signal frequency \( \omega \) is too high, the parasitic caps will no longer be a small-signal open.

The result is that the amplifier gain is reduced—the values of the parasitic capacitors determine the high-end amplifier bandwidth \( f_H \).
Mid-band Gain

Q: So, to find the mid-band gain of this amplifier:

we must find the analyze this small signal circuit:
to determine:

\[ A_v(\omega) = \frac{V_o(\omega)}{V_i(\omega)} \]

and then plotting the magnitude:

we determine mid-band gain \( A_m \), right?

A: You could do all that, but there is an easier way.

Recall the midband gain is the value of \( |A_v(\omega)| \) for frequencies within the amplifier bandwidth. For those frequencies, the AC coupling capacitors (i.e., COUS) are approximate AC short-circuits (i.e., very low impedance).
Likewise, for the signal frequencies within the amplifier bandwidth, the parasitic BJT capacitances are approximate AC open-circuits (i.e., very high impedance).

Thus, we can apply these approximations to the capacitors in our small-signal circuit:

Now simplifying this circuit (look, no capacitors!):
Q: Hey wait! Isn’t this the same small-signal circuit that we analyzed earlier, where we found that:

\[ v_o(t) = -200 v_i(t) \]

A: It is exactly!

All of the small-signal analysis that we performed previously (i.e., the circuits with no capacitors!) actually provided us with the mid-band amplifier gain.

Taking the Fourier transform of the equation above:

\[ V_o(\omega) = -200 V_i(\omega) \]
\[ = e^{j\pi} 200 V_i(\omega) \]

Thus, the midband gain of this amplifier is:

\[ A_M = -200 = e^{j\pi} 200 \]
Low-Frequency Response

Q: OK, I see how to determine mid-band gain, but what about determining amplifier bandwidth?

It seems like I have no alternative but to analyze the exact small-signal circuit (explicitly considering all capacitances):

\[ A_o(\omega) = \frac{V_o(\omega)}{V_i(\omega)} \]

and then plot the magnitude:

\[ |A_M| \]
And then from the plot determine the amplifier bandwidth (i.e., determine \( f_L \) and \( f_H \))?

**A:** You could do all that, but there is an easier way.

An amplifier frequency response \( A_v(\omega) \) (i.e., its eigen value!) can generally be expressed as the product of three distinct terms:

\[
A_v(\omega) = F_L(\omega) A_M F_H(\omega)
\]

The middle term is the of course the mid-band gain—a number that is not frequency dependent.

The function \( F_L(\omega) \) describes the low-frequency response of the amplifier—from it we can determine the lower cutoff frequency \( f_L \).

Conversely, the function \( F_H(\omega) \) describes the high-frequency response of the amplifier—from it we can determine the upper cutoff frequency \( f_H \).

**Q:** So just how do we determine these functions \( F_L(\omega) \) and \( F_H(\omega) \)??

**A:** The low-frequency response \( F_L(\omega) \) is dependent only on the large capacitors (COUS) in the amplifier circuit. In other
words the parasitic capacitances have no affect on the low-frequency response.

Thus, we simply “ignore” the parasitic capacitances when determining $F_L(\omega)$!

For example, say we include the COUS in our common-emitter example, but ignore $C_\mu$ and $C_\pi$. The resulting small-signal circuit is:

To simplify this analysis, we first determine the Thevenin’s equivalent circuit of the portion of the circuit connected to the base.

We start by finding the open-circuit voltage:
And the short-circuit output current is:

\[
I_o^{sc}(\omega) = V_i(\omega) \left( \frac{1}{j\omega C_i} \right) = V_i(\omega) (j\omega C_i)
\]

And thus the Thevenin's equivalent source is:

\[
V_{th}(\omega) = V_{th}^0(\omega) = V_i(\omega) \left( \frac{j\omega (0.37) C_i}{1 + j\omega (0.37) C_i} \right)
\]
\[
Z_{th}(\omega) = \frac{V_{oc}^{oc}(\omega)}{I_{oc}^{sc}(\omega)} = \frac{j\omega (0.37) C_i}{1 + j\omega (0.37) C_i} \left( \frac{1}{j\omega C_i} \right) = \frac{(0.37)}{1 + j\omega (0.37) C_i}
\]

Likewise, the two parallel elements on the emitter terminal can be combined:

\[
Z_E(\omega) = \frac{1}{j\omega C_e} = \frac{1}{1 + j\omega C_e}
\]

Thus, the small-signal circuit is now:

From KVL:
\[ 0 + V_{th} - I_b (Z_{th} + 0.5) - (\beta + 1) I_b Z_E = 0 \]

\[ \Rightarrow I_b = \frac{V_{th}}{Z_{th} + 0.5 + 101Z_E} \]

From Ohm's Law:

\[ V_{be} = 0.5I_b \]

Therefore:

\[ V_o(\omega) = -200V_{be}(1) \]

\[ = -200(0.5) \frac{V_{th}(\omega)}{Z_{th} + 0.5 + 101Z_E} \]

\[ = V_{th}(\omega) \left( \frac{-100}{Z_{th} + 0.5 + 101Z_E} \right) \]

Inserting the expressions for the Thevenin's equivalent source, as well as \( Z_E \).

\[ V_o(\omega) = V_{th}(\omega) \left( \frac{-100}{Z_{th} + 0.5 + 101Z_E} \right) \]

\[ = V_i(\omega) \left( \frac{j\omega (0.37) C_i}{1 + j\omega (0.37) C_i} \right) \left( \frac{-200}{1 + j\omega (0.37) C_i + 1 + \frac{202}{1 + j\omega C_E}} \right) \]
Now, it can be shown that:

\[ \frac{1}{2(0.37)} \frac{1}{1 + j\omega(0.37)C_i} + \frac{202}{1 + j\omega C_E} \approx \frac{j\omega (C_E/203)}{1 + j\omega (C_E/203)} \]

Therefore:

\[ V_o(\omega) = V_i(\omega) \left( \frac{j\omega (0.37)C_i}{1 + j\omega (0.37)C_i} \right) \left( \frac{j\omega (C_E/203)}{1 + j\omega (C_E/203)} \right)(-200) \]

And so:

\[ A_o(\omega) = \left( \frac{j\omega (0.37)C_i}{1 + j\omega (0.37)C_i} \right) \left( \frac{j\omega (C_E/203)}{1 + j\omega (C_E/203)} \right)(-200) \]

Now, since we are ignoring the parasitic capacitances, the function \( F_H(\omega) \) that describes the high frequency response is:

\[ F_H(\omega) = 1 \]

And so:

\[ A_o(\omega) = F_L(\omega) A_M \quad F_H(\omega) = F_L(\omega) A_M \]

By inspection, we see for this example:

\[ A_M = -200 \quad \leftarrow \text{We knew this already!} \]
And:
\[ F_L(\omega) = \left( \frac{j\omega(0.37)C_i}{1+j\omega(0.37)C_i} \right) \left( \frac{j\omega(C_E/203)}{1+j\omega(C_E/203)} \right) \]

Now, let's define:
\[ \omega_{p1} = \frac{1}{0.37C_i} = \frac{2.7}{C_i} \quad \text{and} \quad \omega_{p2} = \frac{203}{C_E} \]

Thus,
\[ F_L(\omega) = \left( \frac{j(\omega/\omega_{p1})}{1+j(\omega/\omega_{p1})} \right) \left( \frac{j(\omega/\omega_{p2})}{1+j(\omega/\omega_{p2})} \right) \]

Now, functions of the type:
\[ \left( \frac{j(\omega/\omega_p)}{1+j(\omega/\omega_p)} \right) \]

are high-pass functions:
\[ \left| \frac{j(\omega/\omega_p)}{1+j(\omega/\omega_p)} \right|^2 = \frac{(\omega/\omega_p)^2}{1+(\omega/\omega_p)^2} \]

with a 3dB break frequency of \( \omega_p \).
Thus:

\[
\frac{(w/w_p)^2}{1+(w/w_p)^2} \begin{cases} 
\approx 1.0 & \text{for } w > w_p \\
0.5 & \text{for } w = w_p \\
0 & \text{for } w = 0
\end{cases}
\]

As a result, we find that the transfer function:

\[
A_v(o) = F_L(o) A_M \\
= \left\{ \frac{j(w/w_{p1})}{1 + j(w/w_{p1})} \right\} \left\{ \frac{j(w/w_{p2})}{1 + j(w/w_{p2})} \right\} (-200)
\]

will be approximately equal to the midband gain \(A_M = -200\) for all frequencies \(w\) that are greater than both \(w_{p1}\) and \(w_{p2}\).
I.E.,

\[ A_{vo}(\omega) \approx A_M = -200 \quad \text{if } \omega > \omega_{p1} \text{ and } \omega > \omega_{p2} \]

Hopefully, it is now apparent (please tell me it is!) that the lower end of the amplifier bandwidth—specified by frequency \( \omega_L \)—is determined by the larger of the two frequencies \( \omega_{p1} \) and \( \omega_{p2} \)!

The larger of the two frequencies is called the dominant pole of the transfer function \( F_L(\omega) \).

For our example—comparing the two frequencies \( \omega_{p1} \) and \( \omega_{p2} \):

\[ \omega_{p1} = \frac{1}{0.37C_i} = \frac{2.7}{C_i} \quad \text{and} \quad \omega_{p2} = \frac{203}{C_E} \]

it is apparent that the larger of the two (the dominant pole!) is likely \( \omega_{p2} \)—that darn emitter capacitor is the key!

Say we want the common-emitter amplifier in this circuit to have a bandwidth that extends down to \( f_L = 100 \text{ Hz} \)

The emitter capacitor must therefore be:
This certainly is a Capacitor Of Unusual Size!
High-Frequency Response

To determine the high-frequency response of our example common-emitter amp, we simply consider explicitly the parasitic capacitances in the small-signal model, while approximating the COUS as small-signal short-circuits:

Now, since we are **ignoring** the COUS, the function $F_L(\omega)$ that describes the low-frequency response is:

$$F_L(\omega) = 1$$

And so:

$$A_v(\omega) = F_L(\omega) A_M F_H(\omega) = A_M F_H(\omega)$$
We will find that the high-frequency response will (approximately) have the form:

\[ F_H(\omega) = \frac{1}{1 + j(\omega/\omega_p)} \left( \frac{1}{1 + j(\omega/\omega_{p1})} \right) \]

Now, functions of the type:

\[ \left( \frac{1}{1 + j(\omega/\omega_p)} \right) \]

are low-pass functions:

\[ \left| \frac{1}{1 + j(\omega/\omega_p)} \right|^2 = \frac{1}{1 + (\omega/\omega_p)^2} \]

with a 3dB break frequency of \( \omega_p \).
Thus:

\[
\frac{1}{1 + (\omega/\omega_p)^2} = \begin{cases} 
\approx 1.0 & \text{for } \omega < \omega_p \\
0.5 & \text{for } \omega = \omega_p \\
0 & \text{for } \omega \to \infty
\end{cases}
\]

As a result, we find that the transfer function:

\[
A_v(\omega) = A_M \, F_H(\omega)
\]

\[
= (-200) \left[ \frac{1}{1 + j(\omega/\omega_p)} \right] \left[ \frac{1}{1 + j(\omega/\omega_p)} \right]
\]

will be approximately equal to the midband gain \( A_M = -200 \) for all frequencies \( \omega \) that are less than both \( \omega_{p3} \) and \( \omega_{p4} \).

I.E.:

\[
A_v(\omega) \approx A_M = -200 \quad \text{if } \omega < \omega_{p3} \text{ and } \omega > \omega_{p4}
\]

Hopefully, it is now apparent (please tell me it is!) that the higher end of the amplifier bandwidth—specified by frequency \( \omega_H \)—is determined by the smaller of the two frequencies \( \omega_{p3} \) and \( \omega_{p3} \)!

The smaller of the two frequencies is called the dominant pole of the transfer function \( F_H(\omega) \).
Generally speaking, to increase the value $\omega_m$, we must increase the DC bias currents of our amplifier design!