5.9 Frequency Response of the Common-Emitter Amp

Reading **Assignment**: 491-503

Amplifiers made with BJTs are similar to amplifiers made with op-amps—the both exhibit **finite bandwidth**.

HO: AMPLIFIER BANDWIDTH

The gain within the bandwidth is usually constant with respect to frequency—we call this value the **mid-band gain**.

HO: MID-BAND GAIN

Large capacitors (e.g., COUS) determine the low-frequency limit of amplifier bandwidth. We can explicitly determine this value be analyzing the **low-frequency** small-signal circuit.

HO: THE LOW-FREQUENCY RESPONSE

Parasitic capacitors (e.g., C_π) determine the high-frequency limit of amplifier bandwidth. We can explicitly determine this value be analyzing the **high-frequency** small-signal circuit.

HO: THE HIGH-FREQUENCY RESPONSE

Amplifier Bandwidth

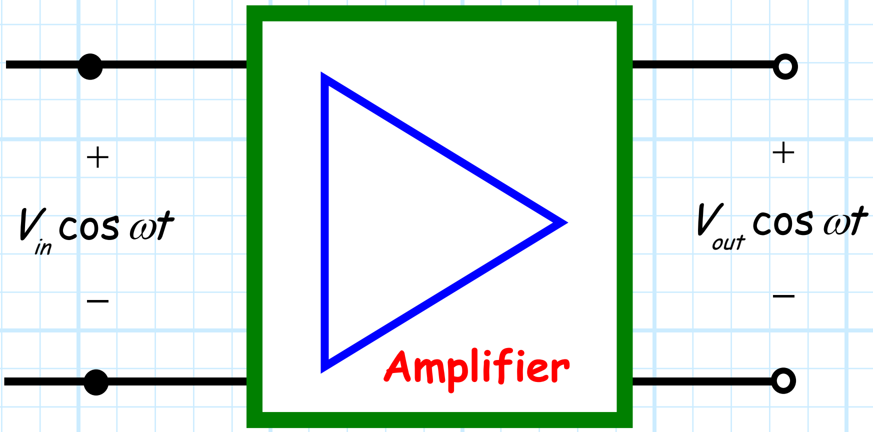
BJT amplifiers are band-limited devices—in other words, they exhibit a **finite bandwidth**.

Q: *???*

A: Say the input to a BJT small-signal amplifier is **the** eigen function of **linear**, time-invariant system:

$$V_{in} \cos \omega t = V_{in} \operatorname{Re}\left\{ e^{-j\omega t} \right\}$$

Since the small-signal BJT amp is (approximately) a **linear** system, the output will likewise be **the** eigen function—an **undistorted** sinusoidal function of precisely the same frequency ω as the input!



Q: *Of course that's true! We know that:*

$$v_{out}(t) = A_{vo}\, v_{in}(t)$$

Therefore the magnitudes of the input and output sinusoids are related as:

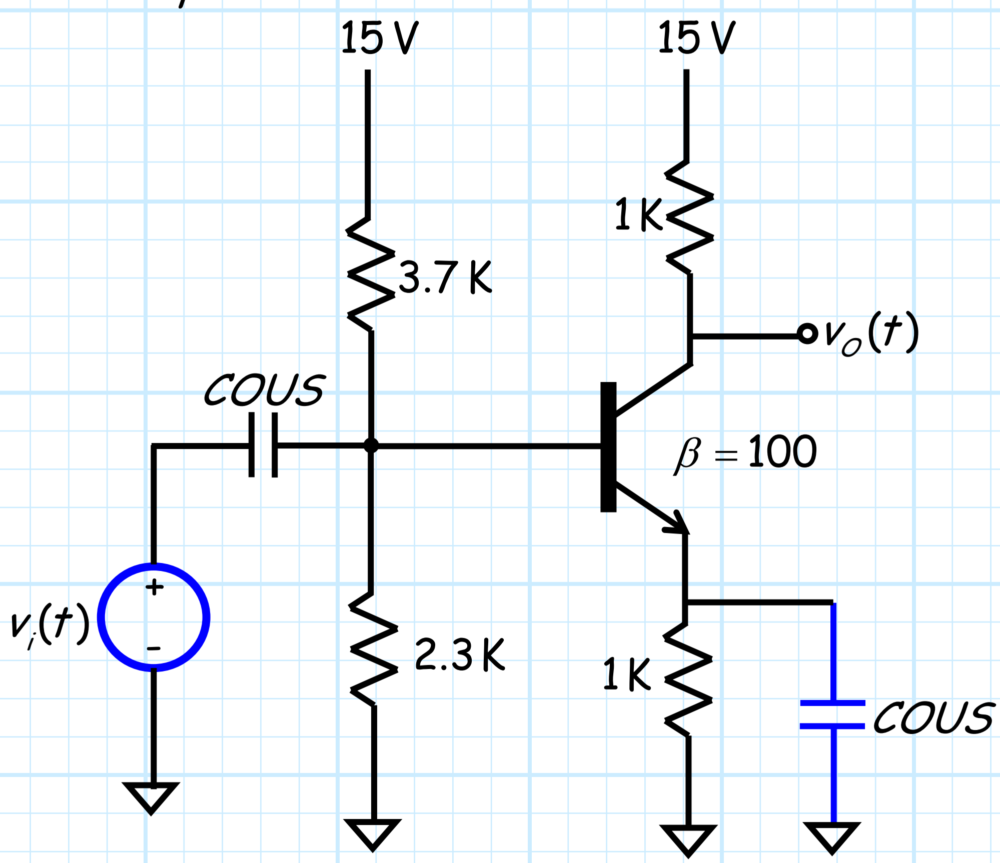
$$V_{out} = A_{vo}\, V_{in}$$

Right?

A: Not **necessarily**!

The small-signal, open-circuit voltage **gain** of a BJT amplifier **depends** on the **frequency** ω of the input signal!

Q: *Huh!?! We determined earlier that the small-signal voltage gain of this amplifier:*



was:

$$A_{vo} = \frac{v_o}{v_i} = -200$$

So then if the small-signal input is:

$$v_i(t) = V_{in} \cos \omega t$$

isn't the small-signal output simply:

$$v_o(t) = -200\, V_{in} \cos \omega t \qquad ?????????$$

A: Maybe—or maybe **not**!

Again, the gain of the amplifier is **frequency dependent**. We find that if ω is **too high** (i.e., large) or **too low** (i.e., small), then the output might be much less than the 200 times larger than the input (e.g., only 127.63 times larger than the input—Doh!).

Now, the signal frequencies ω for which

$$v_o(t) = -200\, V_{in} \cos \omega t$$

is an **accurate** statement, are frequencies that are said to lie **within the bandwidth** of this amplifier (ω is just right!).

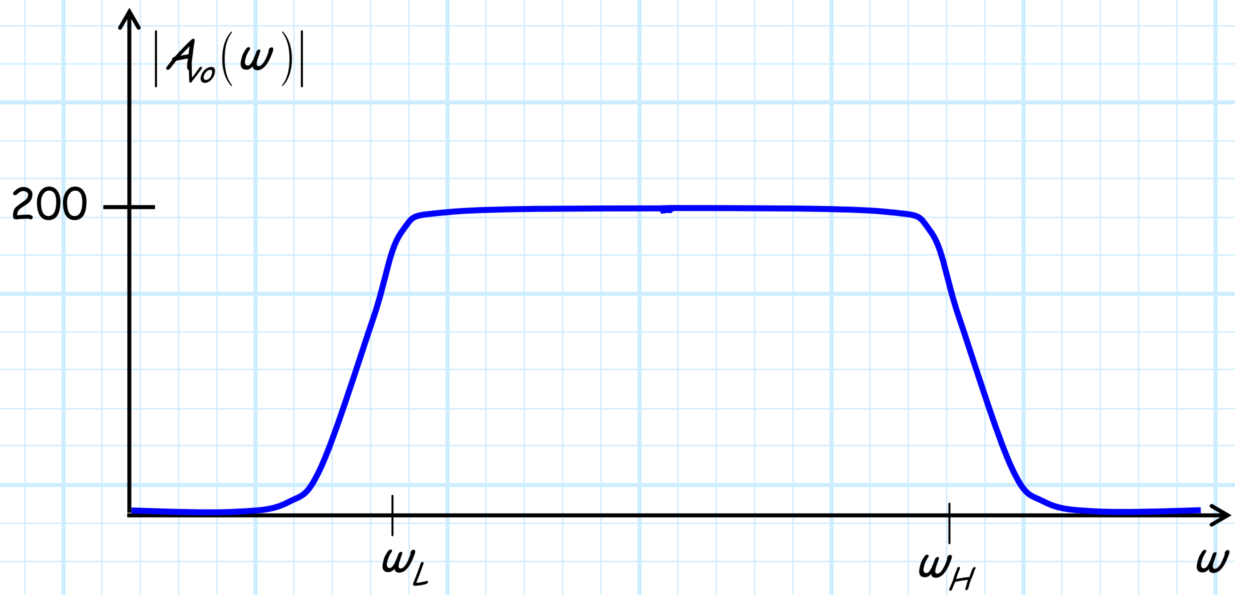
Conversely, frequencies ω for which:

$$v_o(t) \neq -200\, V_{in} \cos \omega t$$

are frequencies ω that lie **outside** this amplifier's bandwidth.

Fortunately, the frequencies that compose an amplifier's bandwidth typically form a **continuum**, such that the frequencies outside this bandwidth are either **higher** or **lower** than all frequencies within the bandwidth.

Perhaps a **plot** would help.



The frequencies **between** ω_L and ω_H thus lie within the **bandwidth** of the amplifier. The gain within the bandwidth is sometimes referred to as the **midband gain**.

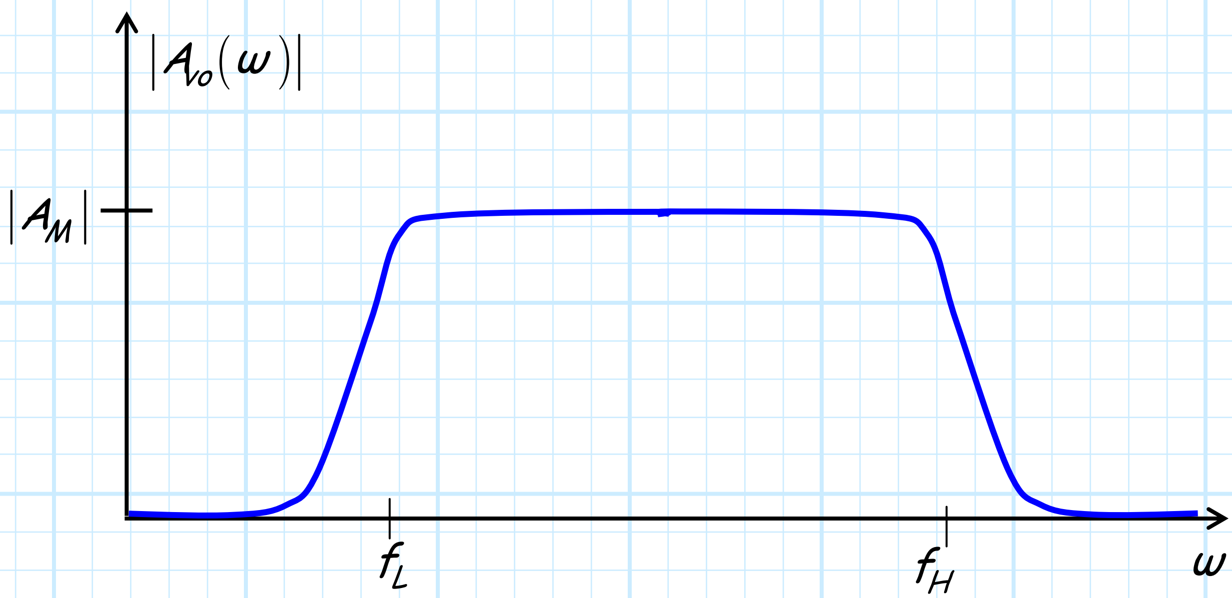
For signals with frequencies less than $\omega_L\,(f_L)$, the amplifier gain will be **less** than the midband gain—likewise for frequencies greater than $\omega_H\,(f_H)$.

Q: *So what then is the value:*

$$A_{vo} = \frac{v_o}{v_i} = -200$$

determined for the example amplifier? It doesn't seem to be a function of frequency!

A: The value -200 calculated for this amplifier is the **midband gain**—it's the gain exhibited for all signals that lie within the amplifier bandwidth. Your **book** at times uses the variable A_M to denote this value:



Q: *So it's actually the **midband gain** that we've been determining from our small-signal circuit analysis (e.g. $A_M = -200$)?*

A: That's **exactly** correct!

Q: *So how do we determine the **frequency dependent gain** $A_{vo}(\omega)$? More specifically, how do we determine midband gain A_M, along with f_L and f_H ?*

A: The function $A_{vo}(\omega)$ is simply the **eigen value** of the **linear operator** relating the small-signal input and the small signal output:

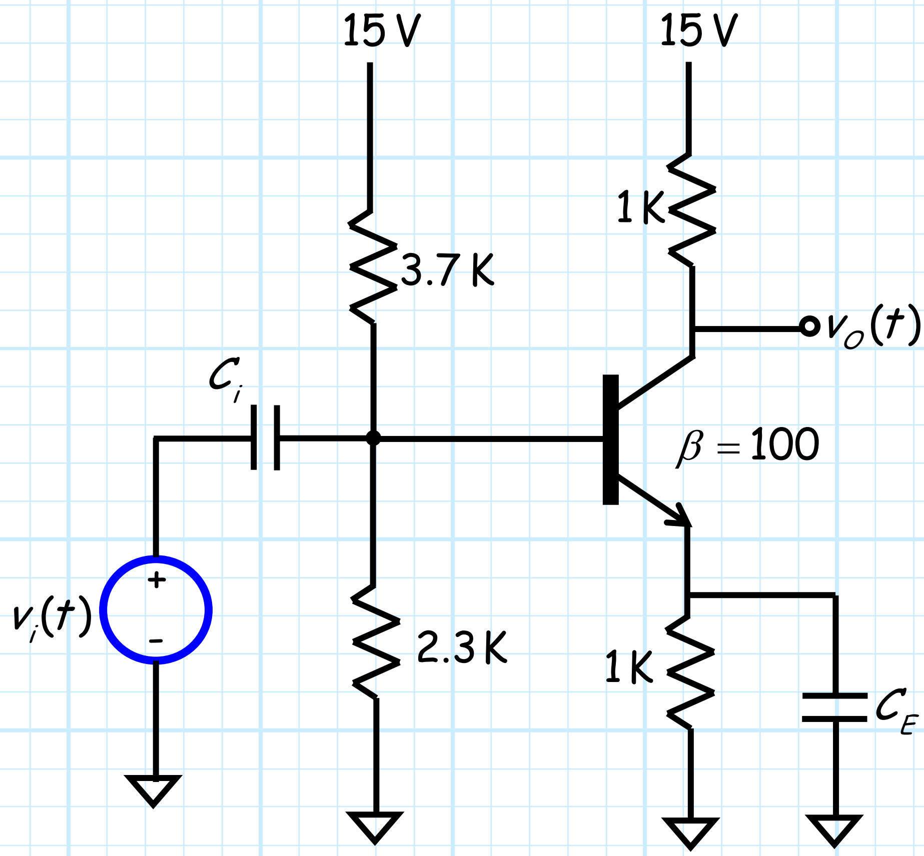
$$v_o(t) = \mathcal{L}\{v_i(t)\} \qquad \Rightarrow \qquad V_o(\omega) = A_{vo}(\omega)\, V_i(\omega)$$

Q: *Yikes! How do we determine the eigen value of this linear operator?*

A: We simply **analyze** the small-signal circuit, determining $V_o(\omega)$ in terms of $V_i(\omega)$.

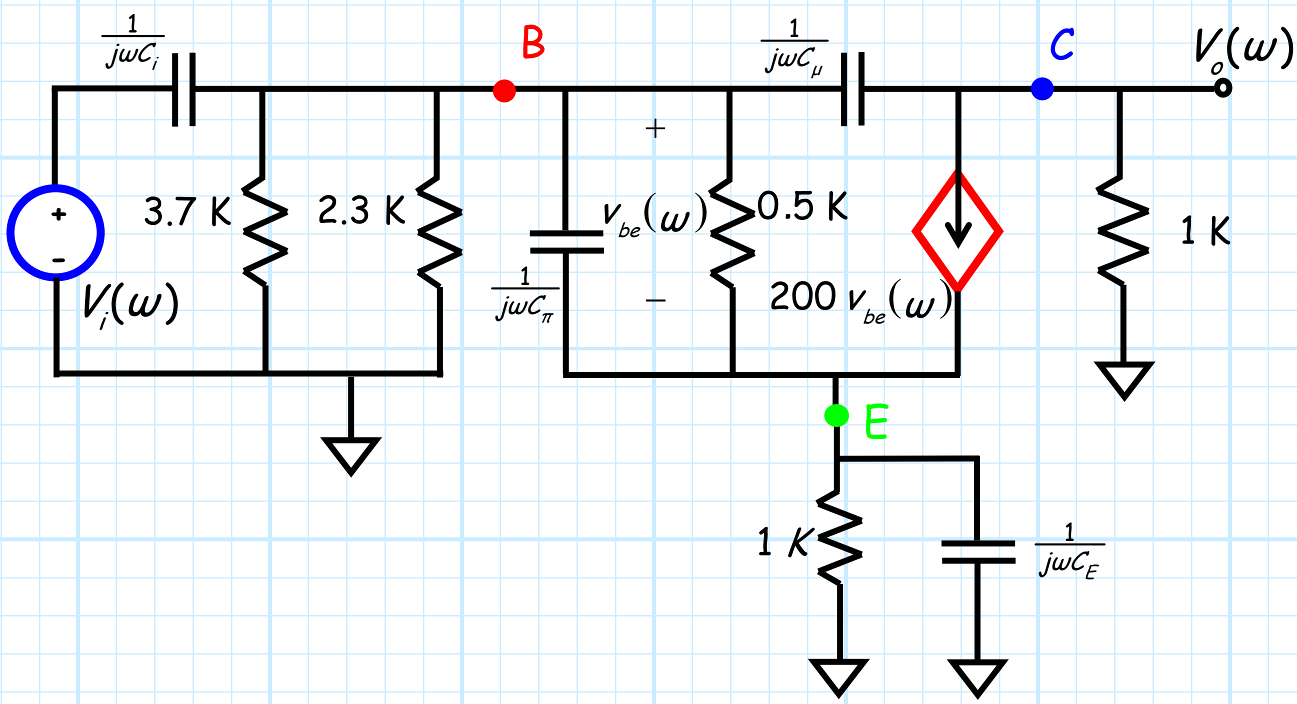
Specifically, we must e**xplicitly** consider the capacitance in the small-signal amplifier—**no longer** can we make **approximations**!

So, instead of **vaguely** labeling large capacitors as **C**apacitors **O**f **U**nusual **S**ize, let's explicitly consider the **exact values** of these large capacitors:



Likewise, we must consider the **parasitic capacitances** of the BJT—specifically C_μ and C_π.

The **small-signal circuit**—when we explicitly consider these capacitances—is thus:



We **analyze** this circuit to determine $V_o(\omega)$, and then the eigen value—the small-signal gain—is determined as:

$$A_{vo}(\omega) = \frac{V_o(\omega)}{V_i(\omega)}$$

Q: *So what again is the meaning of $V_i(\omega)$ and $V_o(\omega)$?*

A: It's the **Fourier transform** of $v_i(t)$ and $v_o(t)$!

$$V_i(\omega) = \int_{-\infty}^{\infty} v_i(t)\, e^{j\omega t}\, dt \qquad \text{and} \qquad V_o(\omega) = \int_{-\infty}^{\infty} v_o(t)\, e^{j\omega t}\, dt$$

Q: So—I can't recall—what's the **relationship** between $v_i(t)$ and $v_o(t)$?

A: If:

$$V_o(\omega) = A_{vo}(\omega)\, V_i(\omega)$$

Then in the time domain, we find that the input and output are related by the **always enjoyable convolution** integral!!!

$$v_o(t) = \int_{-\infty}^{\infty} g(t-t')\, v_i(t')\, dt'$$

where the **impulse response** of the amplifier is of course:

$$g(t) = \int_{-\infty}^{\infty} A_{vo}(\omega)\, e^{-j\omega t}\, d\omega$$

Q: *What the heck? What happened to solutions like:*

$$v_o(t) = -200\, v_i(t) \quad ??$$

A: This result implies that the **impulse response** of the amplifier is:

$$g(t) = -200\, \delta(t)$$

Such that:

$$\begin{aligned} v_o(t) &= \int_{-\infty}^{\infty} g(t-t')\, v_i(t')\, dt' \\ &= -200 \int_{-\infty}^{\infty} \delta(t-t')\, v_i(t')\, dt' \\ &= -200\, v_i(t) \end{aligned}$$

Q: *You say that the result:*

$$v_o(t) = -200\, v_i(t)$$

"implies" *that the impulse response of the amplifier is:*

$$g(t) = -200\, \delta(t)$$

Are you saying the impulse response of the common-emitter example is ***not*** *this function?*

A: It is definitely **not** that function. The impulse response

$$g(t) = -200\, \delta(t)$$

is **ideal**—the impulse response of an amplifier with an **infinite bandwidth**!

Q: *So* ***all*** *our small-signal analysis up to this point has been incorrect and* ***useless****???*

A: Not at all! The small-signal gain we have been evaluating up to this point (e.g., -200) is the amplifier **midband gain** A_M .

As long as the small-signal input $v_i(t)$ **resides completely** within the amplifier **bandwidth**, then the output will be:

$$v_o(t) = -200\, v_i(t)$$

The **problem** occurs when the input signal lies—at least partially—**outside** the amplifiers bandwidth.

In that case, we find that:

$$v_o(t) \neq -200\, v_i(t)$$

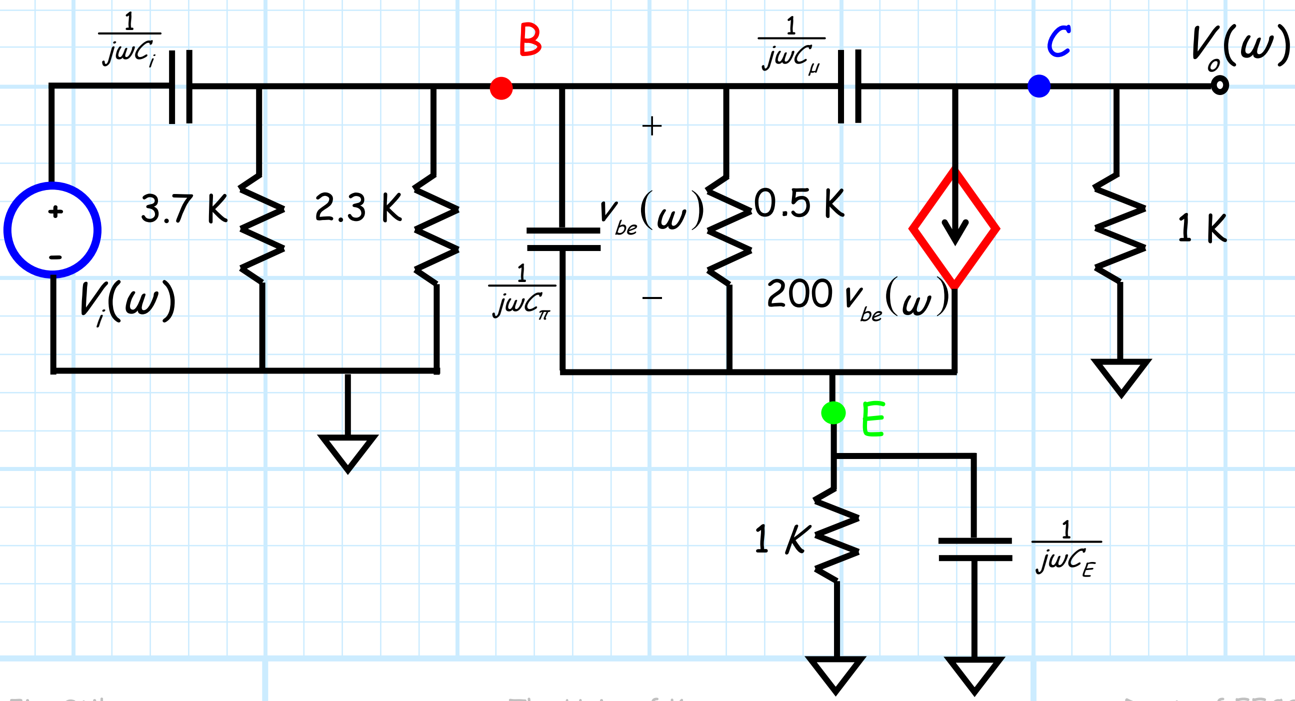
And instead:

$$v_o(t) = \int_{-\infty}^{\infty} g(t-t')\, v_i(t')\, dt'$$

where:

$$g(t) = \int_{-\infty}^{\infty} A_{vo}(\omega)\, e^{-j\omega t}\, dw$$

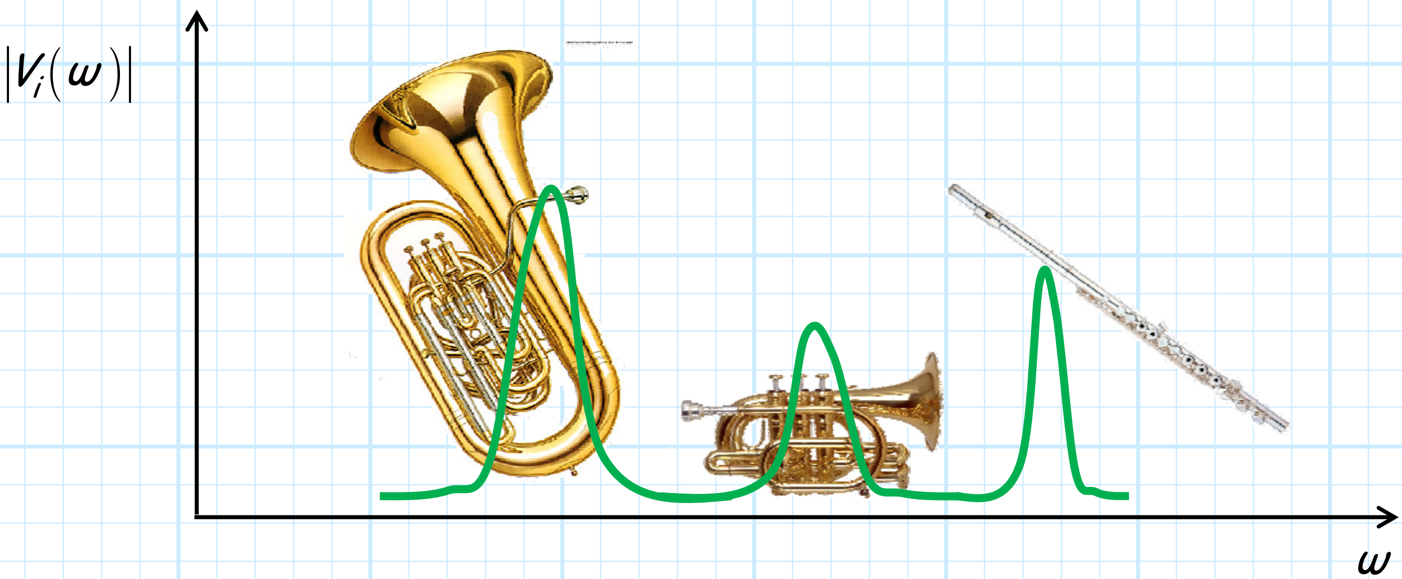
and the eigen value $A_{vo}(\omega) = V_o(\omega)/V_i(\omega)$ is determined by evaluating this small-signal circuit:



Q: *What do you mean when you say that a signal lies "**within** the amplifier bandwidth" or "**outside** the amplifier bandwidth? **How can we tell?***

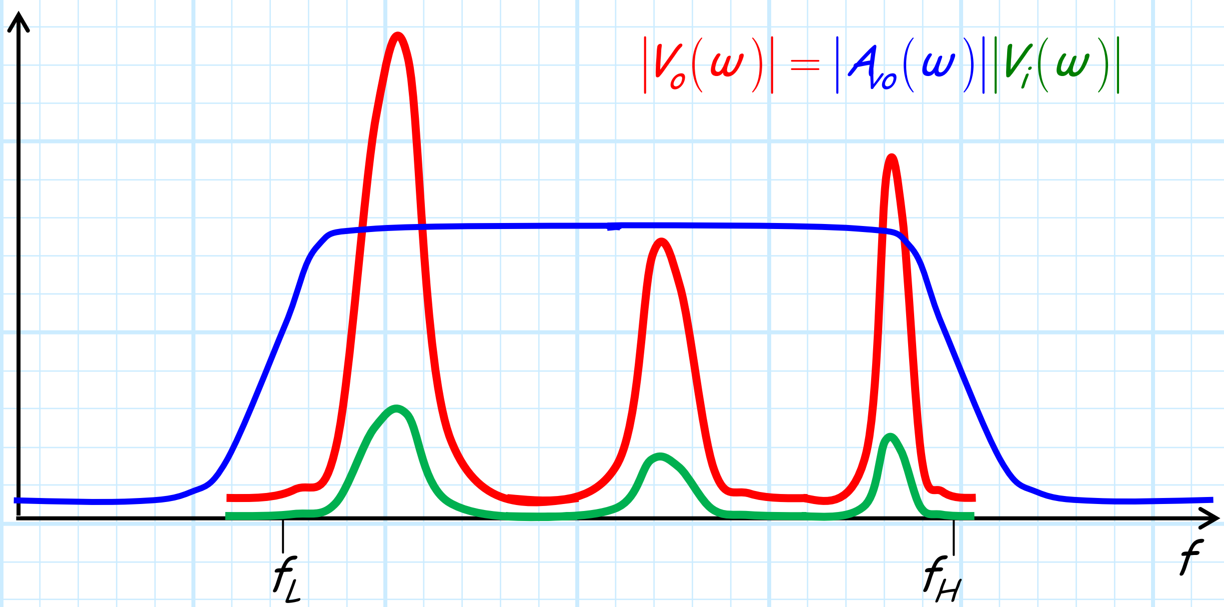
A: Use the Fourier Transform!

If we plot the magnitude of the Fourier Transform $V_i(\omega)$ of the input signal $v_i(t)$, we can see the **spectrum** of the input signal:



For example, if you are attempting to amplify a signal representing the audio of **symphonic music**, the spectrum $|V_i(\omega)|$ will include **low-frequency** signals (e.g., from the tubas), **mid-range** frequency signals (e.g., from the trumpets), and **high-frequency** signals (e.g., from the flutes).

Now, if it is your desire to reproduce **exactly** this music at the output of your amplifier, then the amplifier bandwidth must be **wide enough** to include **all** these spectral components !



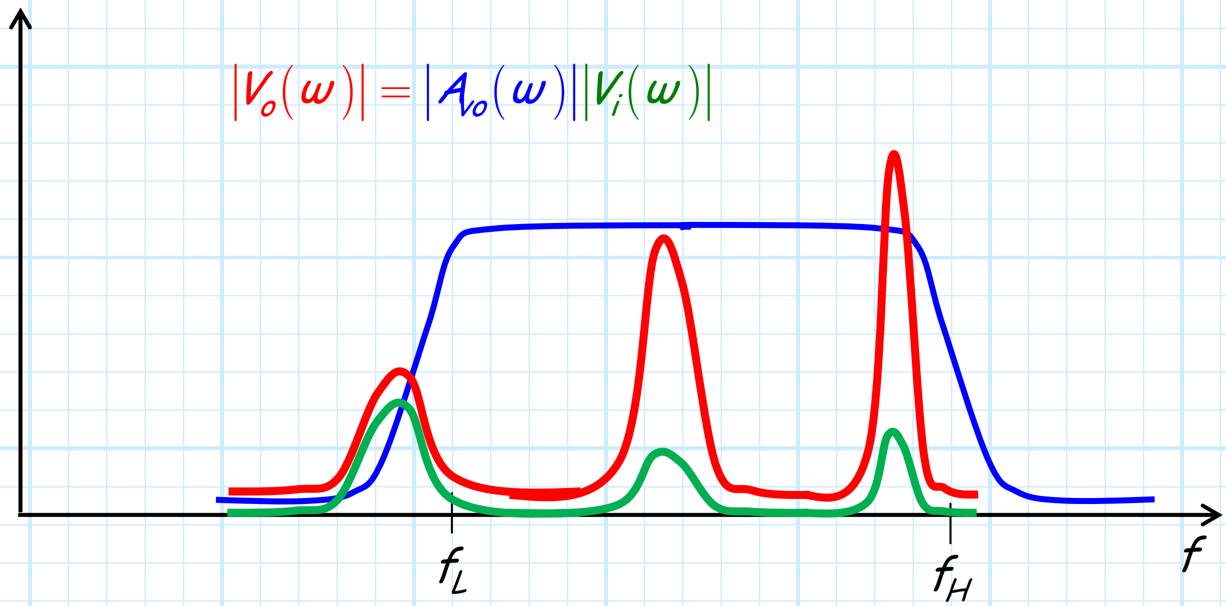
For the case above, the input signal resides **completely** within the bandwidth of the amplifier (i.e., between f_L and f_H), and so we find (for $A_M = -200$) that:

$$|V_o(\omega)| = (200)|V_i(\omega)|$$

and:

$$v_o(t) = -200\ v_i(t)$$

However, if the input spectrum resides (at least partially) **outside** the amplifier bandwidth, e.g.:



then we find that:

$$|V_o(\omega)| \neq (200)|V_i(\omega)|$$

and:

$$v_o(t) \neq -200\, v_i(t) \quad !!!!$$

Instead, we find the more general (and more **difficult!**) expressions:

$$V_o(\omega) = A_{vo}(\omega)\, V_i(\omega)$$

and:

$$v_o(t) = \int_{-\infty}^{\infty} g(t-t')\, v_i(t')\, dt'$$

where the **impulse response** of the amplifier is:

$$g(t) = \int_{-\infty}^{\infty} A_{vo}(\omega)\, e^{-j\omega t} d\omega$$

Q: *So just what **causes** this amplifier to have a finite bandwith?*

A: For mid-band frequencies f_m (i.e., between $f_L < f_m < f_H$), we will find that the Capacitors Of Unusual Size exhibit an impedance that is pretty small—approximately an **AC short** circuit:

$$\left| Z_{COUS}(\omega_m) \right| = \frac{1}{\omega_m C_{ous}} \cong 0$$

Likewise, the **tiny parasitic capacitances** C_μ and C_π exhibit an impedance that is very large for mid-band frequencies—approximately an **open** circuit:

$$\left| Z_{C_\pi}(\omega_m) \right| = \frac{1}{\omega_m C_\pi} \cong \infty$$

However, when the signal frequency ω drops too **low**, the COUS will **no longer** be a small-signal short.

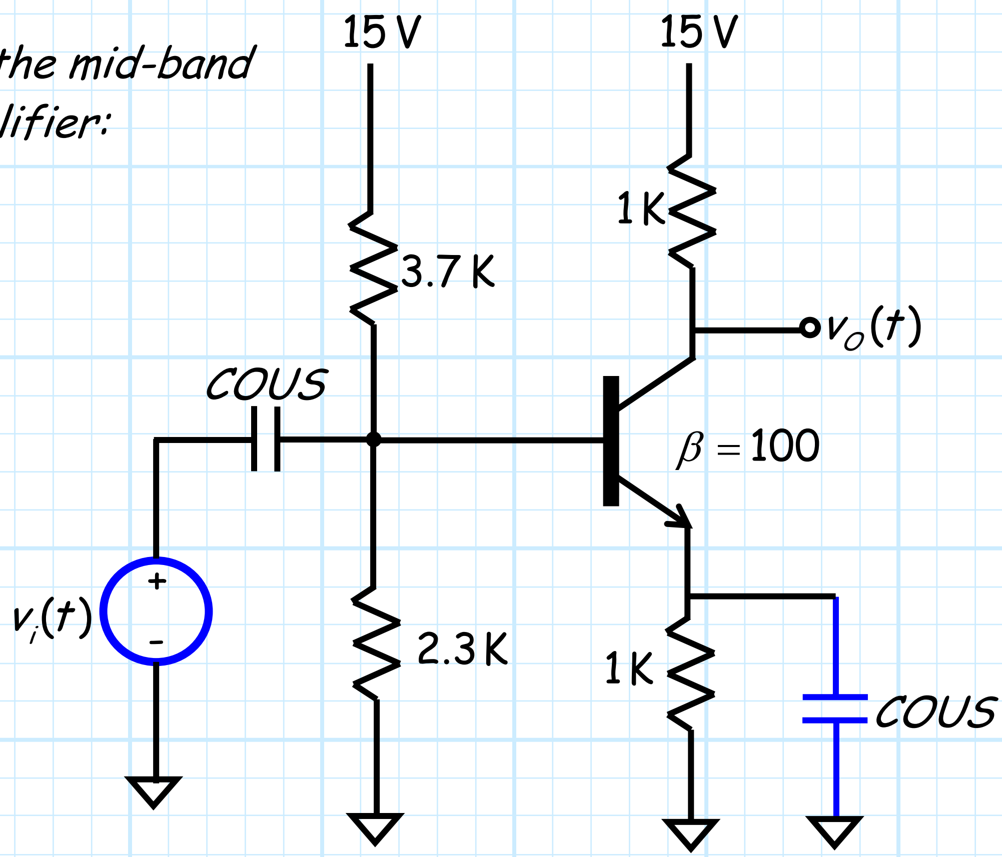
The result is that the amplifier **gain is reduced**—the values of the COUS determine the **low-end amplifier bandwidth** f_L.

Likewise, when the signal frequency ω is **too high**, the parasitic caps will **no longer** be a small-signal **open**.

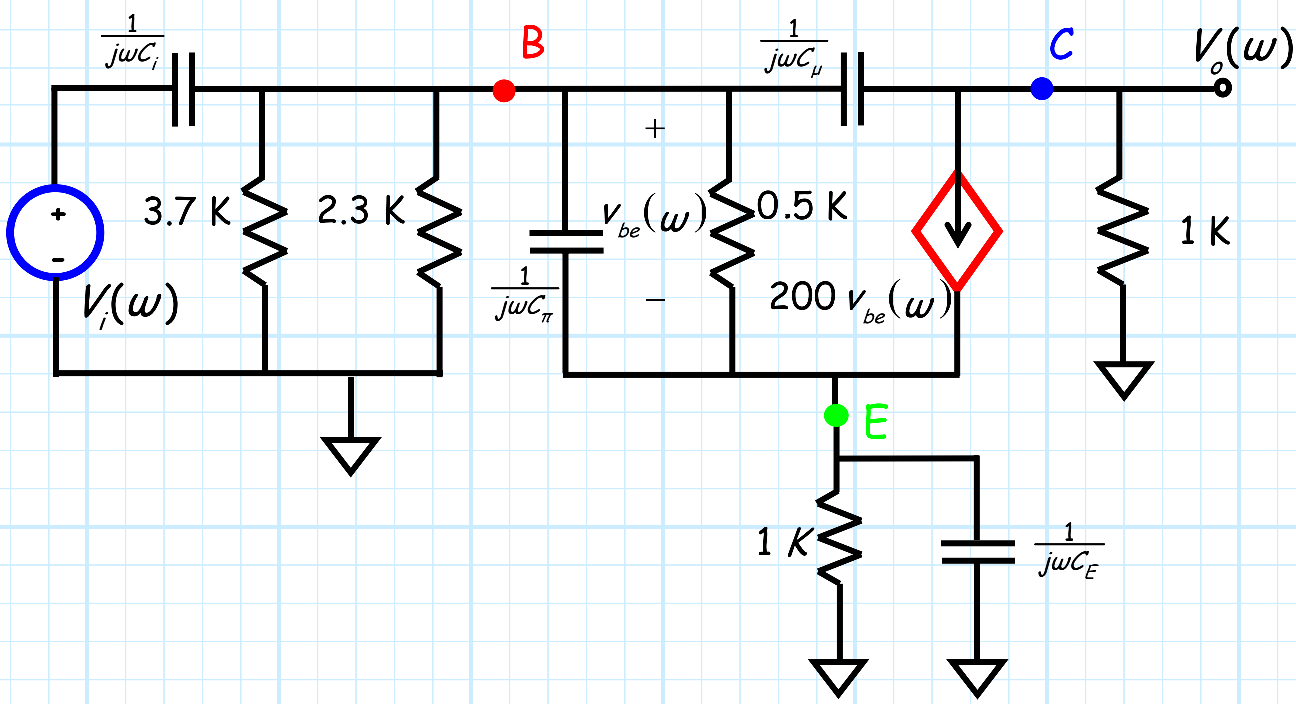
The result is that the amplifier **gain is reduced**—the values of the **parasitic capacitors** determine the **high-end amplifier bandwidth** f_H.

Mid-band Gain

Q: *So, to find the mid-band gain of* ***this*** *amplifier:*



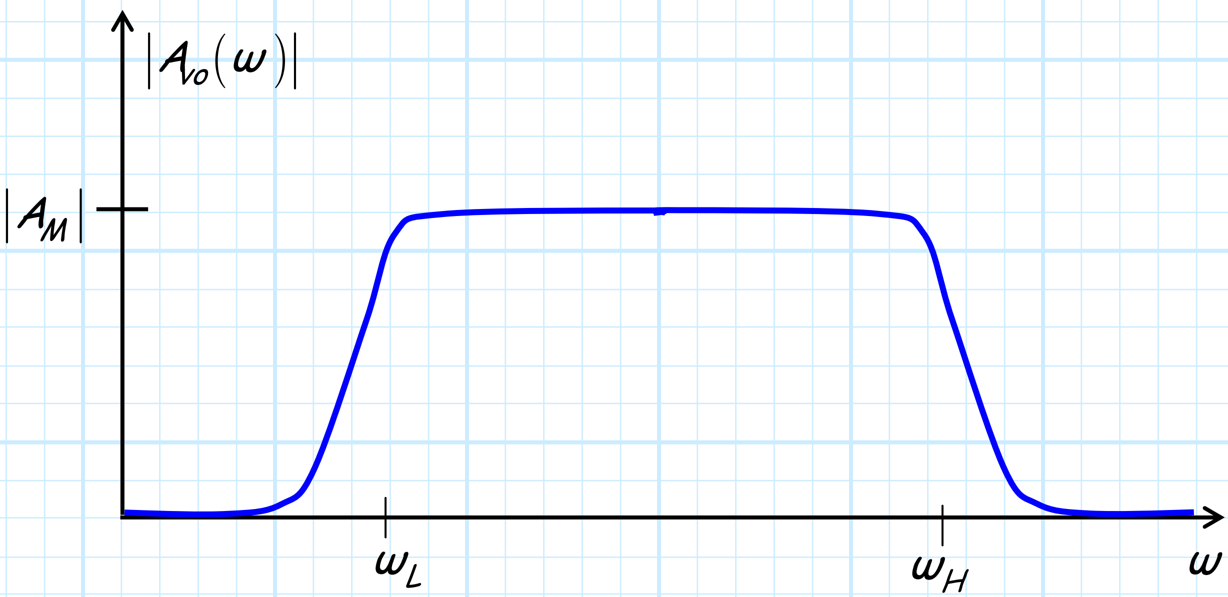
we must find the analyze this small signal circuit:



to determine:

$$A_{vo}(\omega) = \frac{V_o(\omega)}{V_i(\omega)}$$

and then plotting the magnitude:



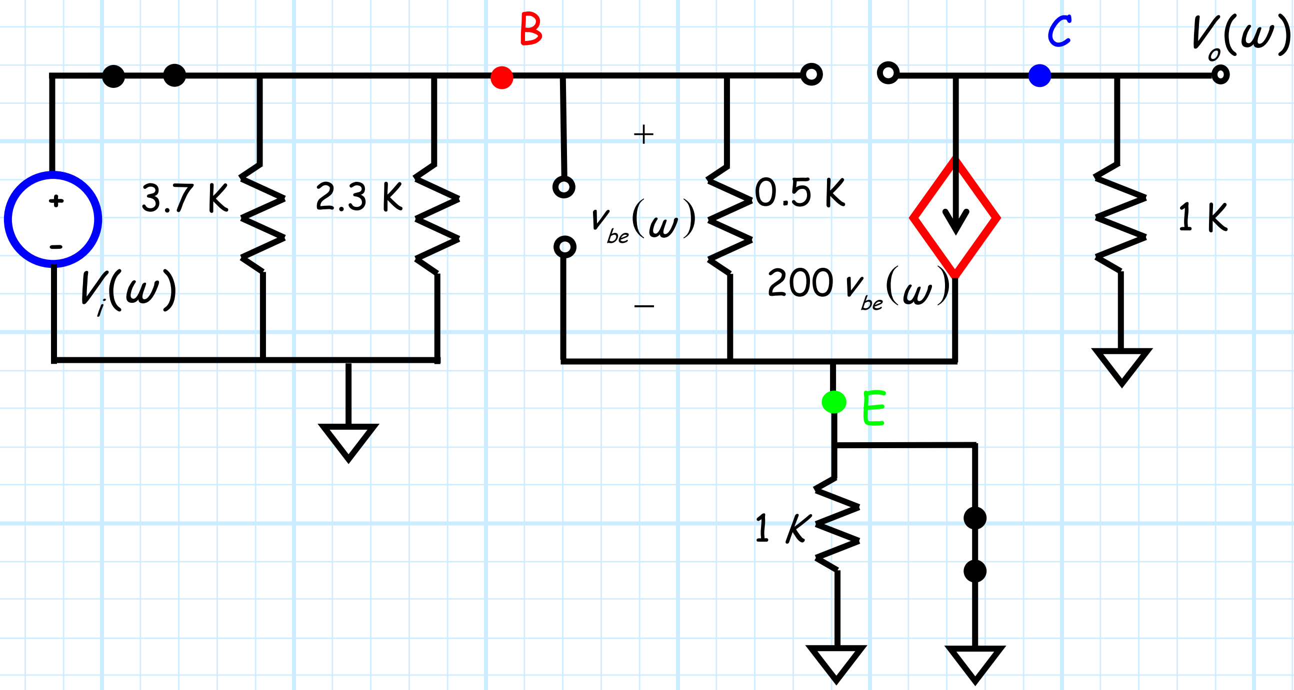
we determine mid-band gain A_M, right?

A: You **could** do all that, but there is an easier way.

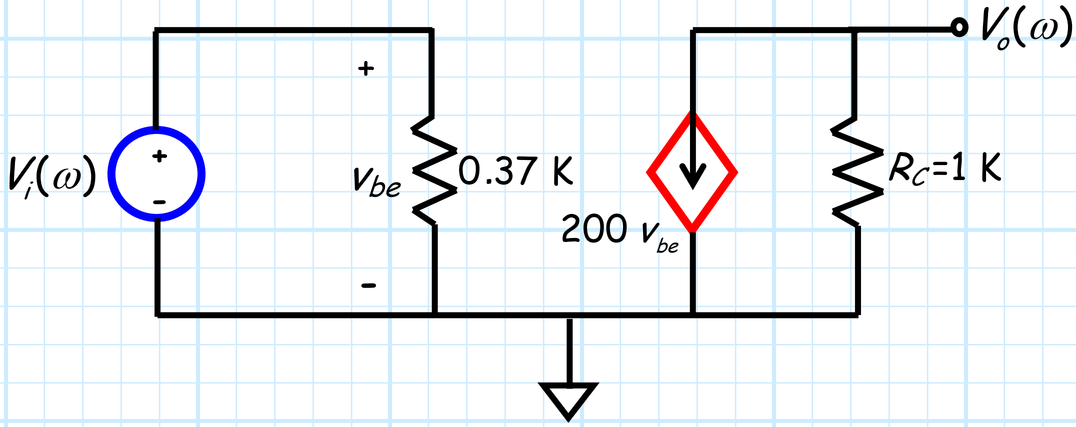
Recall the midband gain is the value af $|A_{vo}(\omega)|$ for frequencies within the amplifier bandwidth. For those frequencies, the AC coupling capacitors (i.e., COUS) are approximate AC short-circuits (i.e., very low impedance) .

Likewise, for the signal frequencies within the amplifier bandwidth, the parasitic BJT capacitances are approximate AC open-circuits (i.e., very high impedance).

Thus, we can apply these approximations to the capacitors in our small-signal circuit:



Now simplifying this circuit (look, no capacitors!):



Q: *Hey wait! Isn't this the same small-signal circuit that we analyzed earlier, where we found that:*

$$v_o(t) = -200\, v_i(t) \quad ??$$

A: It is exactly!

All of the small-signal analysis that we performed **previously** (i.e., the circuits with **no capacitors**!) actually provided us with the **mid-band** amplifier gain.

Taking the Fourier transform of the equation above:

$$\begin{aligned} V_o(\omega) &= -200\, V_i(\omega) \\ &= e^{j\pi} 200\, V_i(\omega) \end{aligned}$$

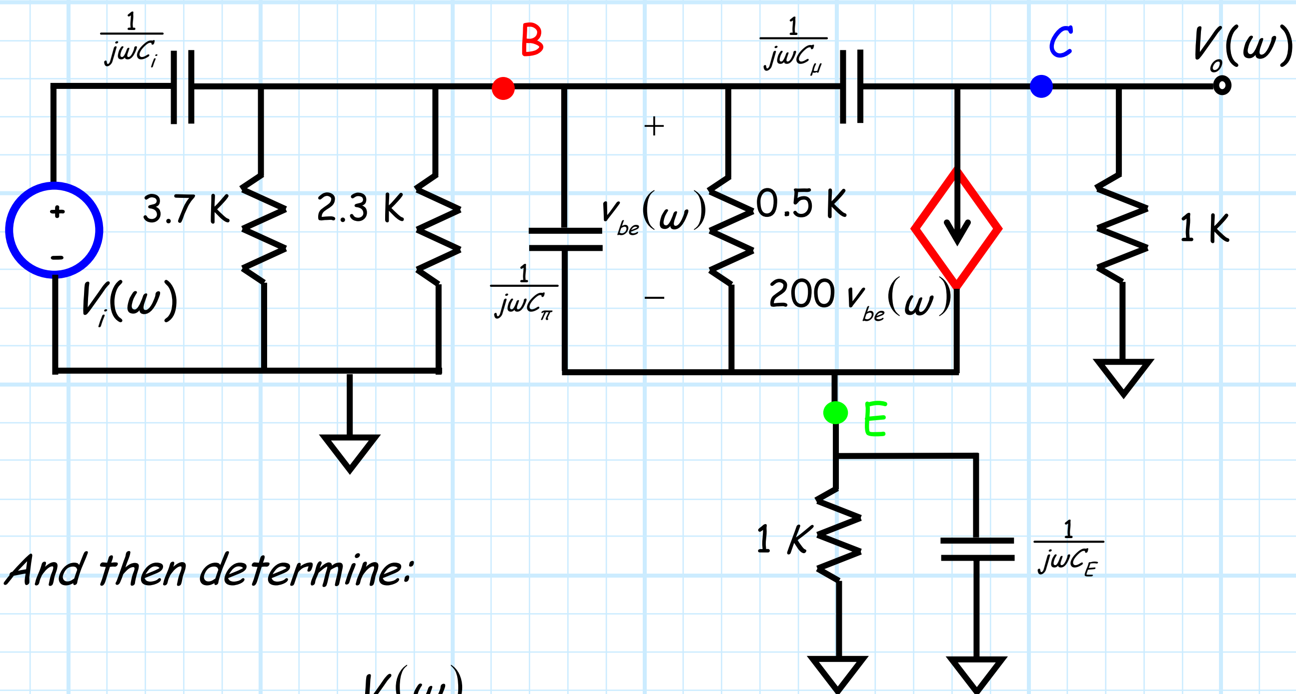
Thus, the midband gain of this amplifier is:

$$A_M = -200 = e^{j\pi} 200$$

Low-Frequency Response

Q: *OK, I see how to determine mid-band gain, but what about determining amplifier **bandwidth**?*

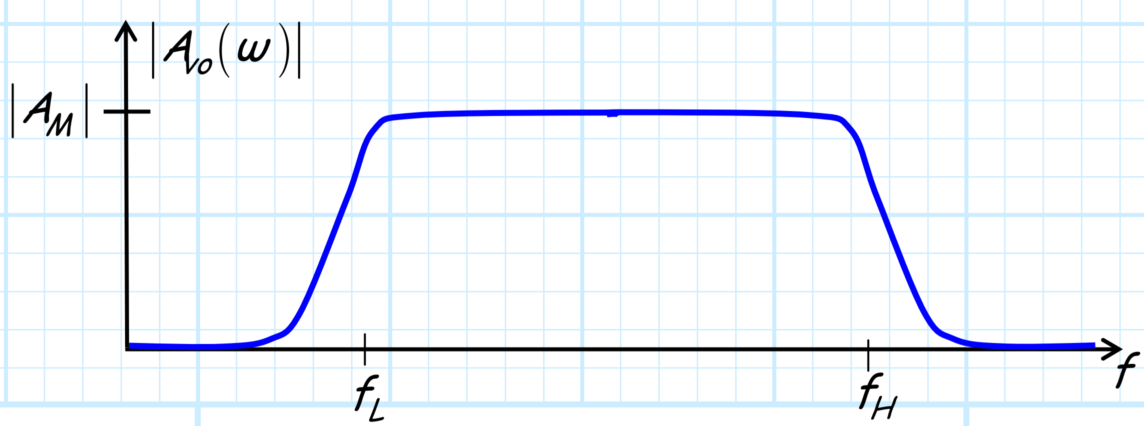
It seems like I have no alternative but to analyze the exact small-signal circuit (explicitly considering all capacitances):



And then determine:

$$A_{vo}(\omega) = \frac{V_o(\omega)}{V_i(\omega)}$$

*and then **plot** the magnitude:*



And then from the plot determine the amplifier bandwidth (i.e., determine f_L and f_H)?

A: You **could** do all that, but there is an easier way.

An amplifier frequency response $A_{vo}(\omega)$ (i.e., its eigen value!) can generally be expressed as the product of **three** distinct terms:

$$A_{vo}(\omega) = F_L(\omega)\, A_M\, F_H(\omega)$$

The middle term is the of course the **mid-band gain**—a number that is not frequency dependent.

The function $F_L(\omega)$ describes the **low-frequency response** of the amplifier—from it we can determine the lower cutoff frequency f_L.

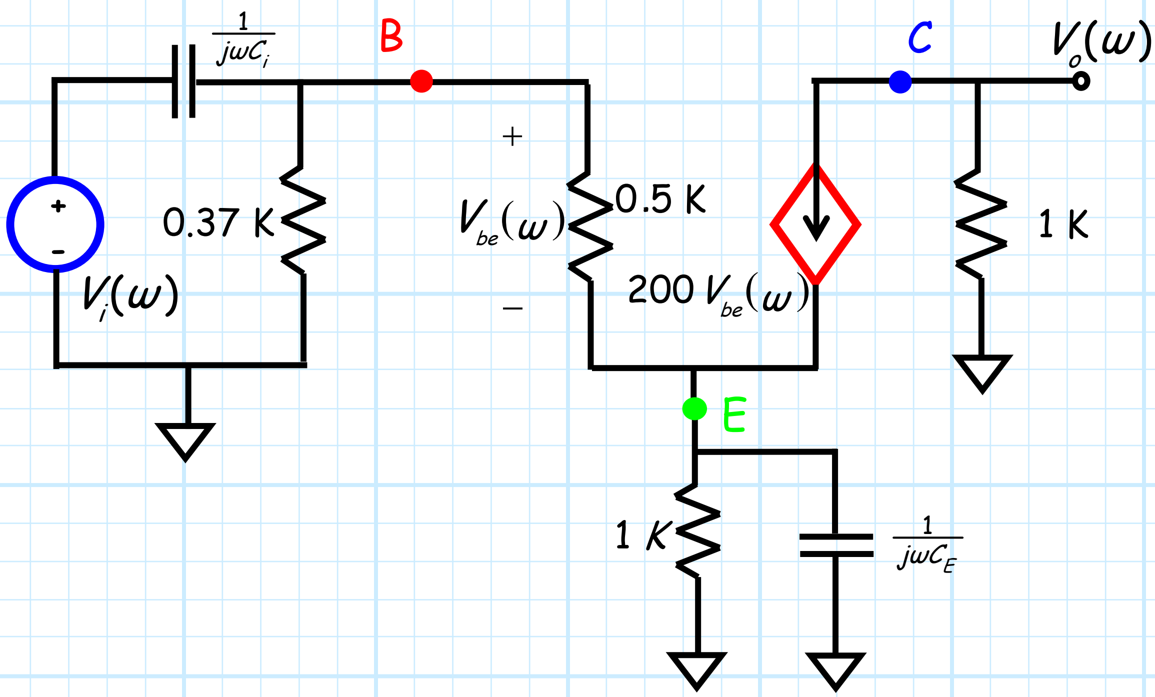
Conversely, the function $F_H(\omega)$ describes the **high-frequency** response of the amplifier—from it we can determine the upper cutoff frequency f_H.

Q: *So just how do we determine these functions $F_L(\omega)$ and $F_H(\omega)$??*

A: The low-frequency response $F_L(\omega)$ is dependent **only** on the large capacitors (COUS) in the amplifier circuit. In other words the parasitic capacitances have **no affect** on the low-frequency response.

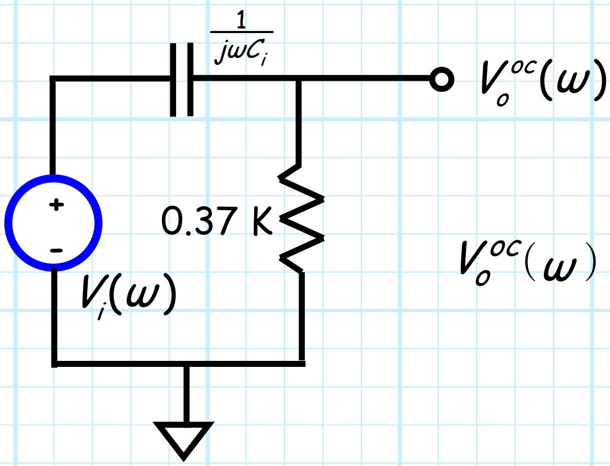
Thus, we simply "ignore" the **parasitic** capacitances when determining $F_L(\omega)$!

For example, say we include the COUS in our common-emitter example, but **ignore** C_μ and C_π. The resulting small-signal circuit is:



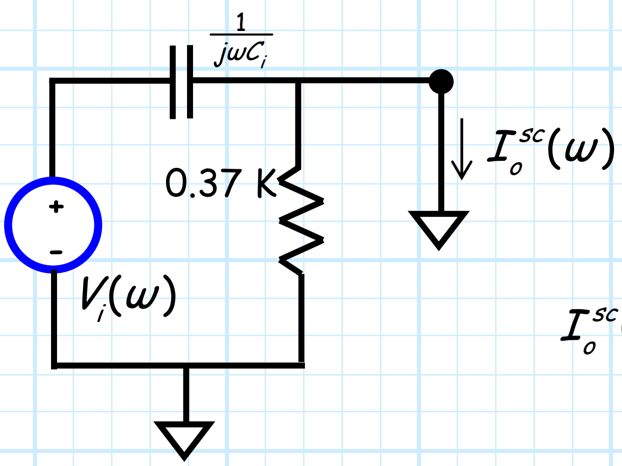
To simplify this analysis, we first determine the **Thevenin's** equivalent circuit of the portion of the circuit connected to the base.

We start by finding the **open-circuit voltage**:



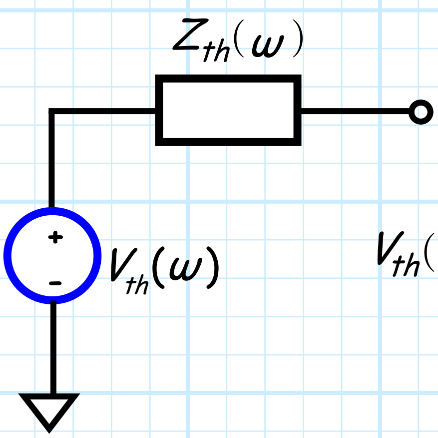
$$V_o^{oc}(\omega) = V_i(\omega)\left(\frac{0.37}{0.37 + \frac{1}{j\omega C_i}}\right)$$
$$= V_i(\omega)\left(\frac{j\omega(0.37)C_i}{1 + j\omega(0.37)C_i}\right)$$

And the **short-circuit** output current is:



$$I_o^{sc}(\omega) = \frac{V_i(\omega)}{\frac{1}{j\omega C_i}} = V_i(\omega)\left(j\omega C_i\right)$$

And thus the **Thevenin's equivalent** source is:



$$V_{th}(\omega) = V_o^{oc}(\omega)$$
$$= V_i(\omega)\left(\frac{j\omega(0.37)C_i}{1 + j\omega(0.37)C_i}\right)$$

$$Z_{th}(\omega) = \frac{V_o^{oc}(\omega)}{I_o^{sc}(\omega)}$$

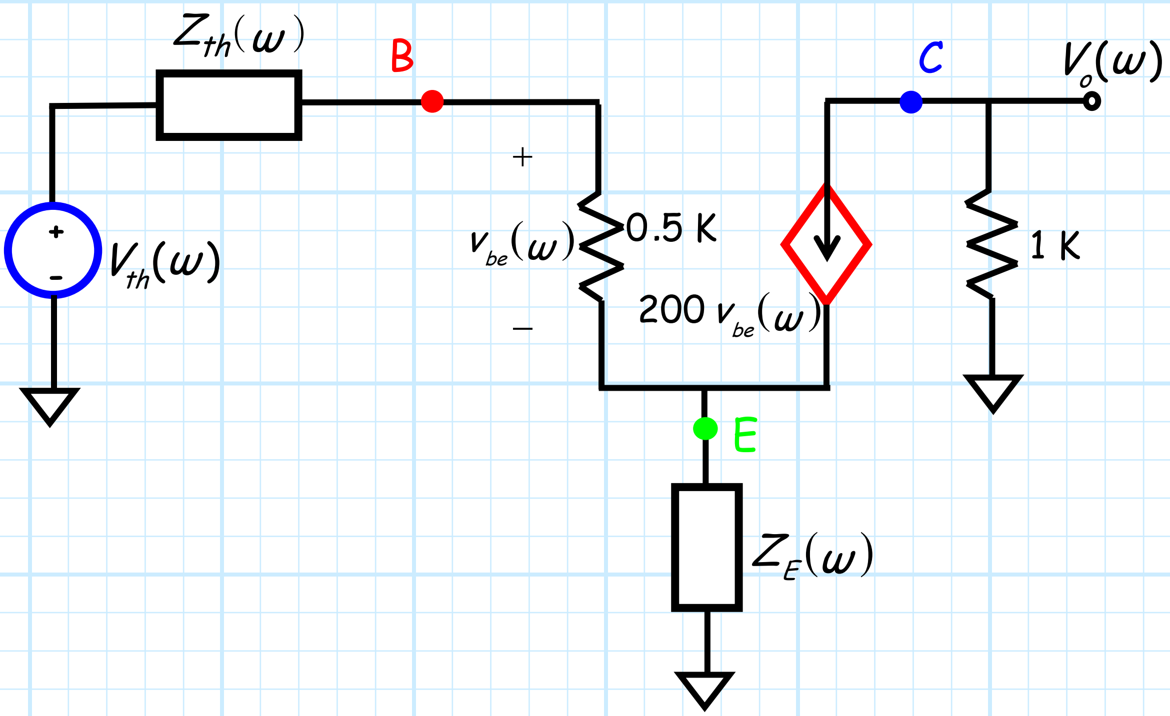
$$= \left(\frac{j\omega(0.37)C_i}{1+j\omega(0.37)C_i}\right)\left(\frac{1}{j\omega C_i}\right)$$

$$= \frac{(0.37)}{1+j\omega(0.37)C_i}$$

Likewise, the two parallel elements on the emitter terminal can be combined:

$$Z_E(\omega) = \frac{\frac{1}{j\omega C_E}}{1+\frac{1}{j\omega C_E}} = \frac{1}{1+j\omega C_E}$$

Thus, the small-signal circuit is **now**:



From KVL:

$$0 + V_{th} - I_b\left(Z_{th} + 0.5\right) - (\beta + 1) I_b Z_E = 0$$

$$\Rightarrow \qquad I_b = \frac{V_{th}}{Z_{th} + 0.5 + 101 Z_E}$$

From Ohm's Law:

$$V_{be} = 0.5 I_b$$

Therefore:

$$\begin{aligned} V_o(\omega) &= -200\, V_{be}\,(1) \\ &= -200(0.5)\frac{V_{th}(\omega)}{Z_{th} + 0.5 + 101 Z_E} \\ &= V_{th}(\omega)\left(\frac{-100}{Z_{th} + 0.5 + 101 Z_E}\right) \end{aligned}$$

Inserting the expressions for the Thevenin's equivalent source, as well as Z_E.

$$\begin{aligned} V_o(\omega) &= V_{th}(\omega)\left(\frac{-100}{Z_{th} + 0.5 + 101 Z_E}\right) \\ &= V_i(\omega)\left(\frac{jw(0.37)C_i}{1 + jw(0.37)C_i}\right)\left(\frac{-200}{\dfrac{2(0.37)}{1 + jw(0.37)C_i} + 1 + \dfrac{202}{1 + jwC_E}}\right) \end{aligned}$$

Now, it can be shown that:

$$\frac{1}{\dfrac{2(0.37)}{1+jw(0.37)C_i}+1+\dfrac{202}{1+jwC_E}} \cong \frac{jw(C_E/203)}{1+jw(C_E/203)}$$

Therefore:

$$V_o(\omega)=V_i(\omega)\left(\frac{jw(0.37)C_i}{1+jw(0.37)C_i}\right)\left(\frac{jw(C_E/203)}{1+jw(C_E/203)}\right)(-200)$$

And so:

$$A_{vo}(\omega)=\left(\frac{jw(0.37)C_i}{1+jw(0.37)C_i}\right)\left(\frac{jw(C_E/203)}{1+jw(C_E/203)}\right)(-200)$$

Now, since we are **ignoring** the parasitic capacitances, the function $F_H(\omega)$ that describes the high frequency response is:

$$F_H(\omega)=1$$

And so:

$$A_{vo}(\omega)=F_L(\omega)\,A_M\,F_H(\omega)=F_L(\omega)\,A_M$$

By inspection, we see for this example:

$A_M=-200$ ← We knew this **already**!

And:

$$F_L(\omega) = \left(\frac{j\omega(0.37)C_i}{1+j\omega(0.37)C_i}\right)\left(\frac{j\omega(C_E/203)}{1+j\omega(C_E/203)}\right)$$

Now, let's define:

$$\omega_{P1} = \frac{1}{0.37C_i} = \frac{2.7}{C_i} \qquad \text{and} \qquad \omega_{P2} = \frac{203}{C_E}$$

Thus,

$$F_L(\omega) = \left(\frac{j(\omega/\omega_{P1})}{1+j(\omega/\omega_{P1})}\right)\left(\frac{j(\omega/\omega_{P2})}{1+j(\omega/\omega_{P2})}\right)$$

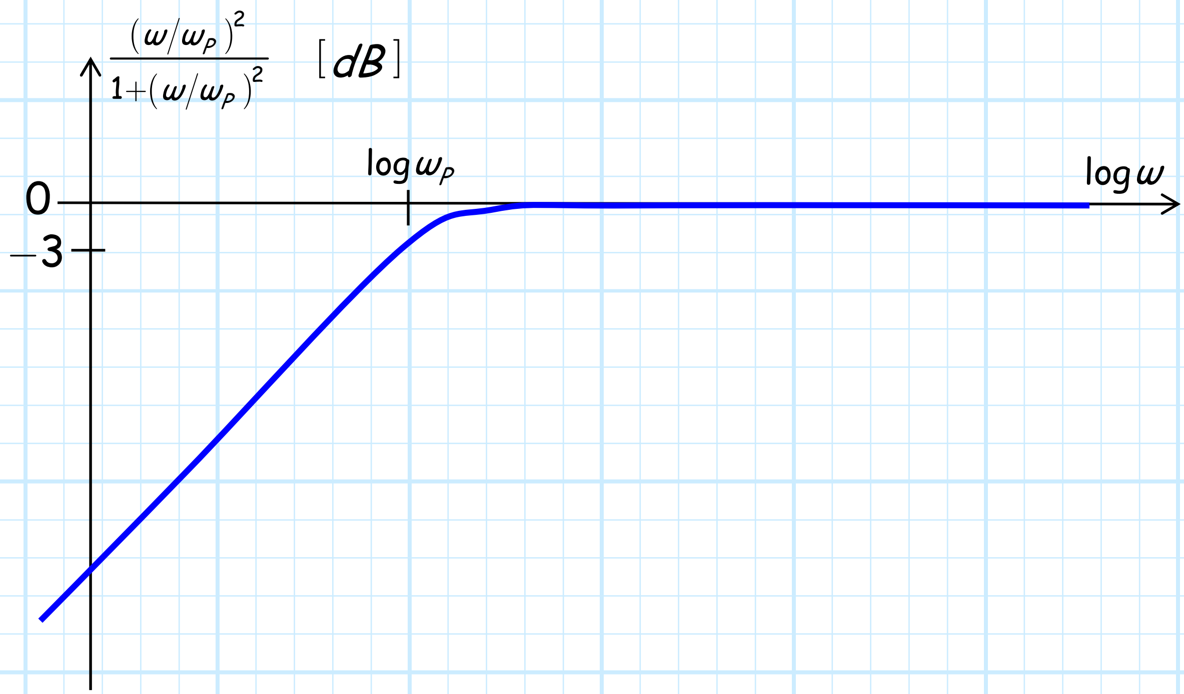
Now, functions of the type:

$$\left(\frac{j(\omega/\omega_P)}{1+j(\omega/\omega_P)}\right)$$

are **high-pass** functions:

$$\left|\frac{j(\omega/\omega_P)}{1+j(\omega/\omega_P)}\right|^2 = \frac{(\omega/\omega_P)^2}{1+(\omega/\omega_P)^2}$$

with a **3dB break frequency** of ω_P.



Thus:

$$\frac{(\omega/\omega_p)^2}{1+(\omega/\omega_p)^2} = \begin{cases} \cong 1.0 & \textit{for } \omega > \omega_p \\ 0.5 & \textit{for } \omega = \omega_p \\ 0 & \textit{for } \omega = 0 \end{cases}$$

As a result, we find that the transfer function:

$$\begin{aligned} A_{vo}(\omega) &= F_L(\omega)\, A_M \\ &= \left(\frac{j(\omega/\omega_{p1})}{1+j(\omega/\omega_{p1})}\right)\left(\frac{j(\omega/\omega_{p2})}{1+j(\omega/\omega_{p2})}\right)(-200) \end{aligned}$$

will be **approximately** equal to the midband gain $A_M = -200$ for all frequencies ω that are greater than **both** ω_{p1} **and** ω_{p2}.

I.E.,:

$$A_{vo}(\omega) \cong A_M = -200 \qquad \text{if } \omega > \omega_{P1} \text{ and } \omega > \omega_{P2}$$

Hopefully, it is **now** apparent (please tell me it is!) that the lower end of the amplifier bandwidth—specified by frequency ω_L—is the determined by the **larger** of the two frequencies ω_{P1} and ω_{P2}!

The **larger** of the two frequencies is called the **dominant pole** of the transfer function $F_L(\omega)$.

For our example—comparing the two frequencies ω_{P1} and ω_{P2}:

$$\omega_{P1} = \frac{1}{0.37C_i} = \frac{2.7}{C_i} \qquad \text{and} \qquad \omega_{P2} = \frac{203}{C_E}$$

it is apparent that the **larger** of the two (the dominant pole!) is likely ω_{P2}—that darn **emitter capacitor** is the key!

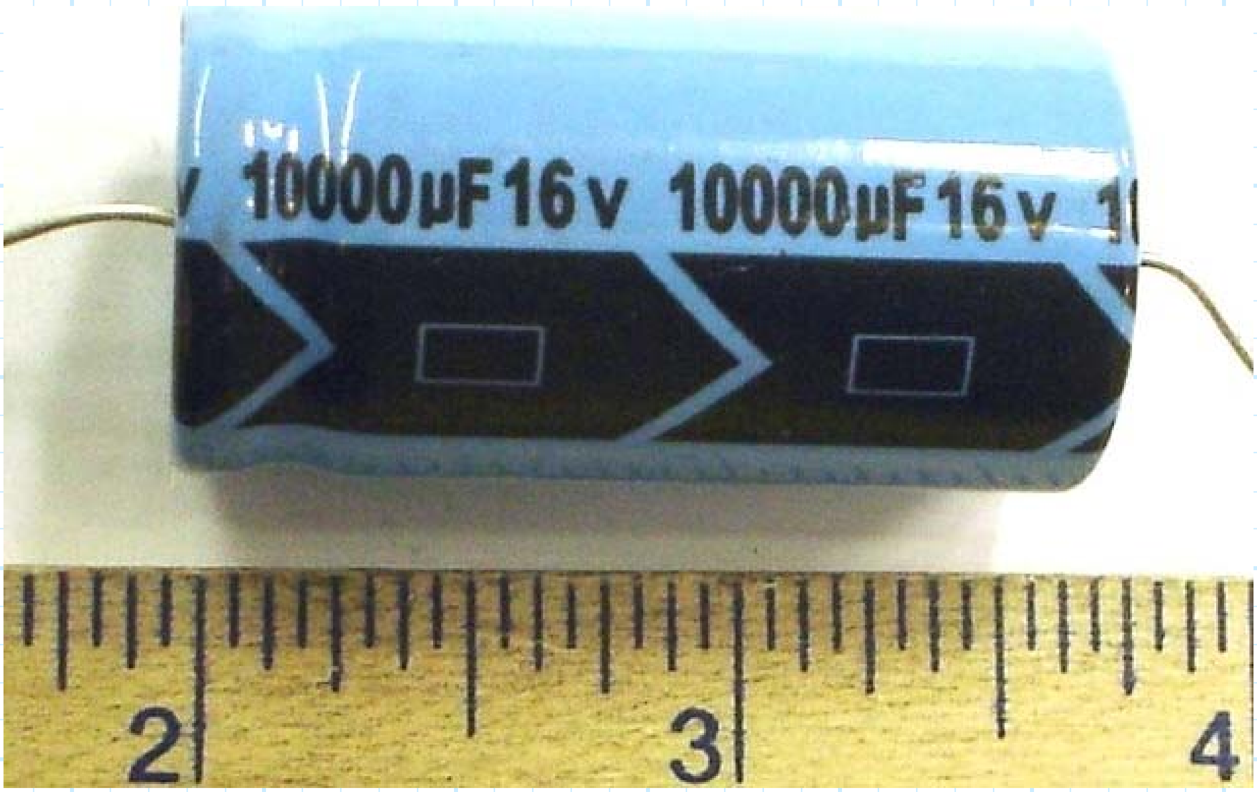
Say we want the common-emitter amplifier in this circuit to have a bandwidth that extends down to $f_L = 100\ Hz$

The emitter capacitor **must** therefore be:

$$2\pi f_L > \omega_{P2} = \frac{203}{C_E}$$

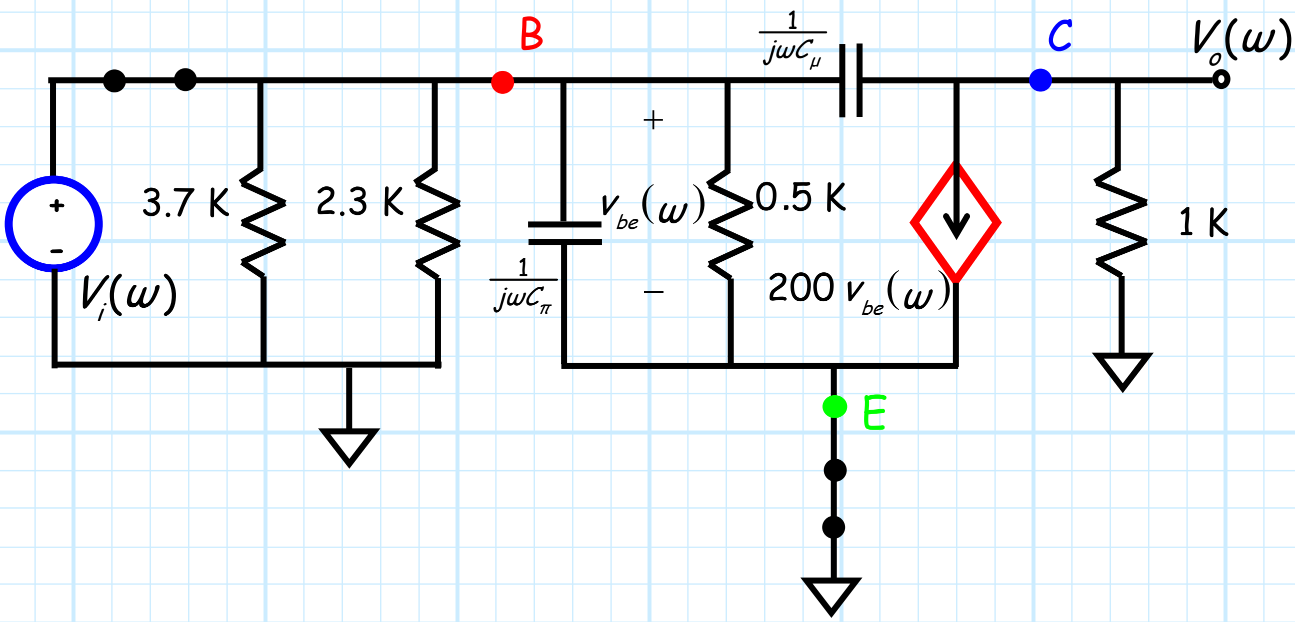
$$\Rightarrow C_E > \frac{203}{2\pi f_L} = \frac{203}{2\pi(1000)} = 32{,}300\mu F \quad !!!!$$

This certainly **is** a **C**apacitor **O**f **U**nusual **S**ize !



High-Frequency Response

To determine the high-frequency response of our example common-emitter amp, we simply consider explicitly the parasitic capacitances in the small-signal model, while approximating the COUS as small-signal short-circuits:



Now, since we are **ignoring** the COUS, the function $F_L(\omega)$ that describes the low- frequency response is:

$$F_L(\omega) = 1$$

And so:

$$A_{vo}(\omega) = F_L(\omega)\ A_M\ F_H(\omega) = A_M\ F_H(\omega)$$

We will find that the high-frequency response will (approximately) have the form.

$$F_H(\omega) = \left(\frac{1}{1 + j(\omega/\omega_{P3})}\right)\left(\frac{1}{1 + j(\omega/\omega_{P4})}\right)$$

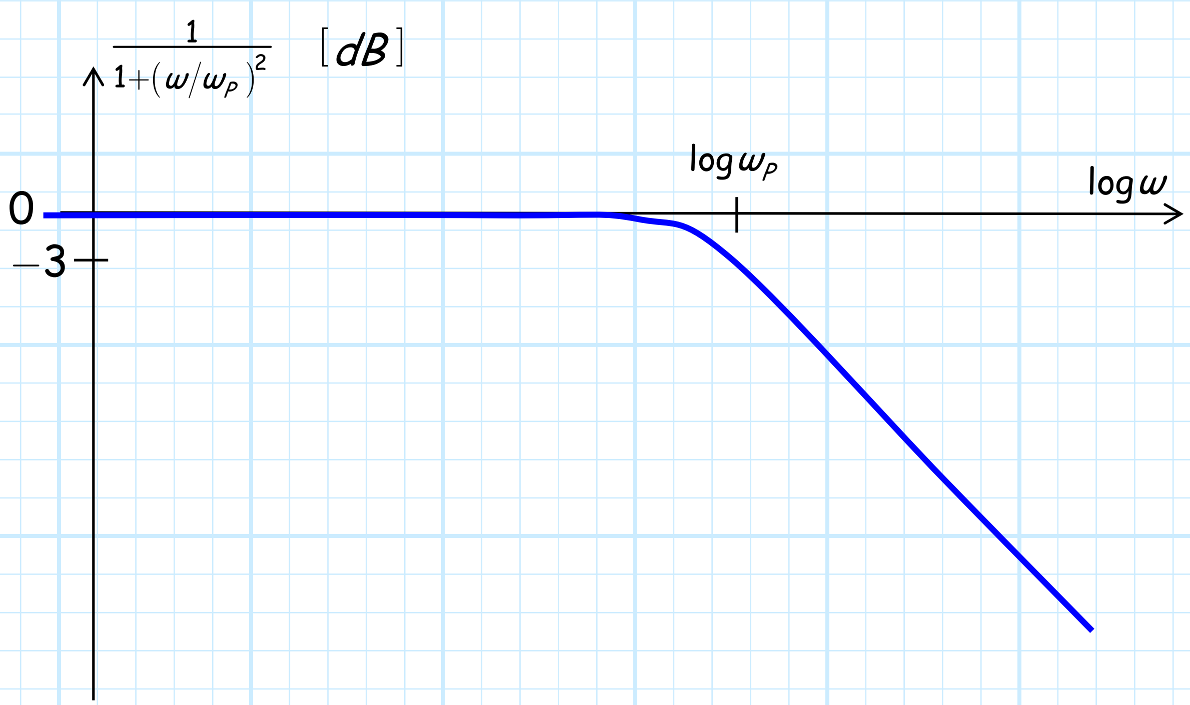
Now, functions of the type:

$$\left(\frac{1}{1 + j(\omega/\omega_P)}\right)$$

are **low-pass** functions:

$$\left|\frac{1}{1 + j(\omega/\omega_P)}\right|^2 = \frac{1}{1 + (\omega/\omega_P)^2}$$

with a **3dB break frequency** of ω_P.



Thus:

$$\frac{1}{1+\left(\omega/\omega_p\right)^2} = \begin{cases} \cong 1.0 & \text{for } \omega < \omega_p \\ 0.5 & \text{for } \omega = \omega_p \\ 0 & \text{for } \omega \rightarrow \infty \end{cases}$$

As a result, we find that the transfer function:

$$\begin{aligned} A_{vo}(\omega) &= A_M\ F_H(\omega) \\ &= (-200)\left(\frac{1}{1+j\left(\omega/\omega_{P3}\right)}\right)\left(\frac{1}{1+j\left(\omega/\omega_{P3}\right)}\right) \end{aligned}$$

will be **approximately** equal to the midband gain $A_M = -200$ for all frequencies ω that are less than **both** ω_{P3} **and** ω_{P4}.

I.E.,:

$$A_{vo}(\omega) \cong A_M = -200 \qquad \text{if } \omega < \omega_{P3} \text{ and } \omega > \omega_{P4}$$

Hopefully, it is **now** apparent (please tell me it is!) that the higher end of the amplifier bandwidth—specified by frequency ω_H—is the determined by the **smaller** of the two frequencies ω_{P3} and ω_{P3}!

The **smaller** of the two frequencies is called the **dominant pole** of the transfer function $F_H(\omega)$.

Generally speaking, to **increase** the value ω_H, we must increase the **DC bias currents** of our amplifier design!