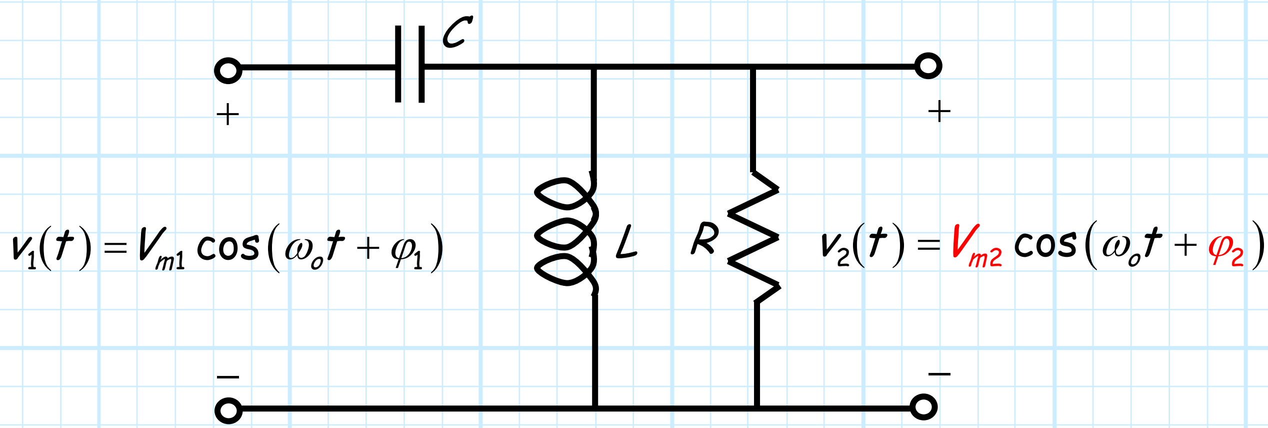
A Complex Representation of Sinusoidal Functions

Q: *So, you say (for example) if a linear two-port circuit is driven by a sinusoidal source with arbitrary frequency ω_o, then the output will be identically sinusoidal, only with a different magnitude and relative phase.*



How do we determine the unknown magnitude V_{m2} and phase φ_2 of this output?

Eigen values are complex

A: Say the input and output are related by the impulse response $g(t)$:

$$v_2(t) = \mathcal{L}\left[v_1(t)\right] = \int_{-\infty}^{t} g(t-t')\, v_1(t')\, dt'$$

We now know that **if** the input were **instead**:

$$v_1(t) = e^{j\omega_0 t}$$

then:

$$v_2(t) = \mathcal{L}\left[e^{j\omega_0 t}\right] = G(\omega_0)\, e^{j\omega_0 t}$$

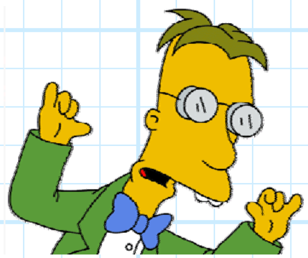
where:

$$G(\omega_0) \doteq \int_{0}^{\infty} g(t)\, e^{-j\omega_0 t}\, dt$$

Thus, we simply multiply the input $v_1(t) = e^{j\omega_0 t}$ by the **complex** eigen value $G(\omega_0)$ to determine the **complex** output $v_2(t)$:

$$v_2(t) = G(\omega_0)\, e^{j\omega_0 t}$$

Complex voltages and currents are your friend!



Q: *You professors drive me crazy with all this math involving **complex** (i.e., real and imaginary) voltage functions. In the lab I can only generate and measure **real-valued** voltages and **real-valued** voltage functions. Voltage is a **real-valued, physical** parameter!*

A: You are quite **correct**.

Voltage **is** a real-valued parameter, expressing electric potential (in Joules) per unit charge (in Coulombs).

Q: *So, all your **complex** formulations and **complex** eigen values and **complex** eigen functions may all be sound **mathematical abstractions**, but aren't they **worthless** to us **electrical engineers** who work in the **"real"** world (pun intended)?*

A: Absolutely not! Complex analysis actually **simplifies** our analysis of real-valued voltages and currents in **linear circuits** (but **only** for linear circuits!).

Remember Euler

The key relationship comes from **Euler's Identity**:

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

Meaning:

$$\mathrm{Re}\left\{e^{j\omega t}\right\} = \cos\omega t$$

Now, consider a **complex value** C. We of course **can** write this complex number in terms of it **real** and **imaginary** parts:

$$C = a + j\,b \qquad \therefore \quad a = \mathrm{Re}\{C\} \quad \text{and} \quad b = \mathrm{Im}\{C\}$$

But, we can **also** write it in terms of its **magnitude** $|C|$ and **phase** φ!

$$\boxed{C = |C|\,e^{j\varphi}}$$

where:

$$|C| = C\,C^* = a^2 + b^2 \qquad \varphi = \tan^{-1}\left[{}^{b}\!/_{a}\right]$$

A complex number has magnitude and phase

Thus, the complex **function** $C\, e^{j\omega_0 t}$ is:

$$
\begin{aligned}
C\, e^{j\omega_0 t} &= |C|\, e^{j\varphi}\, e^{j\omega_0 t} \\
&= |C|\, e^{j\omega_0 t+\varphi} \\
&= |C| \cos(\omega_0 t + \varphi) + j\,|C| \sin(\omega_0 t + \varphi)
\end{aligned}
$$

Therefore we find:

$$|C| \cos(\omega_0 t + \varphi) = \text{Re}\{C\, e^{j\omega_0 t}\}$$

Now, consider again the **real-valued** voltage function:

$$v_1(t) = V_{m1} \cos(\omega t + \varphi_1)$$

This function is of course **sinusoidal** with a magnitude V_{m1} and phase φ_1.

Using what we have learned above, we can **likewise** express this real function as:

$$v_1(t) = V_{m1} \cos(\omega t + \varphi_1) = \text{Re}\{V_1\, e^{j\omega t}\}$$

where V_1 is the **complex number**: $V_1 = V_{m1}\, e^{j\varphi_1}$

But what is the output signal?

Q: *I see! A **real-valued sinusoid** has a magnitude and phase, just like **complex number**.*

*A **single** complex number (V) can be used to specify **both** of the fundamental (real-valued) parameters of our sinusoid (V_m, φ).*

What I don't see is how this helps us in our circuit analysis.

After all:

$$v_2(t) \neq G(\omega_o)\ Re\left\{V_1 e^{j\omega_o t}\right\}$$

*What then is the **real-valued** output $v_2(t)$ of our two-port network when the input $v_1(t)$ is the **real-valued** sinusoid:*

$$\begin{aligned} v_1(t) &= V_{m1}\cos(\omega_o t + \varphi_1) \\ &= \text{Re}\left\{V_1 e^{j\omega_o t}\right\} \end{aligned} \quad ???$$

The math will reveal the answer!

A: Let's go back to our **original** convolution integral:

$$v_2(t) = \int_{-\infty}^{t} g(t-t')\, v_1(t')\, dt'$$

If:

$$v_1(t) = V_{m1}\cos(\omega_o t + \varphi_1)$$
$$= \text{Re}\left\{V_1\, e^{j\omega_o t}\right\}$$

then:

$$v_2(t) = \int_{-\infty}^{t} g(t-t')\, Re\left\{V_1\, e^{j\omega_o t'}\right\} dt'$$

Now, since the impulse function $g(t)$ is **real-valued** (this is really important!) it **can** be shown that:

$$v_2(t) = \int_{-\infty}^{t} g(t-t')\, Re\left\{V_1\, e^{j\omega_o t'}\right\} dt'$$
$$= Re\left\{\int_{-\infty}^{t} g(t-t')\, V_1\, e^{j\omega_o t'}\, dt'\right\}$$

The output signal

Now, applying what we have **previously** learned;

$$v_2(t) = Re\left\{\int_{-\infty}^{t} g(t-t')\, V_1\, e^{j\omega_o t'}\, dt'\right\}$$

$$= Re\left\{V_1 \int_{-\infty}^{t} g(t-t')\; e^{j\omega_o t'}\, dt'\right\}$$

$$= Re\left\{V_1\, G(\omega_0)\, e^{j\omega_o t}\right\}$$

Thus, we **finally** can conclude the real-valued output $v_2(t)$ due to the **real-valued input**:

$$v_1(t) = V_{m1}\cos(\omega_o t + \varphi_1) = \mathrm{Re}\left\{V_1\, e^{j\omega_o t}\right\}$$

is:

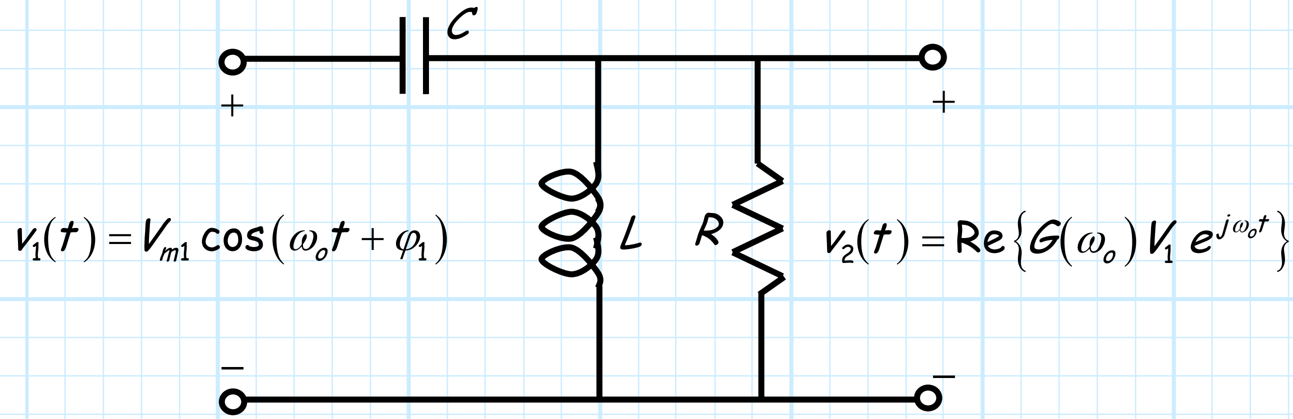
$$v_2(t) = Re\left\{V_2\, e^{j\omega_o t}\right\} = V_{m2}\cos(\omega_o t + \varphi_2)$$

where:

$$V_2 = G(\omega_o)\, V_1$$

The **really important** result here is the last one!

The Eigen value of the Linear operator is its "Frequency Response"



The magnitude and phase of the **output** sinusoid (expressed as **complex** value V_2) is related to the magnitude and phase of the **input** sinusoid (expressed as **complex** value V_1) by the system **eigen value** $G(\omega_o)$:

$$\frac{V_2}{V_1} = G(\omega_o)$$

Therefore we find that **really** often in electrical engineering, we:

1. Use sinusoidal (i.e., eigen function) sources.

2. Express the voltages and currents created by these sources as complex **values** (i.e., **not** as real functions of time)!

Make sure you know what complex voltages and currents physically represent!

For **example**, we might say “ $V_3 = 2.0$”, meaning:

$$V_3 = 2.0 = 2.0\, e^{j0} \quad \Rightarrow \quad v_3(t) = \text{Re}\left\{2.0\, e^{j0} e^{j\omega_o t}\right\} = 2.0 \cos \omega_o t$$

Or “$I_L = -3.0$”, meaning:

$$I_L = -2.0 = 3.0\, e^{j\pi} \quad \Rightarrow \quad i_L(t) = \text{Re}\left\{3.0\, e^{j\pi} e^{j\omega_o t}\right\} = 3.0 \cos\left(\omega_o t + \pi\right)$$

Or “$V_s = j$”, meaning:

$$V_s = j = 1.0\, e^{j\left(\pi/2\right)} \quad \Rightarrow \quad v_s(t) = \text{Re}\left\{1.0\, e^{j\left(\pi/2\right)} e^{j\omega_o t}\right\} = 1.0 \cos\left(\omega_o t + \pi/2\right)$$

Summarizing

* Remember, if a linear circuit is excited by a sinusoid (e.g., **eigen function** $\exp\left[j\omega_0 t\right]$), then the **only** unknowns are the magnitude and phase of the sinusoidal **currents** and **voltages** associated with **each element** of the circuit.

* These unknowns are **completely** described by complex values, as complex values **likewise** have a magnitude and phase.

* We can always **"recover"** the **real-valued** voltage or current function by multiplying the complex value by $\exp\left[j\omega_0 t\right]$ and then taking the real part, but typically we don't—after all, **no** new or unknown information is revealed by this operation!