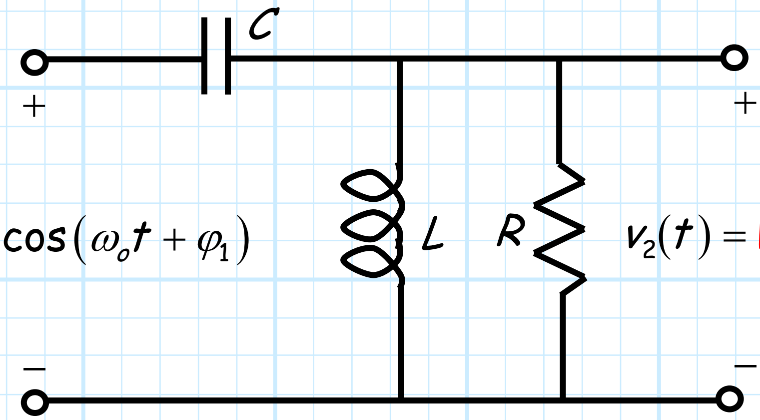
A Complex Representation of Sinusoidal Functions

Q: *So, you say (for example) if a linear two-port circuit is driven by a sinusoidal source with arbitrary frequency* ω_o*, then the output will be identically sinusoidal, only with a different magnitude and relative phase.*

$$v_1(t) = V_{m1} \cos\left(\omega_o t + \varphi_1\right)$$



$$v_2(t) = V_{m2} \cos\left(\omega_o t + \varphi_2\right)$$

How do we determine the unknown magnitude V_{m2} *and phase* φ_2 *of this output?*

A: Say the input and output are related by the impulse response $g(t)$:

$$v_2(t) = \mathcal{L}\left[v_1(t)\right] = \int_{-\infty}^{t} g(t-t')\, v_1(t')\, dt'$$

We now know that **if** the input were **instead**:

$$v_1(t) = e^{j\omega_0 t}$$

then:

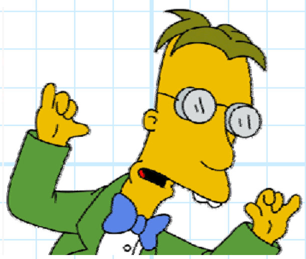
$$v_2(t) = \mathcal{L}\left[e^{j\omega_0 t}\right] = G(\omega_0)\, e^{j\omega_0 t}$$

where:

$$G(\omega_0) \doteq \int_0^{\infty} g(t)\, e^{-j\omega_0 t}\, dt$$

Thus, we simply multiply the input $v_1(t) = e^{j\omega_0 t}$ by the **complex** eigen value $G(\omega_0)$ to determine the **complex** output $v_2(t)$:

$$v_2(t) = G(\omega_0)\, e^{j\omega_0 t}$$



Q: *You professors drive me crazy with all this math involving* ***complex*** *(i.e., real and imaginary) voltage functions. In the lab I can only generate and measure* ***real-valued*** *voltages and* ***real-valued*** *voltage functions. Voltage is a* ***real-valued, physical*** *parameter!*

A: You are quite **correct**.

Voltage **is** a real-valued parameter, expressing electric potential (in Joules) per unit charge (in Coulombs).

Q: *So, all your* ***complex*** *formulations and* ***complex*** *eigen values and* ***complex*** *eigen functions may all be sound* ***mathematical abstractions****, but aren't they* ***worthless*** *to us* ***electrical engineers*** *who work in the* ***"real"*** *world (pun intended)?*

A: Absolutely not! Complex analysis actually **simplifies** our analysis of real-valued voltages and currents in **linear circuits** (but **only** for linear circuits!).

The key relationship comes from **Euler's Identity**:

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

Meaning:

$$\text{Re}\left\{e^{j\omega t}\right\} = \cos\omega t$$

Now, consider a **complex value** C. We of course **can** write this complex number in terms of it **real** and **imaginary** parts:

$$C = a + j\,b \qquad \therefore \quad a = \text{Re}\{C\} \quad \text{and} \quad b = \text{Im}\{C\}$$

But, we can **also** write it in terms of its **magnitude** $|C|$ and **phase** φ!

$$\boxed{C = |C|\,e^{j\varphi}}$$

where:

$$|C| = C\,C^* = a^2 + b^2$$

$$\varphi = \tan^{-1}\left[b/a\right]$$

Thus, the complex **function** $C\,e^{j\omega_0 t}$ is:

$$
\begin{aligned}
C\, e^{j\omega_0 t} &= |C|\, e^{j\varphi}\, e^{j\omega_0 t} \\
&= |C|\, e^{j\omega_0 t+\varphi} \\
&= |C| \cos(\omega_0 t + \varphi) + j\,|C| \sin(\omega_0 t + \varphi)
\end{aligned}
$$

Therefore we find:

$$|C| \cos(\omega_0 t + \varphi) = \text{Re}\left\{ C\, e^{j\omega_0 t} \right\}$$

Now, consider again the **real-valued** voltage function:

$$v_1(t) = V_{m1} \cos(\omega t + \varphi_1)$$

This function is of course **sinusoidal** with a magnitude V_{m1} and phase φ_1. Using what we have learned above, we can **likewise** express this real function as:

$$
\begin{aligned}
v_1(t) &= V_{m1} \cos(\omega t + \varphi_1) \\
&= \text{Re}\left\{ V_1\, e^{j\omega t} \right\}
\end{aligned}
$$

where V_1 is the **complex number**:

$$V_1 = V_{m1}\, e^{j\varphi_1}$$

Q: *I see! A **real-valued sinusoid** has a magnitude and phase, just like **complex number**. A **single** complex number (V) can be used to specify **both** of the fundamental (real-valued) parameters of our sinusoid (V_m, φ).*

What I don't see is how this helps us in our circuit analysis. After all:

$$v_2(t) \neq G(\omega_o)\ Re\{V_1 e^{j\omega_o t}\}$$

*What then is the **real-valued** output $v_2(t)$ of our two-port network when the input $v_1(t)$ is the **real-valued** sinusoid:*

$$\begin{aligned} v_1(t) &= V_{m1}\cos(\omega_o t + \varphi_1) \\ &= \text{Re}\{V_1 e^{j\omega_o t}\} \end{aligned} \quad ???$$

A: Let's go back to our **original** convolution integral:

$$v_2(t) = \int_{-\infty}^{t} g(t - t')\, v_1(t')\, dt'$$

If:

$$\begin{aligned} v_1(t) &= V_{m1}\cos(\omega_o t + \varphi_1) \\ &= \text{Re}\{V_1 e^{j\omega_o t}\} \end{aligned}$$

then:

$$v_2(t) = \int_{-\infty}^{t} g(t - t')\, Re\{V_1 e^{j\omega_o t'}\}\, dt'$$

Now, since the impulse function $g(t)$ is **real-valued** (this is really important!) it **can** be shown that:

$$v_2(t) = \int_{-\infty}^{t} g(t-t')\, Re\left\{V_1\, e^{j\omega_o t'}\right\} dt'$$
$$= Re\left\{\int_{-\infty}^{t} g(t-t')\, V_1\, e^{j\omega_o t'} dt'\right\}$$

Now, applying what we have **previously** learned;

$$v_2(t) = Re\left\{\int_{-\infty}^{t} g(t-t')\, V_1\, e^{j\omega_o t'} dt'\right\}$$
$$= Re\left\{V_1 \int_{-\infty}^{t} g(t-t')\ e^{j\omega_o t'} dt'\right\}$$
$$= Re\left\{V_1\, G(\omega_0)\, e^{j\omega_o t}\right\}$$

Thus, we **finally** can conclude the real-valued output $v_2(t)$ due to the **real-valued input**:

$$v_1(t) = V_{m1} \cos(\omega_o t + \varphi_1)$$
$$= \mathrm{Re}\left\{V_1\, e^{j\omega_o t}\right\}$$

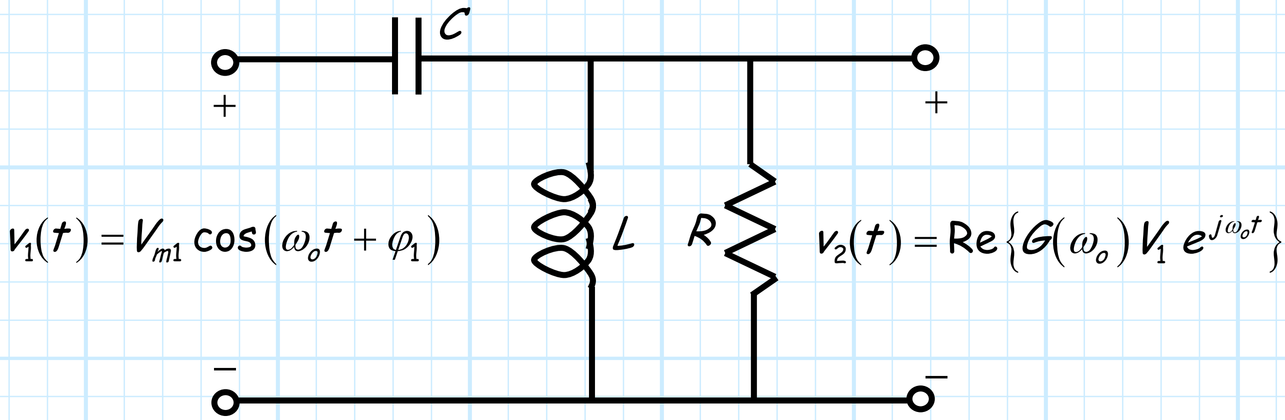
is:

$$v_2(t) = Re\left\{V_2\ e^{j\omega_o t}\right\}$$
$$= V_{m2}\, cos(\omega_o t + \varphi_2)$$

where:

$$V_2 = G(\omega_o)\, V_1$$

The **really important** result here is the last one!



The magnitude and phase of the **output** sinusoid (expressed as **complex** value V_2) is related to the magnitude and phase of the **input** sinusoid (expressed as **complex** value V_1) by the system **eigen value** $G(\omega_o)$:

$$\frac{V_2}{V_1} = G(\omega_o)$$

Therefore we find that **really** often in electrical engineering, we:

1. Use sinusoidal (i.e., eigen function) sources.

2. Express the voltages and currents created by these sources as complex **values** (i.e., **not** as real functions of time)!

For example, we might say " $V_3 = 2.0$", meaning:

$$V_3 = 2.0 = 2.0\, e^{j0} \quad \Rightarrow \quad v_3(t) = \text{Re}\{2.0\, e^{j0} e^{j\omega_o t}\} = 2.0\cos\omega_o t$$

Or "$I_L = -3.0$", meaning:

$$I_L = -2.0 = 3.0\, e^{j\pi} \quad \Rightarrow \quad i_L(t) = \text{Re}\left\{3.0\, e^{j\pi} e^{j\omega_o t}\right\} = 3.0 \cos(\omega_o t + \pi)$$

Or "$V_s = j$", meaning:

$$V_s = j = 1.0\, e^{j\left(\pi/2\right)} \quad \Rightarrow \quad v_s(t) = \text{Re}\left\{1.0\, e^{j\left(\pi/2\right)} e^{j\omega_o t}\right\} = 1.0 \cos\left(\omega_o t + \pi/2\right)$$

* Remember, if a linear circuit is excited by a sinusoid (e.g., **eigen function** $\exp[j\omega_0 t]$), then the **only** unknowns are the magnitude and phase of the sinusoidal **currents** and **voltages** associated with **each element** of the circuit.

* These unknowns are **completely** described by complex values, as complex values **likewise** have a magnitude and phase.

* We can always **"recover"** the **real-valued** voltage or current function by multiplying the complex value by $\exp[j\omega_0 t]$ and then taking the real part, but typically we don't—after all, **no** new or unknown information is revealed by this operation!