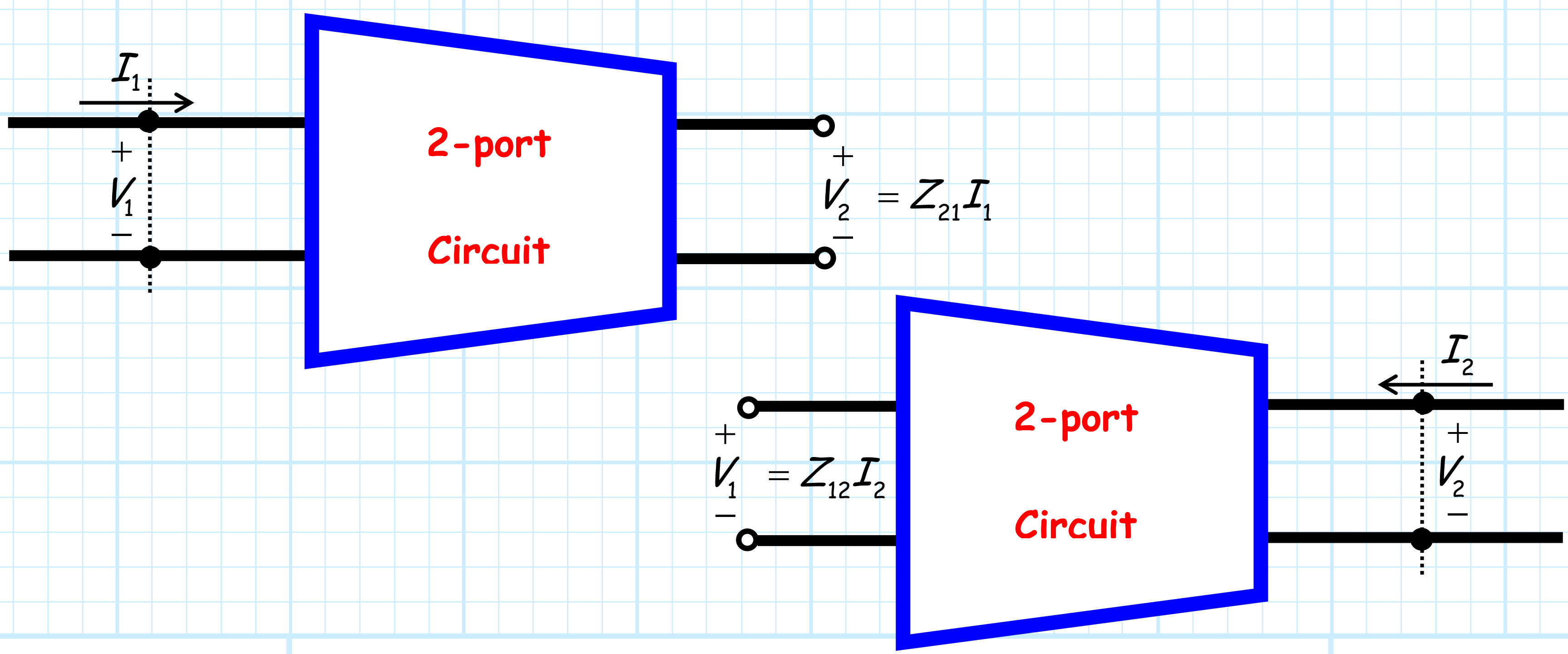
Amplifier Gain

One interesting characteristic of an amplifier is that it is a **unilateral** device—it makes a big difference **which end** you use as the **input**!

Most passive linear circuits (e.g., using only R, L and C) are **reciprocal**. With respect to a 2-port device, **reciprocity** means:

$$Z_{12}(\omega) = Z_{21}(\omega) \qquad \text{and} \qquad Y_{12}(\omega) = Y_{21}(\omega)$$

For example, consider these two **open-circuit voltage** measurements:



Most linear circuits are reciprocal…

If this linear two-port circuit is also **reciprocal**, then when the two currents I_1 and I_2 are equal, so too will be the resulting **open-circuit** voltages V_1 and V_2!

Thus, a **reciprocal** 2-port circuit will have the property:

$$V_1 = V_2 \quad \text{when} \quad I_1 = I_2$$

Note this would likewise mean that:

$$\frac{V_2}{I_1} = \frac{V_1}{I_2}$$

And since (because of the **open-circuits**!):

$$V_2 = Z_{21} I_1 \qquad \text{and} \qquad V_1 = Z_{12} I_2$$

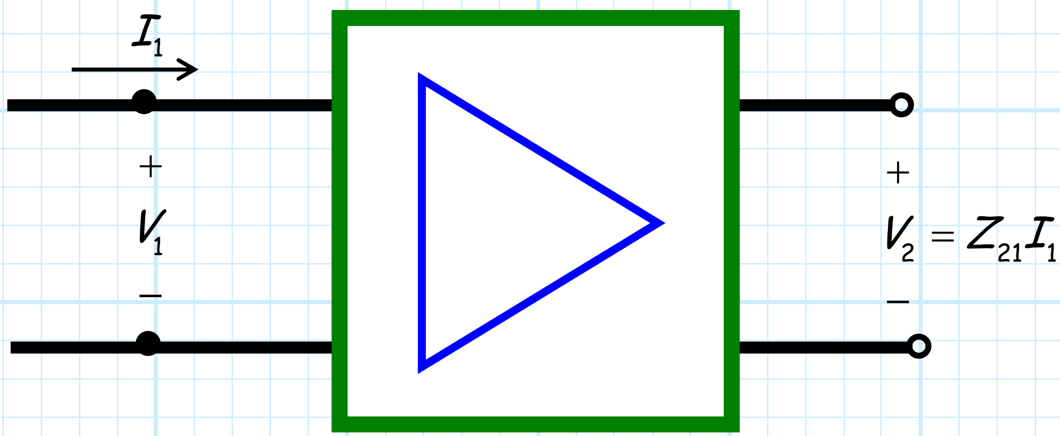
We can conclude from this "experiment" that these trans-impedance parameters of a **reciprocal** 2-port device are **equal**:

$$Z_{12}(\omega) = Z_{21}(\omega)$$

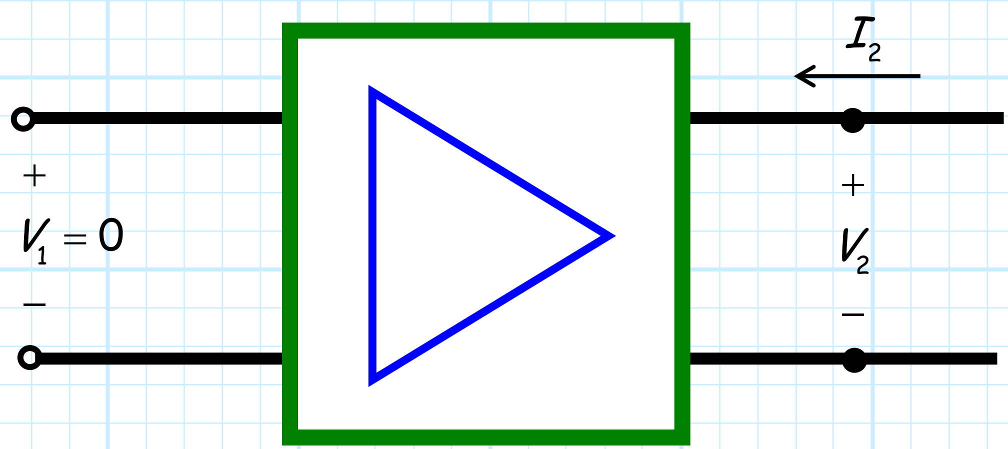
...but amplifiers are not!

Contrast this with an **amplifier**.

A **current** on the input port will indeed produce a **voltage** on an open-circuited output:



However, **amplifiers are not reciprocal**. Placing the same current at the **output** will **not** create the equal voltage on the input—in fact, it will produce **no voltage at all**!



Amps are unilateral: an input and output

Since for this **open-circuited** input port we know that:

$$Z_{12} = \frac{V_1}{I_2},$$

the fact that voltage produced at the input port is zero ($V_1 = 0$) means the trans-impedance parameter Z_{12} is likewise **zero** (or nearly so) for unilateral amplifiers:

$$Z_{12}(\omega) = 0 \qquad \text{(for amplifiers)}$$

Thus, the two equations describing an amplifier (a two-port device) **simplify nicely**.

Here's the simplification

Beginning with:

$$V_1 = Z_{11}\, I_1 + Z_{12}\, I_2$$

$$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$$

Now since $Z_{12} = 0$, we find:

$$V_1 = Z_{11}\, I_1$$

$$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$$

Q: *Gee; I'm sort of **unimpressed** by this simplification—I was hoping the result would be a little more—simple.*

A: Actually, the two equations above represent a **tremendous** simplification—it completely **decouples** the input port from the output, and it allows us to assign very real **physical interpretations** to the remaining impedance parameters!

To see all these benefits (**try** to remain calm), we will now make a few changes in the **notation**.

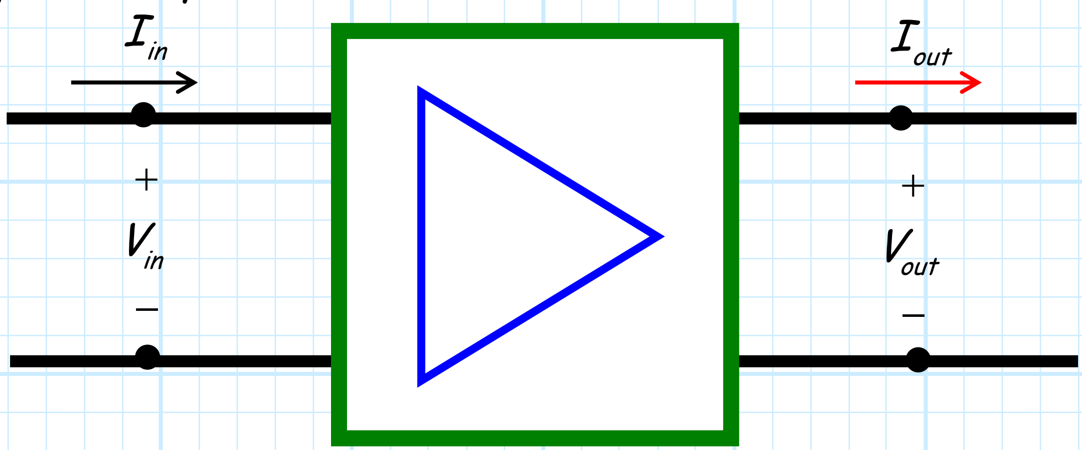
A slight change in notation

First we explicitly denote voltage V_1 as V_{in}, and likewise V_2 as V_{out} (the same with currents I).

Additionally, we change the current **definition** at the output port, **reversing** the direction of positive current as flowing **outward** from the output port. Thus:

$$I_{out} = -I_2$$

And so, a **tidy** summary:



$$V_{in} = Z_{11}\, I_{in}$$

$$V_{out} = Z_{21}\, I_{in} - Z_{22}\, I_{out}$$

The input is independent of the output!

From this summary, it is evident that the relationship between the **input** current and **input** voltage is determined by impedance parameter Z_{11}—and Z_{11} **only**:

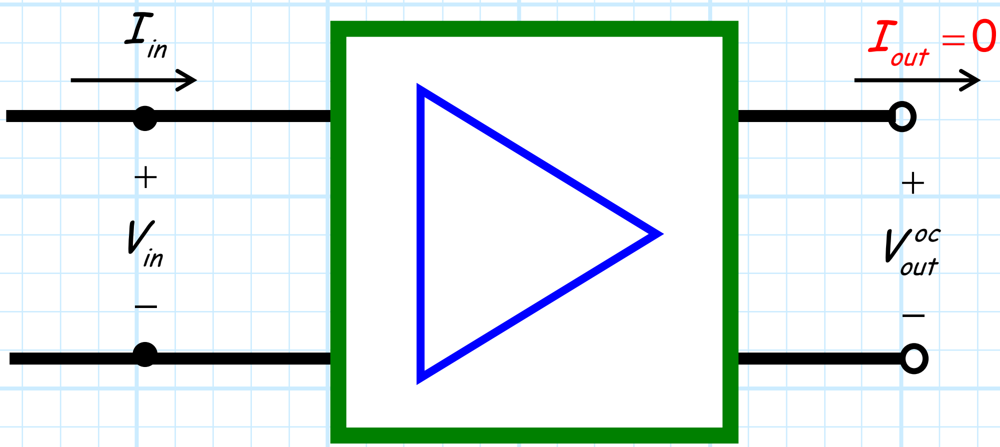
$$Z_{11} = \frac{V_{in}}{I_{in}}$$

Thus, the impedance parameter Z_{11} is known as the **input impedance** Z_{in} of an (unilateral!) amplifier:

$$Z_{in}(\omega) \doteq \frac{V_{in}(\omega)}{I_{in}(\omega)} = Z_{11}(\omega)$$

The open-circuit output voltage

Now, consider the case where the output port of the amplifier is **open-circuited** ($I_{out} = 0$):



The (**open-circuit**) output voltage is therefore simply:

$$
\begin{aligned}
V_{out} &= Z_{21} I_{in} - Z_{22} I_{out} \\
&= Z_{21} I_{in} - Z_{22} (0) \\
&= Z_{21} I_{in}
\end{aligned}
$$

The **open-circuit output voltage** is thus proportional to the **input current**.

Open-circuit trans-impedance

The proportionality constant is the impedance parameter Z_{21}—a value otherwise known as the **open-circuit trans-impedance** Z_m:

$$Z_m(\omega) \doteq \frac{V_{out}^{oc}(\omega)}{I_{in}(\omega)} = Z_{21}(\omega)$$

Thus, an (unilateral!) amplifier can be described as:

$$V_{in} = Z_{in}\, I_{in}$$

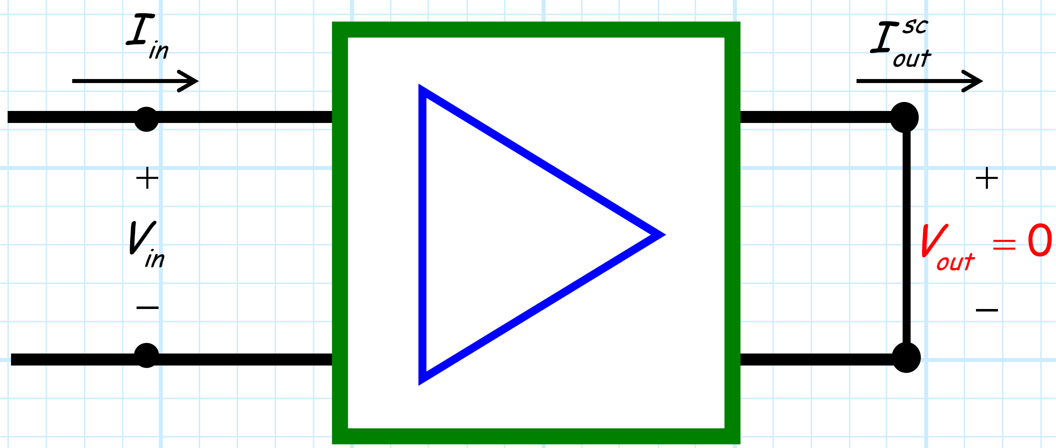
$$V_{out} = Z_m\, I_{in} - Z_{22}\, I_{out}$$

Short-circuit output current

Q: *What about impedance parameter Z_{22}; does it have any **physical** meaning?*

A: It sure does!

Consider now the result of **short-circuiting** the amplifier **output** ($\therefore V_{out} = 0$) :



Since $V_{out} = 0$:

$$V_{out} = 0 = Z_m I_{in} - Z_{22} I_{out}^{sc}$$

we can quickly determine the **short-circuit output current**:

$$I_{out}^{sc} = \frac{Z_m I_{in}}{Z_{22}}$$

The output impedance

Q: *I'm **not** seeing the significance of this result!?*

A: Let's rearrange to determine Z_{22}:

$$Z_{22} = \frac{Z_m \, I_{in}}{I_{out}^{sc}}$$

Note the **numerator**—it is the **open-circuit voltage** $V_{out}^{oc} = Z_m \, I_{in}$, and so:

$$Z_{22} = \frac{Z_m \, I_{in}}{I_{out}^{sc}} = \frac{V_{out}^{oc}}{I_{out}^{sc}}$$

Of course, **you** remember that the ratio of the **open-circuit voltage** to **short-circuit current** is the **output impedance** of a source:

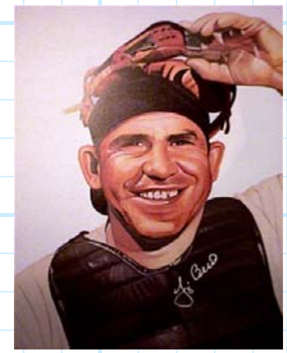
$$Z_{out} \doteq \frac{V_{out}^{oc}}{I_{out}^{sc}} = Z_{22}$$

These equations look familiar!

Thus, the **output impedance** of an (unilateral) amplifier is the impedance parameter Z_{22}, and so:

$$V_{in} = Z_{in}\, I_{in}$$

$$V_{out} = Z_m\, I_{in} - Z_{out}\, I_{out}$$



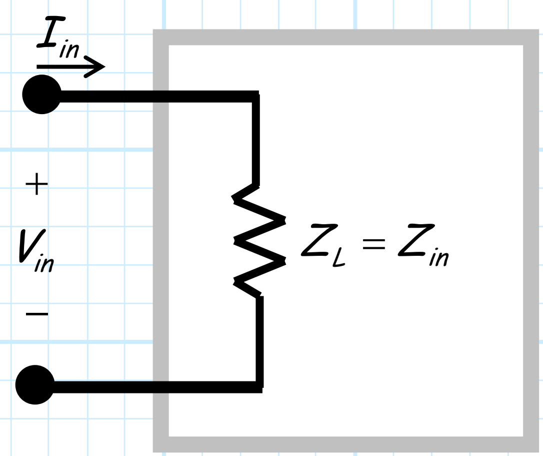
Q: *It's déjà vu all over again; haven't we seen equations like this **before**?*

A: Yes! Recall the first (i.e., input) equation:

$$V_{in} = Z_{in}\, I_{in}$$

is that of a simple **load impedance**:

$$\boxed{V_{in} = Z_{in} I_{in}}$$



Looks like a Thevenin's source

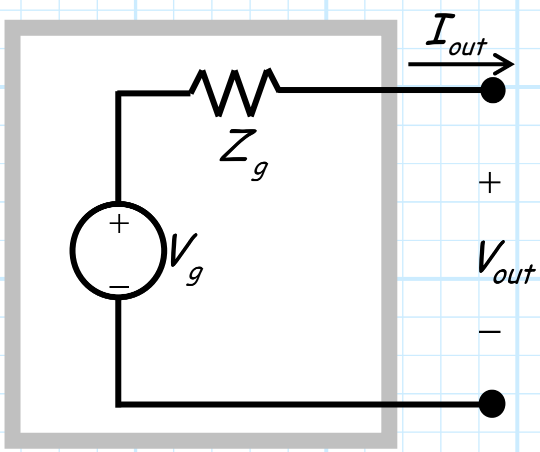
And the **second** (i.e., output) amplifier equation:

$$V_{out} = Z_m I_{in} - Z_{out} I_{out}$$

is of the form of a **Thevenin's source**:

where:

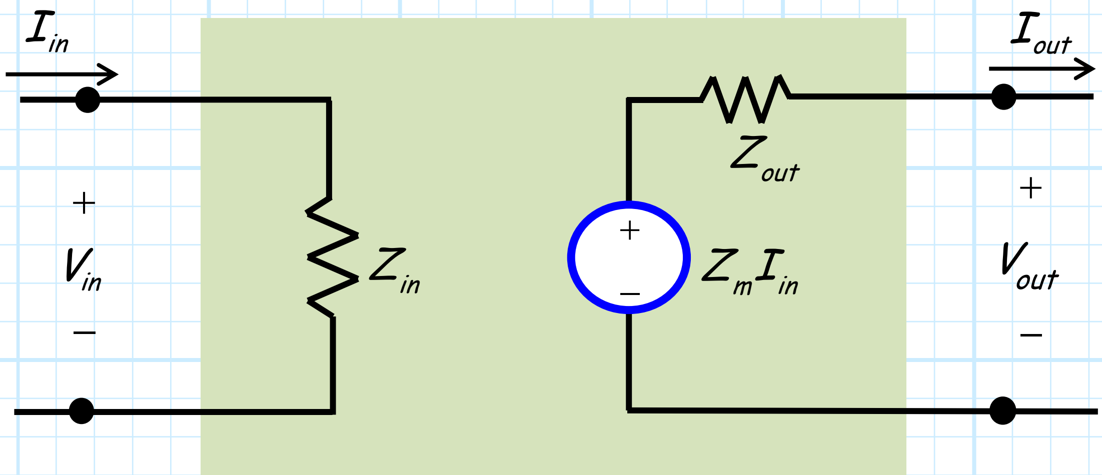
$$V_g = Z_m I_{in} \qquad \text{and} \qquad Z_g = Z_{out}$$



$$V_{out} = V_g - Z_g I_{out}$$

An equivalent circuit model

We can combine these two observations to form an **equivalent circuit model** of an (unilateral) amplifier:



$$V_{in} = Z_{in} I_{in}$$

$$V_{out} = Z_m I_{in} - Z_{out} I_{out}$$

Note in this model, the output of the amp is a **dependent** Thevenin's source—dependent on the **input current**!

Let's make the model more useful

Q: *So, do we **always** use **this** equivalent circuit to model an amplifier?*

A: Um, actually **no**.

The **truth** is that we EE's rarely use this equivalent circuit (not that there's anything wrong with it!).

Instead, the equivalent circuit we use involves a **slight modification** of the model above.

Relate the input voltage to output voltage

To see this modification, we insert the **first** (i.e., input) equation, expressed as:

$$I_{in} = \frac{V_{in}}{Z_{in}}$$

into the **second** (i.e., output) equation:

$$\begin{aligned} V_{out} &= Z_m \, I_{in} - Z_{out} \, I_{out} \\ &= \left(\frac{Z_m}{Z_{in}}\right) V_{in} - Z_{out} \, I_{out} \end{aligned}$$

Thus, the **open-circuit output voltage** can alternatively be expressed in terms of the **input voltage**!

$$V_{out}^{oc} = \left(\frac{Z_m}{Z_{in}}\right) V_{in}$$

Note the ratio Z_m / Z_{in} is unitless (a coefficient!).

Open-circuit voltage gain

This coefficient is known as the **open-circuit voltage gain** A_{vo} of an amplifier:

$$A_{vo}(\omega) \doteq \frac{V_{out}^{oc}}{V_{in}} = \frac{Z_m}{Z_{in}} = \frac{Z_{21}}{Z_{11}}$$

The **open-circuit voltage gain** $A_{vo}(\omega)$ is perhaps the most **important** of all amplifier parameters.

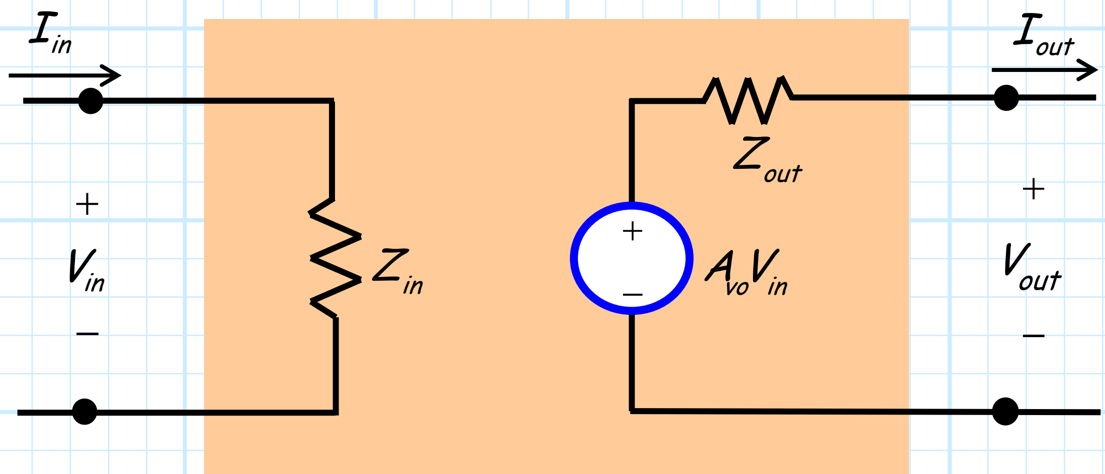
To see why, consider the amplifier equations in terms of this voltage gain:

$$V_{in} = Z_{in}\, I_{in}$$

$$V_{out} = A_{vo} V_{in} - Z_{out}\, I_{out}$$

A more "useful" equivalent circuit

The **equivalent circuit** described by these equations is:



In this circuit model, the output Thevenin's source is again dependent—but now it's dependent on the input **voltage**!

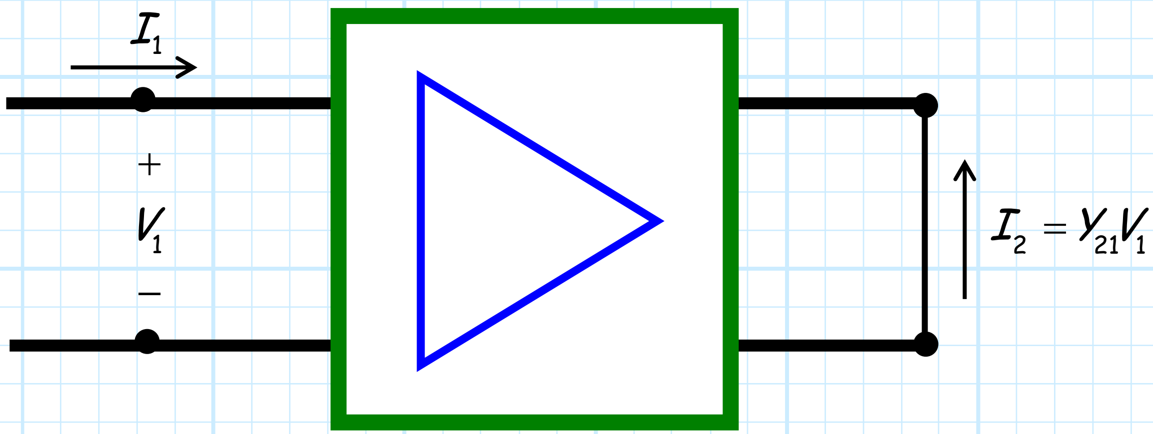
Thus, in this model, the input **voltage** and output **voltage** are more **directly related**.

Now let's consider admittance parameters

Q: *Are these the **only** two was to model a unilateral amplifier?*

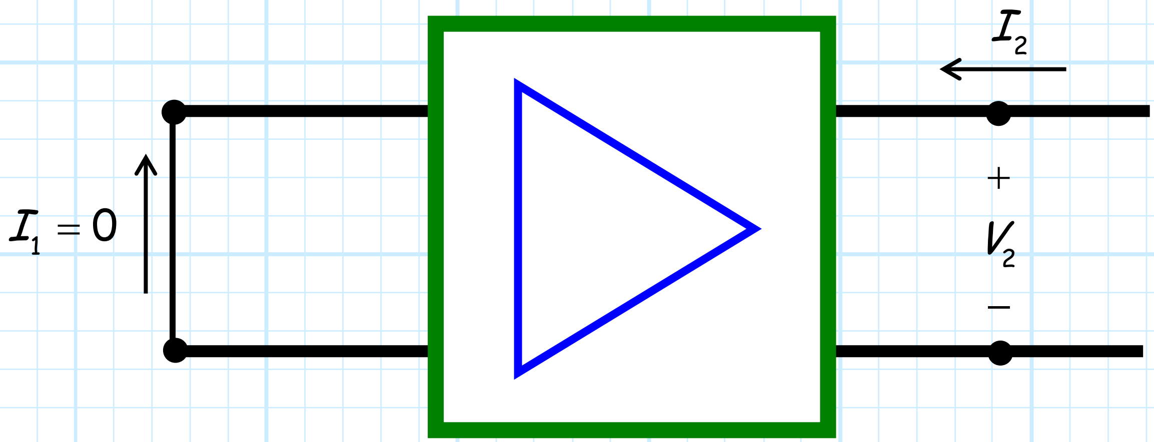
A: Hardly! Consider now **admittance parameters**.

A **voltage** on the **input** port of an amplifier will indeed produce a **short-circuit output current**:



The unilateral amplifier

However, since amplifiers are **not** reciprocal, placing the **same** voltage at the output will **not** create the equal current at the input—in fact, it will produce **no current at all**!



This again shows that amplifiers are **unilateral** devices, and so we find that the trans-admittance parameter Y_{12} is **zero**:

$$Y_{12}(\omega) = 0 \qquad \text{(for amplifiers)}$$

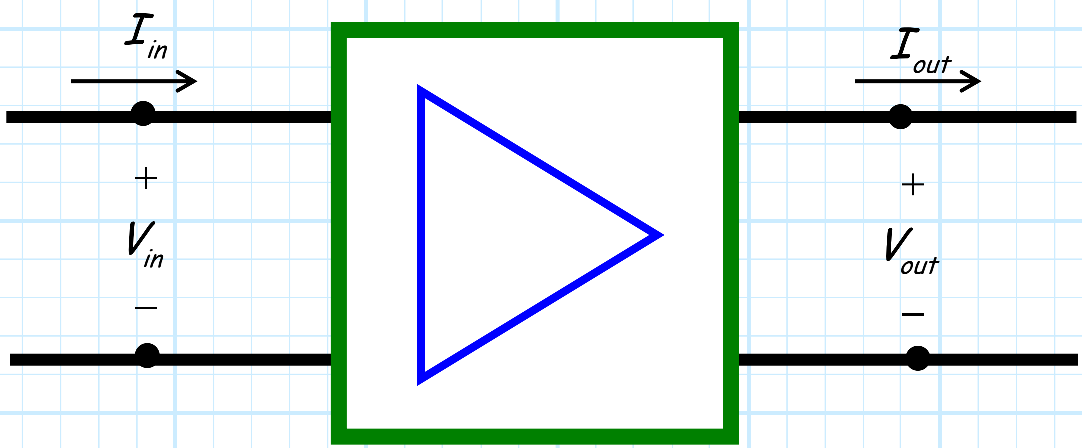
In terms of our new notation

Thus, the two equations using **admittance parameters** simplify to:

$$I_1 = Y_{11} V_1$$

$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

with the same definitions of input and output current/voltage used previously:



$$I_{in} = Y_{11} V_{in}$$

$$I_{out} = -Y_{21} V_{in} - Y_{22} V_{out}$$

Input admittance

As with impedance parameters, it is apparent from this result that the **input** port is **independent** from the **output**.

Specifically, an **input admittance** can be defined as:

$$Y_{in}(\omega) \doteq \frac{I_{in}(\omega)}{V_{in}(\omega)} = Y_{11}(\omega)$$

Note that the **input admittance** of an amplifier is simply the **inverse** of the **input impedance**:

$$Y_{in}(\omega) = \frac{I_{in}(\omega)}{V_{in}(\omega)} = \frac{1}{Z_{in}(\omega)}$$

And from this we can conclude that for a **unilateral** amplifier (but **only** because it's unilateral!):

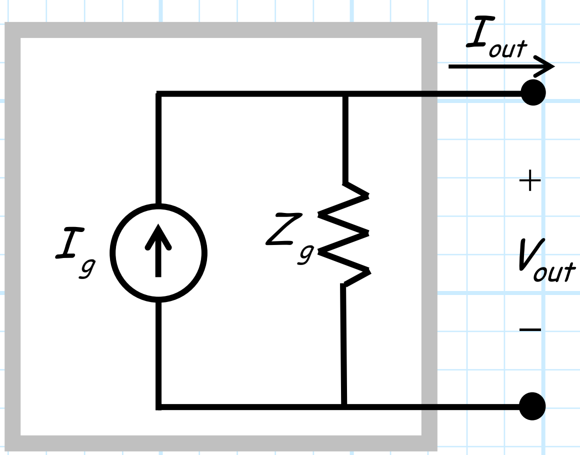
$$Y_{11} = \frac{1}{Z_{11}}$$

Looks like a Norton's source!

Likewise, the **second** amplifier equation:

$$I_{out} = -Y_{21} V_{in} - Y_{22} V_{out}$$

is of the form of a **Norton's** source:



$$I_{out} = I_g - V_{out} / Z_g$$

where:

$$I_g = -Y_{21} V_{in} \quad \text{and} \quad Z_g = \frac{1}{Y_{22}}$$

Short-circuit trans-admittance

More specifically, we can define a **short-circuit trans-admittance**:

$$Y_m \doteq -Y_{21}$$

and an **output impedance**:

$$Z_{out} \doteq \frac{1}{Y_{22}}$$

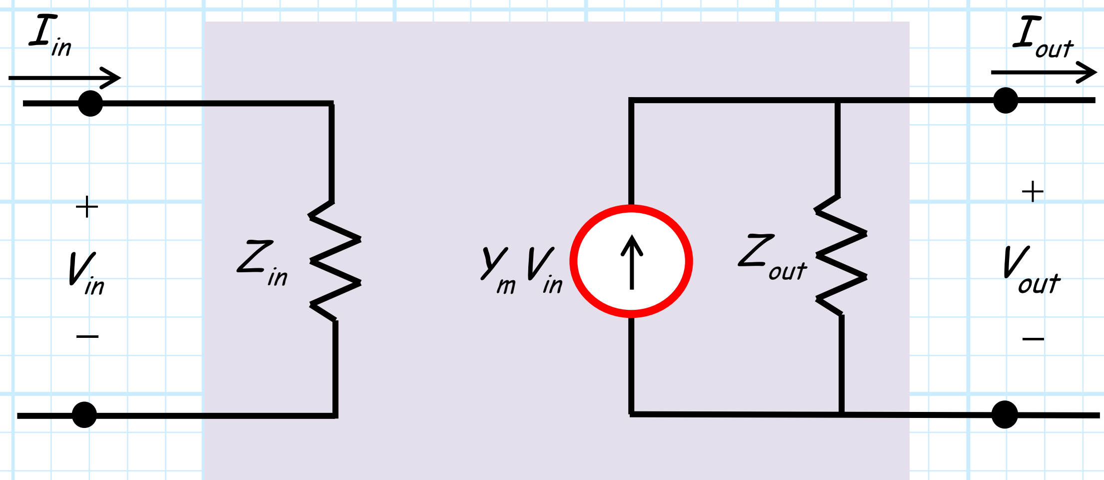
so that the amplifier equations are **now**:

$$I_{in} = V_{in} / Z_{in}$$

$$I_{out} = Y_m V_{in} - V_{out} / Z_{out}$$

Yet another equivalent circuit model

The equivalent circuit described by **these** equations is:



Note in this model, the output of the amp is a **dependent** Norton's source—dependent on the **input voltage**.

However, this particular amplifier model is likewise **seldom** used.

Short-circuit current gain

Instead, we again insert the **input** equation:

$$V_{in} = \frac{I_{in}}{y_{11}}$$

Into the **output** equation:

$$\begin{aligned} I_{out} &= -y_{21} V_{in} - V_{out} / Z_{out} \\ &= -\left(\frac{y_{21}}{y_{11}}\right) I_{in} - V_{out} / Z_{out} \end{aligned}$$

Note the ratio $-y_{21}/y_{11}$ is unitless (a coefficient!).

This coefficient is known as the **short-circuit current gain** A_{is} of an amplifier:

$$A_{is}(\omega) \doteq \frac{I_{out}^{sc}}{I_{in}} = \frac{y_m}{y_{11}} = -\frac{y_{21}}{y_{11}}$$

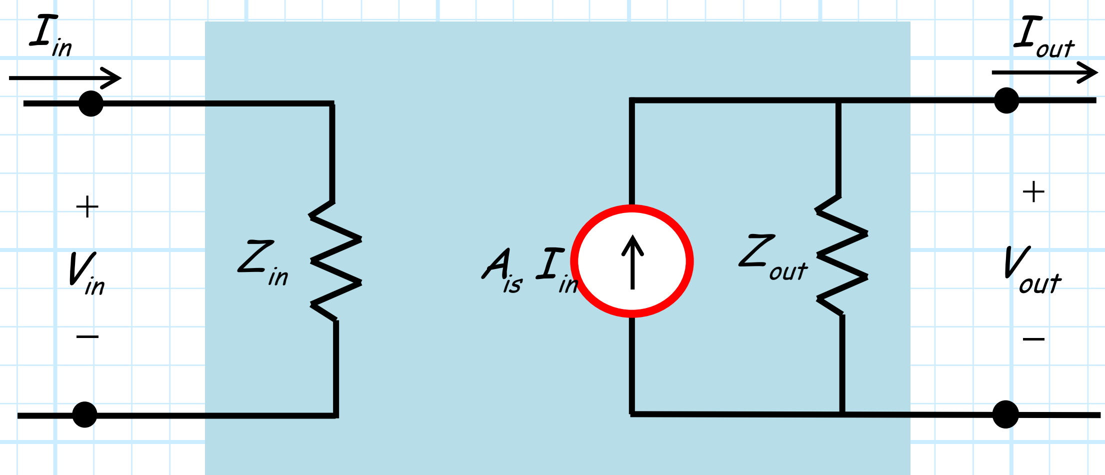
A useful equivalent circuit model

Thus, we can also express the amplifier port equations as:

$$I_{in} = V_{in} / Z_{in}$$

$$I_{out} = A_{is}\, I_{in} - V_{out} / Z_{out}$$

So, the equivalent circuit described by **these** equations is the **last** of **four** we shall consider:



In this circuit model, the output Norton's source is again dependent—but now it's dependent on the input **current**!

Thus, in this model, the input **current** and output **current** are more directly related.