Analysis of Circuits Driven by Arbitrary Functions

Q: *What happens if a linear circuit is excited by some function that is* ***not*** *an "eigen function"? Isn't limiting our analysis to sinusoids* ***too restrictive****?*

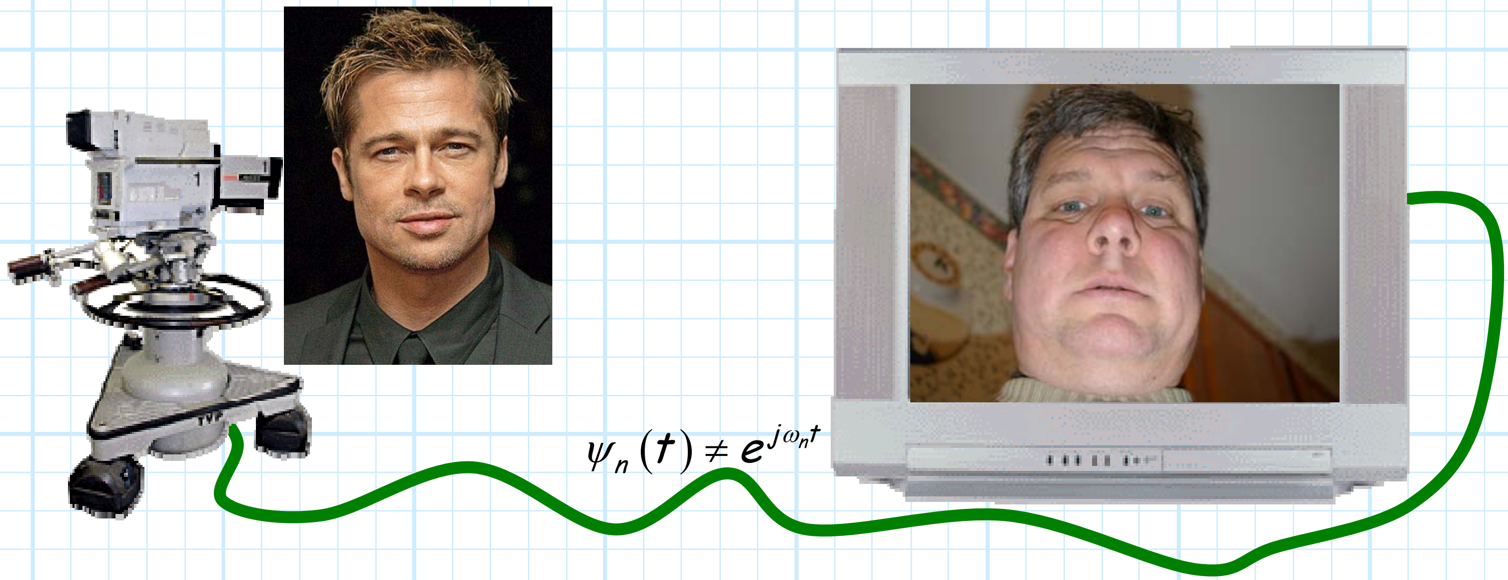
A: Not as restrictive as you might think.

Because sinusoidal functions are the eigen-functions of linear, time-invariant systems, they have become **fundamental** to much of our electrical engineering infrastructure—particularly with regard to **communications**.

For example, every radio and TV station is assigned its **very own eigen function** (i.e., its own frequency ω)!

Eigen functions: without them communication would be impossible

It is **very** important that we use eigen functions for electromagnetic communication, otherwise the **received** signal might look **grotesquely** different from the one that was **transmitted**!



With sinusoidal functions (being eigen functions and all), we **know** that receive function will have **precisely** the same form as the one transmitted (albeit quite a bit **smaller**).

→ Thus, our assumption that a linear circuit is excited by a sinusoidal function is often a very **accurate** and **practical** one!

What if the signal is not sinusoidal?

Q: *Still, we often find a circuit that is **not** driven by a sinusoidal source. How would we analyze **this** circuit?*

A: Recall the property of **linear operators**:

$$\mathcal{L}\left[a\, y_1 + b\, y_2\right] = a\, \mathcal{L}\left[y_1\right] + b\, \mathcal{L}\left[y_2\right]$$

We now know that we can **expand** the function:

$$v(t) = a_0\, \psi_0(t) + a_1\, \psi_1(t) + a_2\, \psi_2(t) + \cdots = \sum_{n=-\infty}^{\infty} a_n\, \psi_n(t)$$

and we found that:

$$\mathcal{L}\left[v(t)\right] = \mathcal{L}\left[\sum_{n=-\infty}^{\infty} a_n\, \psi_n(t)\right] = \sum_{n=-\infty}^{\infty} a_n\, \mathcal{L}\left[\psi_n(t)\right]$$

Let's choose Eigen functions as our basis

We found that any linear operation $\mathcal{L}[\psi_n(t)]$ is greatly simplified **if** we choose as our basis function the **eigen function** of linear systems:

$$\psi_n(t) = \begin{cases} e^{j\omega_n t} & \text{for} \quad 0 \leq t \leq T \\ 0 & \text{for} \quad t < 0, t > T \end{cases} \qquad \text{where} \qquad \omega_n = n\left(\frac{2\pi}{T}\right)$$

so that:

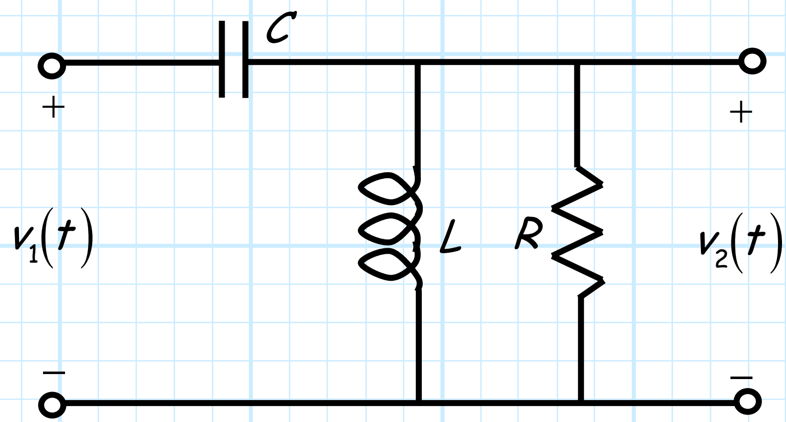
$$\mathcal{L}[\psi_n(t)] = G(\omega_n)\, e^{j\omega_n t}$$

And so:

$$\mathcal{L}\left[v(t)\right] = \mathcal{L}\left[\sum_{n=-\infty}^{\infty} a_n\, e^{j\omega_n t}\right] = \sum_{n=-\infty}^{\infty} a_n\, \mathcal{L}\left[e^{j\omega_n t}\right] = \sum_{n=-\infty}^{\infty} a_n\, G(\omega_n)\, e^{j\omega_n t}$$

Just follow these steps…

Thus, for the example:



We follow these **analysis steps**:

1. **Expand** the input function $v_1(t)$ using the basis functions $\psi_n(t) = exp[j\omega_n t]$:

$$v_1(t) = V_{01}\, e^{j\omega_0 t} + V_{11}\, e^{j\omega_1 t} + V_{21}\, e^{j\omega_2 t} + \cdots = \sum_{n=-\infty}^{\infty} V_{n1}\, e^{j\omega_n t}$$

where:

$$V_{n1} = \frac{1}{T}\int_0^T v_1(t)\, e^{-j\omega_n t}\, dt$$

...and the output is determined

2. Evaluate the **eigen values** of the linear system:

$$G(\omega_n) = \int_0^\infty g(t)\, e^{-j\omega_n t}\, dt$$

3. Perform the **linear operation** (the convolution integral) that relates $v_2(t)$ to $v_1(t)$:

$$\begin{aligned} v_2(t) &= \mathcal{L}\left[v_1(t)\right] \\ &= \mathcal{L}\left[\sum_{n=-\infty}^{\infty} V_{n1}\, e^{j\omega_n t}\right] \\ &= \sum_{n=-\infty}^{\infty} V_{n1}\, \mathcal{L}\left[e^{j\omega_n t}\right] \\ &= \sum_{n=-\infty}^{\infty} V_{n1}\, G(\omega_n)\, e^{j\omega_n t} \end{aligned}$$

A Summary

Summarizing:

$$v_2(t) = \sum_{n=-\infty}^{\infty} V_{n2}\, e^{j\omega_n t}$$

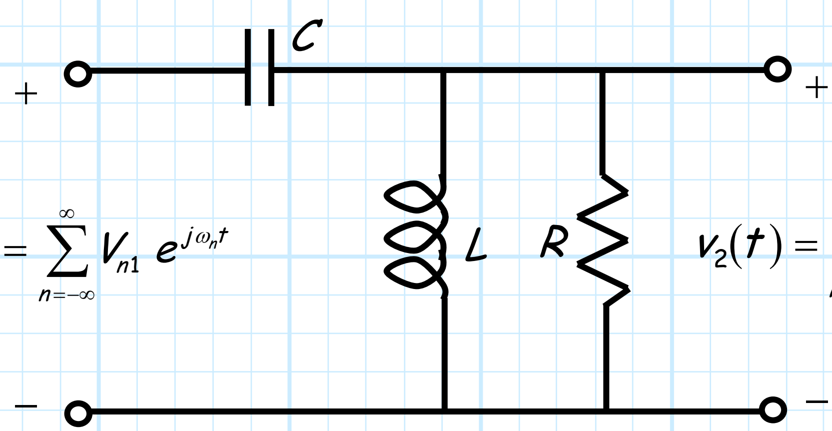
where:

$$V_{n2} = G(\omega_n)\, V_{n1}$$

and:

$$V_{n1} = \frac{1}{T}\int_0^T v_1(t)\, e^{-j\omega_n t}\, dt \qquad\qquad G(\omega_n) = \int_0^\infty g(t)\, e^{-j\omega_n t}\, dt$$

$$v_1(t) = \sum_{n=-\infty}^{\infty} V_{n1}\, e^{j\omega_n t}$$



$$v_2(t) = \sum_{n=-\infty}^{\infty} G_1(\omega_n)\, V_{n1}\, e^{j\omega_n t}$$

As stated earlier, the signal expansion used here is the **Fourier Series**.

The Fourier Transform

Say that the **timewidth** T of the signal $v_1(t)$ becomes **infinite**. In this case we find our analysis becomes:

$$v_2(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} V_2(\omega)\, e^{j\omega t} d\omega$$

where:

$$V_2(\omega) = G(\omega)\, V_1(\omega)$$

and:

$$V_1(\omega) = \int_{-\infty}^{+\infty} v_1(t)\, e^{-j\omega t} dt \qquad G(\omega) = \int_{-\infty}^{+\infty} g(t)\, e^{-j\omega t}\, dt$$

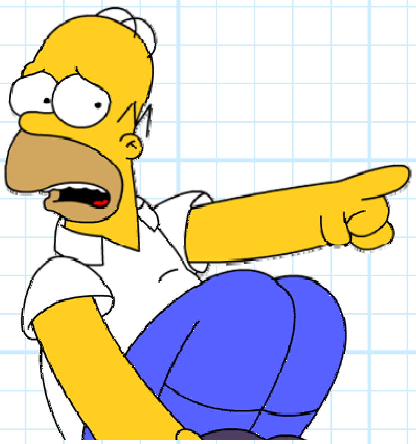
The signal expansion in this case is the **Fourier Transform.**

We find that as $T \rightarrow \infty$ the number of **discrete** system eigen values $G(\omega_n)$ become so numerous that they form a **continuum**—$G(\omega)$ is a **continuous** function of frequency ω.

We thus call the function $G(\omega)$ the **eigen spectrum** or **frequency response** of the circuit.

This still looks very difficult!

Q: *You **claim** that all this fancy mathematics (e.g., eigen functions and eigen values) make analysis of linear systems and circuits much **easier**, yet to apply these techniques, we must **determine** the eigen values or eigen spectrum:*



$$G(\omega_n) = \int_0^{\infty} g(t)\, e^{-j\omega_n t}\, dt \qquad G(\omega) = \int_{-\infty}^{+\infty} g(t)\, e^{-j\omega t}\, dt$$

***Neither** of these operations look **at all** easy.*

*And in addition to performing the integration, we must **somehow** determine the **impulse function** $g(t)$ of the linear system as well !*

*Just how are we supposed to do **that**?*

It's not nearly as difficult as it appears!

A: An insightful question!

Determining the impulse response $g(t)$ and then the frequency response $G(\omega)$ **does** appear to be **exceedingly** difficult—and for many linear systems it indeed **is**!

However, much to our great **relief**, we can determine the eigen spectrum $G(\omega)$ of linear circuits **without** having to perform a difficult integration.

In fact, we **don't** even need to know the impulse response $g(t)$**!**