Eigen Values of the Laplace Transform

Well, I fibbed **a little** when I stated that the Eigen function of linear, time-invariant systems (circuits) is:

$$\mathcal{L}\left\{e^{j\omega t}\right\} = G(\omega)\, e^{j\omega t}$$

Instead, the more **general** Eigen function is:

$$\mathcal{L}\left\{e^{st}\right\} = G(s)\, e^{st}$$

Where s is a **complex** (i.e., real and imaginary) frequency of the form:

$$s = \sigma + j\,\omega$$

such that:

$$e^{s\,t} = e^{(\sigma + j\,\omega)\,t} = e^{\sigma\,t} e^{j\,\omega\,t}$$

Note then, **if** $\sigma = 0$, the Eigen function $e^{s\,t}$ becomes the previously described Eigen function $e^{j\omega\,t}$!

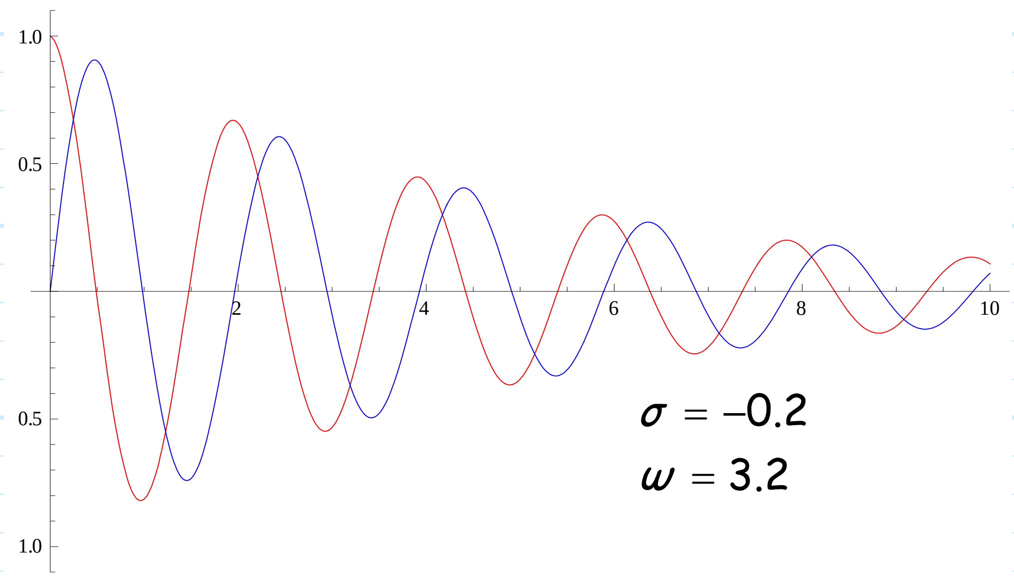
What does this function mean?

Q: *Yikes! I understand e^{st} even* ***less*** *than I understood $e^{j\omega t}$! What does this function mean?*

A: Remember, the function e^{st} is a **complex** function—it is actually an expression of **two real-value** functions.

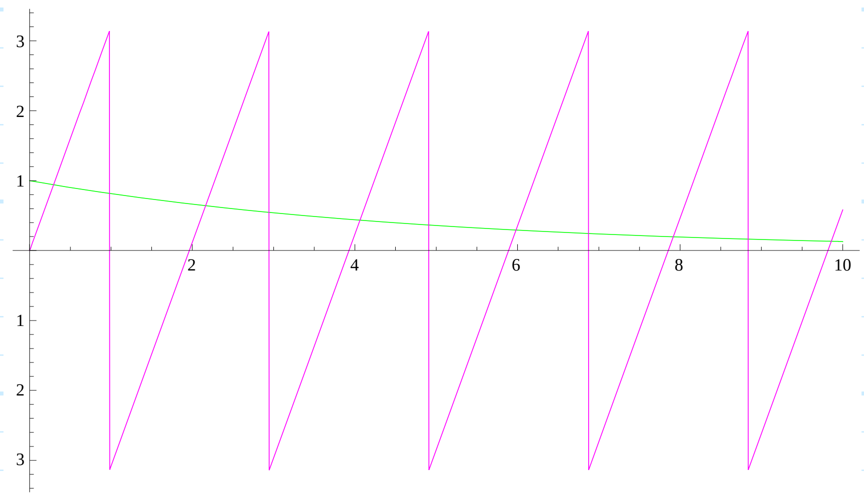
These two real-valued functions could be its **real** and **imaginary** components:

$$
\begin{aligned}
e^{st} &= e^{\sigma t} e^{+j\omega t} \\
&= e^{\sigma t}\left(\cos\omega t + j\sin\omega t\right) \\
&= e^{\sigma t}\cos\omega t + j\, e^{\sigma t}\sin\omega t
\end{aligned}
$$



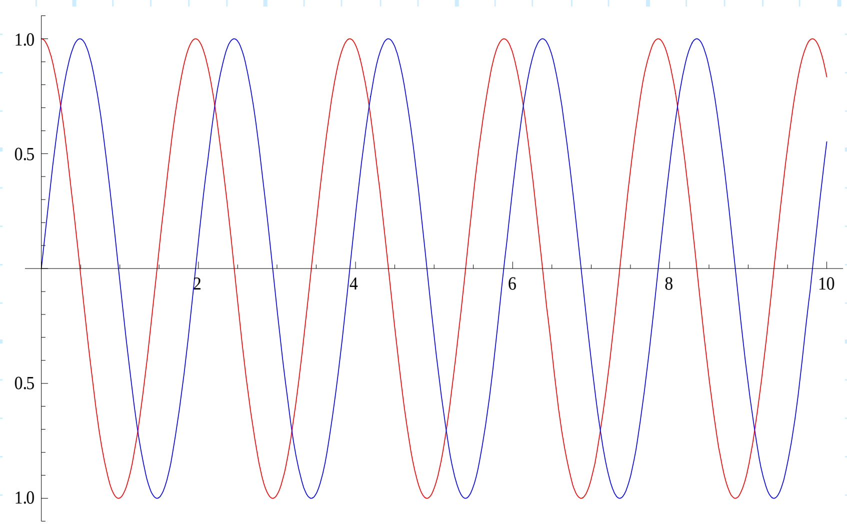
Magnitude and phase

Or, the two real-valued functions could alternatively be the complex values **magnitude** and **phase**:



$\sigma = -0.2$
$\omega = 3.2$

If $\sigma = 0$, then $e^{st} = e^{+j\omega t}$, and we're **back** to the time-harmonic Eigen function:



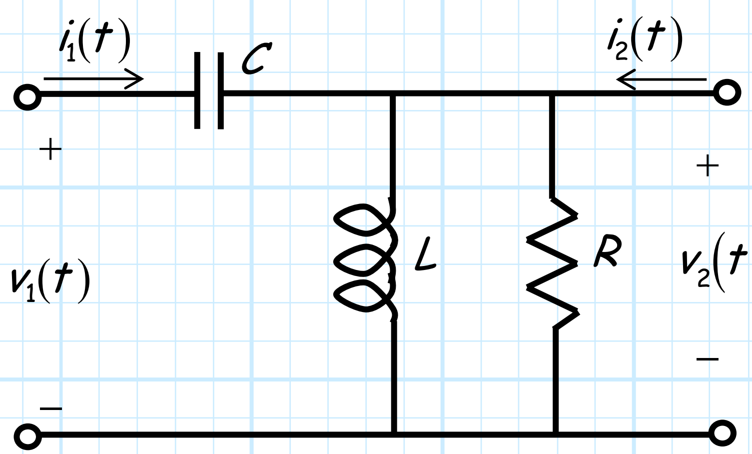
$\sigma = 0.0$
$\omega = 3.2$

Can we use this as a basis?

Q: *What about **basis functions**? Can we use these Eigen function to **expand** a signal?*

A: Sure! Instead of the Fourier Transform, the result of expanding a signal with basis function e^{st} is the **Laplace Transform**.

For example, **again** consider the following linear circuit:



$$v_2(t) = \mathcal{L}\{v_1(t)\} = \int_{-\infty}^{\infty} g(t-t')\, v_1(t')\, dt'$$

A summary

Using the Laplace transform, we can determine the **output** voltage $v_2(t)$ by:

1. Expand the **input** signal $v_1(t)$ using the basis function e^{st}:

$$V_1(s) = \int_0^{+\infty} v_1(t)\, e^{-st}\, dt \quad \text{(or use a \textbf{look-up table}!)}$$

2. Determine the **Eigen value** of the linear operator relating $v_1(t)$ to $v_2(t)$:

$$v_2(t) = \mathcal{L}\{v_1(t)\} = \int_{-\infty}^{\infty} g(t-t')\, v_1(t')\, dt'$$

$$\Rightarrow \qquad V_2(s) = G(s)\, V_1(s)$$

where:

$$G(s) = \int_{-\infty}^{+\infty} g(t)\, e^{-st}\, dt$$

3. Determine $v_2(t)$ from the **inverse** Laplace transform of $V_2(s)$ (**definitely** use a look-up table!).

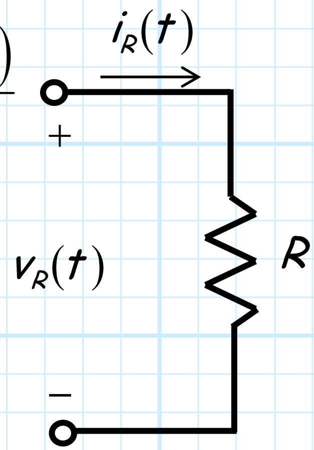
The Eigen values of circuit elements

Q: *But how do we **determine** $G(s)$?*

A: It's just pretty darn **simple**!

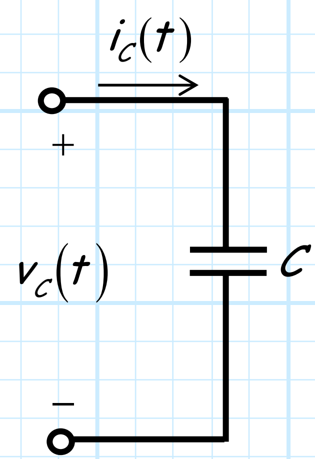
Again, we determine the Eigen value of each linear operator of our **three** linear circuit elements—only **this** time we use the Eigen function e^{st}!

$$i_R(t) = \mathcal{L}_{\mathcal{Y}}^{R}\left[v_R(t)\right] = \frac{v_R(t)}{R}$$



$$\mathcal{L}_{\mathcal{Y}}^{R}\left[e^{st}\right] = \frac{e^{st}}{R}$$

$$I_R(s) = \frac{V_R(s)}{R}$$



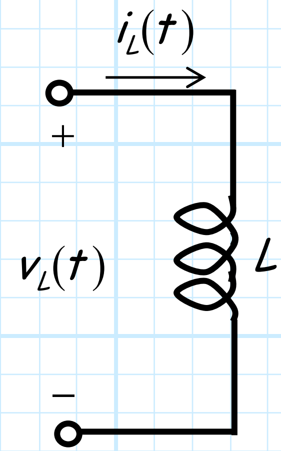
$$i_C(t) = \mathcal{L}_{\mathcal{Y}}^{C}\left[v_C(t)\right] = C\,\frac{d\,v_C(t)}{d\,t}$$

$$\mathcal{L}_{\mathcal{Y}}^{C}\left[e^{st}\right] = C\,\frac{d\,e^{st}}{d\,t} = sC\,e^{st}$$

$$I_C(s) = sC\,V_C(s)$$

Just apply your circuits knowledge!

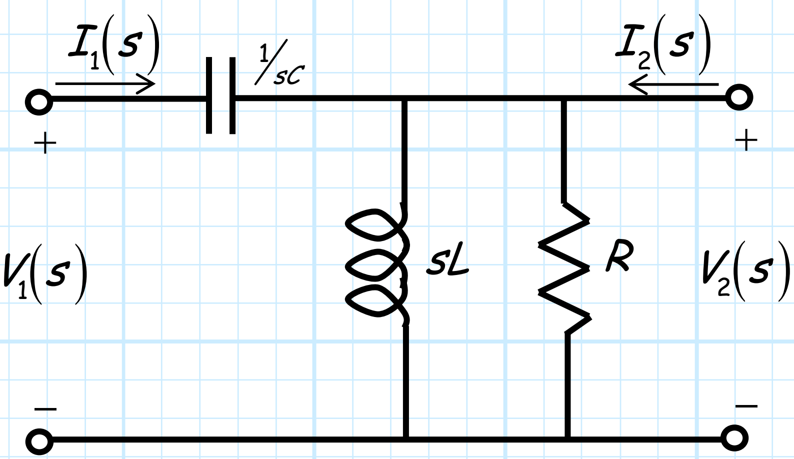
$$\mathcal{L}_{\mathcal{Y}}^{L}\left[v_L(t)\right] = i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t')\,dt'$$



$$\mathcal{L}_{\mathcal{Y}}^{L}\left[e^{st}\right] = \frac{1}{L}\int_{-\infty}^{t} e^{st}\,dt' = \frac{1}{sL}e^{st}$$

$$I_L(s) = \frac{V_L(s)}{s\,L}$$

As a result we can determine the Eigen value $G(s)$ of a linear circuit by applying our **circuit theory**:



$$\frac{V_2(s)}{V_1(s)} = G_{21}(s) = \frac{s\,L \| R}{\frac{1}{sC} + sL \| R}$$