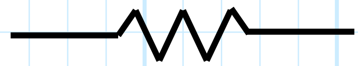
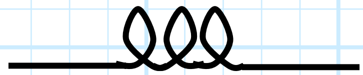
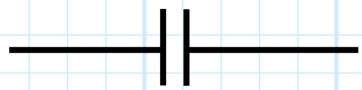
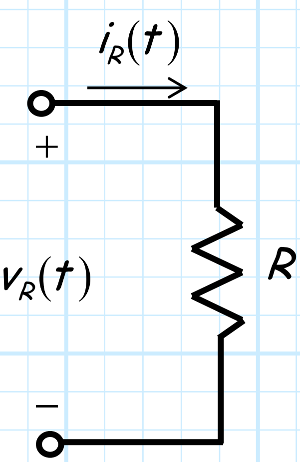
Linear Circuit Elements

Most microwave devices can be described or modeled in terms of the **three** standard circuit elements:

1. RESISTANCE (R) 
2. INDUCTANCE (L) 
3. CAPACITANCE (C) 

For the purposes of circuit analysis, each of these three elements are **defined** in terms of the **mathematical** relationship between the difference in electric potential $v(t)$ between the two terminals of the device (i.e., the **voltage** across the device), and the **current** $i(t)$ flowing through the device.

We find that for these three circuit elements, the relationship between $v(t)$ and $i(t)$ can be expressed as a linear operator!

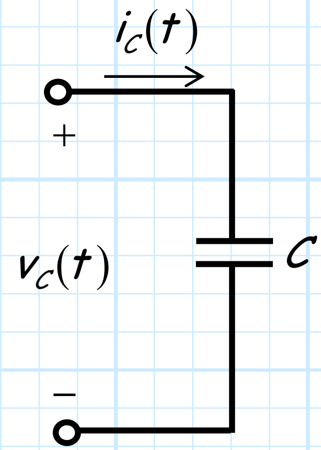


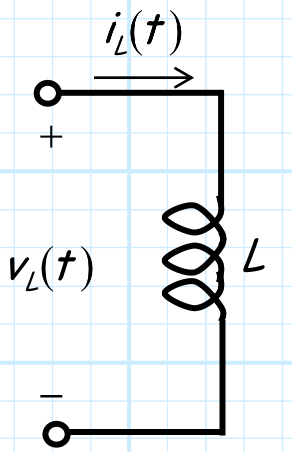
$$\mathcal{L}_{\mathcal{Y}}^{R}\left[v_R(t)\right] = i_R(t) = \frac{v_R(t)}{R}$$

$$\mathcal{L}_{\mathcal{Z}}^{R}\left[i_R(t)\right] = v_R(t) = R\, i_R(t)$$

$$\mathcal{L}_{\mathcal{Y}}^{C}\left[v_C(t)\right] = i_C(t) = C\,\frac{d\,v_C(t)}{d\,t}$$

$$\mathcal{L}_{\mathcal{Z}}^{C}\left[i_C(t)\right] = v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(t')\,dt'$$





$$\mathcal{L}_{\mathcal{Y}}^{L}\left[v_L(t)\right] = i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t')\,dt'$$

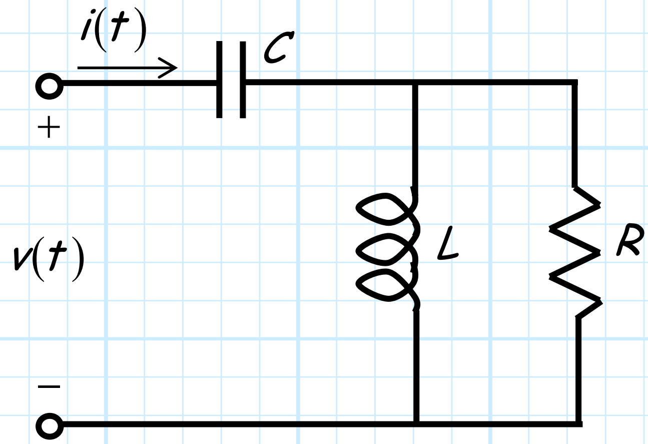
$$\mathcal{L}_{\mathcal{Z}}^{L}\left[i_L(t)\right] = v_L(t) = L\,\frac{d\,i_L(t)}{d\,t}$$

Since the circuit behavior of these devices can be expressed with **linear** operators, these devices are referred to as **linear circuit elements**.

Q: *Well, that's simple enough, but what about an element formed from a* ***composite*** *of these fundamental elements?*

For ***example****, for example, how are $v(t)$ and $i(t)$ related in the circuit below??*

$$\mathcal{L}_{\mathcal{Z}}\left[i(t)\right] = v(t) = ???$$



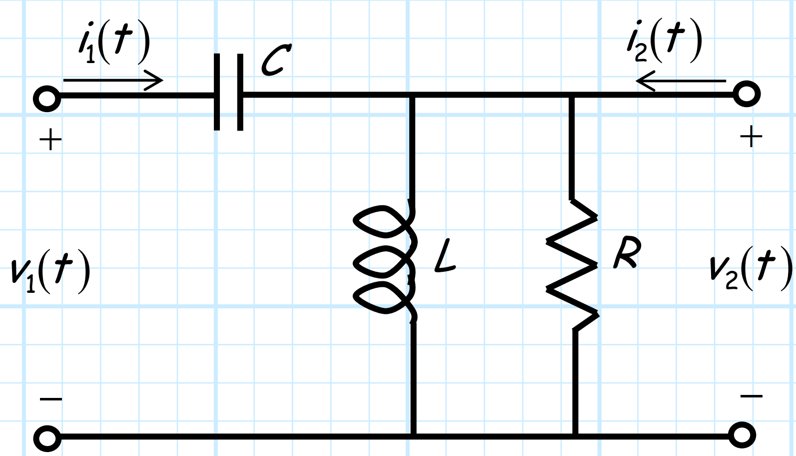
A: It turns out that **any** circuit constructed **entirely** with linear circuit elements is **likewise** a linear system (i.e., a linear circuit).

As a result, we know that that there **must** be some linear operator that relates $v(t)$ and $i(t)$ in your example!

$$\mathcal{L}_{\mathcal{Z}}\left[i(t)\right] = v(t)$$

The circuit above provides a good example of a **single-port** (a.k.a. **one-port**) network.

We can of course construct networks with **two or more** ports; an example of a **two-port network** is shown below:



Since this circuit is **linear**, the relationship between **all** voltages and currents can likewise be expressed as **linear operators, e.g.**:

$$\mathcal{L}_{21}\left[v_1(t)\right] = v_2(t)$$

$$\mathcal{L}_{\mathcal{Z}21}\left[i_1(t)\right] = v_2(t)$$

$$\mathcal{L}_{\mathcal{Z}22}\left[i_2(t)\right] = v_2(t)$$

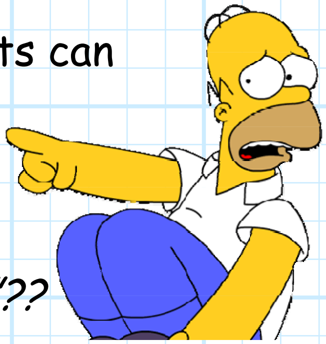
Q: *Yikes! What would these linear operators for this circuit **be**? How can we **determine** them?*

A: It turns out that linear operators for **all** linear circuits can all be expressed in precisely the **same** form! For example, the linear operators of a single-port network are:

$$v(t) = \mathcal{L}_{\mathcal{Z}}\left[i(t)\right] = \int_{-\infty}^{t} g_{\mathcal{Z}}(t-t')\; i(t')\,dt'$$

$$i(t) = \mathcal{L}_{\mathcal{Y}}\left[v(t)\right] = \int_{-\infty}^{t} g_{\mathcal{Y}}(t-t')\; v(t')\,dt'$$

In other words, the linear operator of linear circuits can always be expressed as a **convolution** integral—a convolution with a **circuit impulse function** $g(t)$.



Q: *But just what **is** this "circuit impulse response"??*

A: An impulse response is simply the **response** of one circuit function (i.e., $i(t)$ or $v(t)$) due to a **specific** stimulus by another.

➡ That specific stimulus is the **impulse function** $\delta(t)$.

The impulse function **can** be defined as:

$$\delta(t) = \lim_{\tau \to 0} \frac{1}{\tau} \frac{\sin\left(\frac{\pi t}{\tau}\right)}{\left(\frac{\pi t}{\tau}\right)}$$

Such that is has the following two **properties**:

1. $\delta(t) = 0 \quad \text{for} \quad t \neq 0$

2. $\int_{-\infty}^{\infty} \delta(t)\, dt = 1.0$

The impulse responses of the **one-port example** are therefore defined as:

$$g_z(t) \doteq v(t)\big|_{i(t)=\delta(t)}$$

and:

$$g_y(t) \doteq i(t)\big|_{v(t)=\delta(t)}$$

Meaning simply that $g_z(t)$ is equal to the **voltage** function $v(t)$ when the circuit is "thumped" with a **impulse current** (i.e., $i(t)=\delta(t)$), and $g_y(t)$ is equal to the **current** $i(t)$ when the circuit is "thumped" with a **impulse voltage** (i.e., $v(t)=\delta(t)$).

Similarly, the relationship between the **input** and the **output** of a **two-port** network can be expressed as:

$$v_2(t)=\mathcal{L}_{21}\left[v_1(t)\right]=\int_{-\infty}^{t} g(t-t')\,v_1(t')\,dt'$$

where:

$$g(t)\doteq v_2(t)\big|_{v_1(t)=\delta(t)}$$

Note that the circuit impulse response must be **causal** (nothing can occur at the output **until** something occurs at the input), so that:

$$g(t)=0 \quad \text{for} \quad t<0$$

Q: *Yikes! I recall evaluating convolution integrals to be messy, difficult and **stressful**. Surely there is an **easier** way to describe linear circuits!?!*

A: Nope! The convolution integral is **all** there is. **However,** we can use our linear systems theory toolbox to greatly **simplify the evaluation** of a convolution integral!