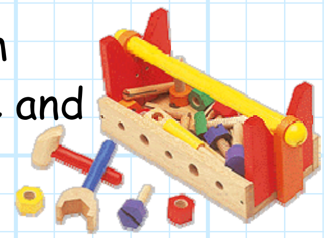
Linear Circuits

Many analog devices and circuits are **linear** (or approximately so).

Let's make sure that we understand what this term means, as if a circuit is linear, we can apply a large and helpful **mathematical** toolbox!



Mathematicians often speak of **operators**, which is "mathspeak" for any mathematical operation that can be applied to a single **element** (e.g., value, variable, vector, matrix, or function).

For example, a **function** $f(x)$ describes an operation **on** variable x (i.e., $f(x)$ is operator on x). E.G.:

$$f(y) = y^2 - 3 \qquad g(t) = 2t \qquad y(x) = |x|$$

Moreover, we find that functions can likewise be operated **on**! For example, **integration** and **differentiation** are likewise mathematical operations—operators that operate **on functions**. E.G.,:

$$\int f(y)\,dy \qquad \frac{d\,g(t)}{dt} \qquad \int_{-\infty}^{\infty} |y(x)|\,dx$$

A special and very important class of operators are **linear operators**.

Linear operators are **denoted** as $\mathcal{L}[y]$, where:

* $\mathcal{L}$ symbolically denotes the mathematical **operation**;

* And y denotes the **element** (e.g., function, variable, vector) being **operated on**.

A **linear** operator is any operator that satisfies the following **two** statements for any and **all** y:

1. $\mathcal{L}[y_1 + y_2] = \mathcal{L}[y_1] + \mathcal{L}[y_2]$

2. $\mathcal{L}[a\,y] = a\,\mathcal{L}[y]$, where a is any **constant**.

From these two statements we can **likewise** conclude that a linear operator has the property:

$$\mathcal{L}[a\,y_1 + b\,y_2] = a\,\mathcal{L}[y_1] + b\,\mathcal{L}[y_2]$$

where both a and b are constants.

Essentially, a linear operator has the property that any weighted sum of solutions is **also** a solution!

For **example**, consider the function:

$$\mathcal{L}[t] = g(t) = 2t$$

At $t = 1$:

$$g(t=1) = 2(1) = 2$$

and at $t = 2$:

$$g(t=2) = 2(2) = 4$$

Now at $t = 1+2 = 3$ we find:

$$\begin{aligned} g(1+2) &= 2(3) \\ &= 6 \\ &= 2+4 \\ &= g(1)+g(2) \end{aligned}$$

More generally, we find that:

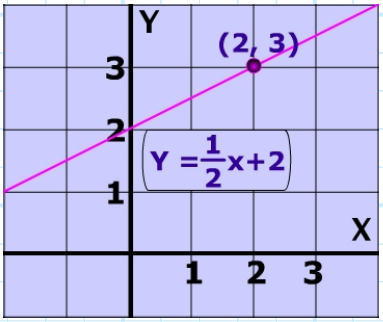
$$\begin{aligned} g(t_1+t_2) &= 2(t_1+t_2) \\ &= 2t_1+2t_2 \\ &= g(t_1)+g(t_2) \end{aligned}$$

and

$$\begin{aligned} g(at) &= 2at \\ &= a\,2t \\ &= a\,g(t) \end{aligned}$$

Thus, we conclude that the function $g(t) = 2t$ is **indeed** a **linear** function!

Now consider **this** function:



$$y(x) = mx + b$$

Q: *But that's the equation of a **line**! That **must** be a linear function, right?*

A: I'm **not sure**—let's find out!

We find that:

$$\begin{aligned} y(ax) &= m(ax) + b \\ &= a\,mx + b \end{aligned}$$

but:

$$\begin{aligned} a\,y(x) &= a(mx + b) \\ &= a\,mx + ab \end{aligned}$$

therefore:

$$y(ax) \neq a\,y(x) \quad !!!$$

Likewise:

$$y(x_1+x_2) = m(x_1+x_2)+b$$
$$= m\,x_1 + m\,x_2 + b$$

but:

$$y(x_1)+y(x_2) = (m\,x_1+b)+(m\,x_2+b)$$
$$= m\,x_1 + m\,x_2 + 2b$$

therefore:

$$y(x_1+x_2) \neq y(x_1)+y(x_2) \quad !!!$$

The equation of a line is **not** a linear function!

Moreover, **you** can show that the functions:

$$f(y) = y^2 - 3 \qquad\qquad y(x) = |x|$$

are likewise **non-linear**.

Remember, linear operators need **not** be **functions**. Consider the derivative operator, which operates **on** functions.

$$\frac{d\,f(x)}{dx}$$

Note that:

$$\frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d\,f(x)}{dx} + \frac{d\,g(x)}{dx}$$

and also:

$$\frac{d}{dx}\left[a\,f(x)\right] = a\,\frac{d\,f(x)}{dx}$$

We thus can conclude that the **derivative** operation is a **linear** operator **on function** $f(x)$:

$$\frac{d\,f(x)}{dx} = \mathcal{L}\left[f(x)\right]$$

You can likewise show that the **integration** operation is likewise a **linear operator**:

$$\int f(y)\,dy = \mathcal{L}\left[f(y)\right]$$

But, you will find that operations such as:

$$\frac{d\,g^2(t)}{dt} \qquad\qquad \int_{-\infty}^{\infty} \left|y(x)\right| dx$$

are **not** linear operators (i.e., they are **non-linear** operators).

We find that **most** mathematical operations are in fact **non-linear**! Linear operators are thus form a small **subset** of all possible mathematical operations.

Q: *Yikes! If linear operators are so* ***rare****, we are we* ***wasting*** *our time learning about them??*

A: **Two** reasons!

Reason 1: In electrical engineering, the behavior of most of our fundamental **circuit elements** are described by **linear operators**—linear operations are prevalent in **circuit analysis**!

Reason 2: To our great relief, the two characteristics of linear operators allow us to **perform** these mathematical operations with **relative ease**!