# Signal Expansions

**Q:** ***How** is performing a **linear** operation easier than performing a **non-linear** one??*

**A:** The "secret" lies is the result:

$$\mathcal{L}\left[a\,y_1 + b\,y_2\right] = a\,\mathcal{L}\left[y_1\right] + b\,\mathcal{L}\left[y_2\right]$$

Note here that the linear operation performed on a relatively **complex** element $a\,y_1 + b\,y_2$ can be determined immediately from the result of operating on the **"simple"** elements $y_1$ and $y_2$.

To see how this might work, let's consider some **arbitrary** function of **time** $v(t)$, a function that exists over some **finite** amount of time $T$ (i.e., $v(t) = 0$ for $t < 0$ and $t > T$).

Say we wish to perform some **linear** operation **on this function**:

$$\mathcal{L}\left[v(t)\right] = ??$$

# Complex signals as collections of simple elements

Depending on the **difficulty** of the operation $\mathcal{L}$, and/or the **complexity** of the function $v(t)$, directly performing this operation could be very **painful** (i.e., approaching impossible).

Instead, we find that we can often **expand** a very complex and **stressful** function in the following way:

$$v(t) = a_0\,\psi_0(t) + a_1\,\psi_1(t) + a_2\,\psi_2(t) + \cdots = \sum_{n=-\infty}^{\infty} a_n\,\psi_n(t)$$

where the values $a_n$ are **constants** (i.e., coefficients), and the functions $\psi_n(t)$ are known as **basis functions**.

# Ms. Nomial's first name is Poly

For example, we could **choose** the basis functions:

$$\psi_n(t) = t^n \quad \text{for} \quad n \geq 0$$

Resulting in a **polynomial** of variable $t$:

$$v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots = \sum_{n=0}^{\infty} a_n t^n$$

This signal expansion is of course know as the **Taylor Series** expansion.

# Choose your basis – but choose wisely

However, there are **many other** useful expansions (i.e., many other useful basis $\psi_n(t)$).

* The key thing is that the basis functions $\psi_n(t)$ are **independent** of the function $v(t)$. That is to say, the basis functions are **selected** by the engineer doing the analysis (i.e., **you**).

* The set of selected basis functions form what's known as a **basis**. With this basis we can **analyze** the function $v(t)$.

* The **result** of this analysis provides the **coefficients** $a_n$ of the signal expansion. Thus, the coefficients **are** directly dependent on the form of function $v(t)$ (as well as the basis used for the analysis). As a result, the set of coefficients $\{a_1, a_2, a_3, \cdots\}$ **completely describe** the function $v(t)$!

# It's simpler to operate on each element

**Q:** *I don't see why this "expansion" of function of $v(t)$ is helpful, it just looks like a lot more **work** to me.*

**A:** Consider what happens when we wish to perform a **linear** operation on this function:

$$\mathcal{L}\left[v(t)\right] = \mathcal{L}\left[\sum_{n=-\infty}^{\infty} a_n \, \psi_n(t)\right] = \sum_{n=-\infty}^{\infty} a_n \, \mathcal{L}\left[\psi_n(t)\right]$$

**Look what happened!**

**Instead** of performing the linear operation on the arbitrary and **difficult** function $v(t)$, we can apply the operation to **each** of the individual basis functions $\psi_n(t)$.

# Choose a basis that makes this "easy"

**Q:** *And that's supposed to be **easier**??*

**A:** It **depends** on the linear operation and on the basis functions $\psi_n(t)$.

**Hopefully**, the operation $\mathcal{L}[\psi_n(t)]$ is **simple** and straightforward.

**Ideally**, the solution to $\mathcal{L}[\psi_n(t)]$ is **already known**!


**Q:** *Oh yeah, like I'm going to get so **lucky**. I'm sure in all my circuit analysis problems evaluating $\mathcal{L}[\psi_n(t)]$ will be long, frustrating, and **painful**.*

**A:** Remember, **you** get to choose the **basis** over which the function $v(t)$ is analyzed.

A **smart** engineer will **choose** a basis for which the operations $\mathcal{L}[\psi_n(t)]$ are simple and **straightforward**!

# This basis is quite popular

**Q:** *But I'm **still** confused. **How** do I choose what basis $\psi_n(t)$ to use, and **how** do I analyze the function $v(t)$ to determine the coefficients $a_n$??*

**A:** Perhaps an **example** would help. Among the **most popular** basis is this one:

$$\psi_n = \begin{cases} e^{j\left(\frac{2\pi n}{T}\right)t} & 0 \le t \le T \\ 0 & t \le 0, t \ge T \end{cases}$$

and:

$$a_n = \frac{1}{T}\int_0^T v(t)\,\psi_n^*(t)\,dt = \frac{1}{T}\int_0^T v(t)\,e^{-j\left(\frac{2\pi n}{T}\right)t}\,dt$$

So therefore:

$$v(t) = \sum_{n=-\infty}^{\infty} a_n\, e^{j\left(\frac{2\pi n}{T}\right)t} \qquad \text{for } 0 \le t \le T$$

The **astute** among you will recognize this signal expansion as the **Fourier Series**!

# It has a very important property!

**Q:** *Yes, just why is Fourier analysis **so** prevalent?*

**A:** The answer reveals itself when we apply a **linear operator** to the signal expansion:

$$\mathcal{L}\left[v(t)\right]=\mathcal{L}\left[\sum_{n=-\infty}^{\infty} a_n\, e^{-j\left(\frac{2\pi n}{T}\right)t}\right]=\sum_{n=-\infty}^{\infty} a_n\, \mathcal{L}\left[e^{-j\left(\frac{2\pi n}{T}\right)t}\right]$$

Note then that we must **simply** evaluate:

$$\mathcal{L}\left[e^{-j\left(\frac{2\pi n}{T}\right)t}\right]$$

for all *n*.

We will find that **performing** almost any linear operation $\mathcal{L}$ on basis functions of this type to be exceeding **simple** (more on this later)!