The Eigen Function of Linear Systems

Recall that that we can express (**expand**) a time-limited signal with a weighted summation of **basis functions**:

$$v(t) = \sum_n a_n \psi_n(t)$$

where $v(t) = 0$ for $t < 0$ and $t > T$.

Say now that we **convolve** this signal with some system **impulse function** $g(t)$:

$$\begin{aligned}
\mathcal{L}\left[v(t)\right] &= \int_{-\infty}^{t} g(t-t')\, v(t')\, dt' \\
&= \int_{-\infty}^{t} g(t-t') \sum_n a_n\, \psi_n(t')\, dt' \\
&= \sum_n a_n \int_{-\infty}^{t} g(t-t')\, \psi_n(t')\, dt'
\end{aligned}$$

Look what happened!

Instead of convolving the general function $v(t)$, we now find that we must simply convolve with the set of basis functions $\psi_n(t)$.

Q: *Huh? You say we must **"simply"** convolve the set of basis functions $\psi_n(t)$. **Why** would this be any simpler?*

A: Remember, **you** get to **choose** the basis $\psi_n(t)$. If you're **smart**, you'll choose a set that makes the convolution integral **"simple"** to perform!

Q: *But don't I first need to **know** the explicit form of $g(t)$ **before** I intelligently choose $\psi_n(t)$??*

A: Not necessarily!

The **key** here is that the convolution integral:

$$\mathcal{L}\left[\psi_n(t)\right] = \int_{-\infty}^{t} g(t-t')\,\psi_n(t')\,dt'$$

is a **linear, time-invariant** operator. Because of this, there exists one **basis** with an **astonishing** property!

These **special** basis functions are:

$$\psi_n(t) = \begin{cases} e^{j\omega_n t} & \text{for} \quad 0 \le t \le T \\ 0 & \text{for} \quad t < 0, t > T \end{cases} \qquad \text{where} \qquad \omega_n = n\left(\frac{2\pi}{T}\right)$$

Now, inserting this function (get ready, here comes the **astonishing** part!) into the convolution integral:

$$\mathcal{L}\left[e^{j\omega_n t}\right] = \int_{-\infty}^{t} g(t-t')\, e^{j\omega_n t'}\, dt'$$

and using the substitution $u = t - t'$, we get:

$$\begin{aligned}\int_{-\infty}^{t} g(t-t')\, e^{j\omega_n t}\, dt' &= \int_{t-(-\infty)}^{t-t} g(u)\, e^{j\omega_n (t-u)}\, (-du) \\ &= e^{j\omega_n t} \int_{+\infty}^{0} g(u)\, e^{-j\omega_n u}\, (-du) \\ &= e^{j\omega_n t} \int_{0}^{\infty} g(u)\, e^{-j\omega_n u}\, du\end{aligned}$$

See! Doesn't **that** astonish!

Q: *I'm astonished only by how **lame** you are. How is this result any **more** "astonishing" than any of the **other** supposedly "useful" things you've been telling us?*

A: Note that the integration in this **result** is **not** a convolution—the integral is simply a **value** that depends on n (but **not** time t):

$$G(\omega_n) \doteq \int_{0}^{\infty} g(t)\, e^{-j\omega_n t}\, dt$$

As a result, convolution with this "special" set of basis functions can **always** be expressed as:

$$\int_{-\infty}^{t} g(t-t')\, e^{j\omega_n t'}\, dt' = \mathcal{L}\left[e^{j\omega_n t} \right] = G(\omega_n)\, e^{j\omega_n t}$$

The **remarkable** thing about this result is that the linear operation on function $\psi_n(t) = exp[j\omega_n t]$ results in precisely the **same** function of **time** t (save the **complex** multiplier $G(\omega_n)$)! I.E.:

$$\mathcal{L}\left[\psi_n(t)\right] = G(\omega_n)\ \ \psi_n(t)$$

Convolution with $\psi_n(t) = exp[j\omega_n t]$ is accomplished by simply **multiplying** the function by the **complex** number $G(\omega_n)$!

Note this is true **regardless** of the impulse response $g(t)$ (the function $g(t)$ affects the **value** of $G(\omega_n)$ **only**)!

Q: *Big deal! Aren't there lots of **other** functions that would satisfy the equation above equation?*

A: Nope. The **only** function where this is true is:

$$\psi_n(t) = e^{j\omega_n t}$$

This function is thus **very** special. We call this function the **eigen function** of linear, time-invariant systems.

Q: *Are you **sure** that there are no other eigen functions??*

A: Well, sort of.

Recall from **Euler's equation** that:

$$e^{j\omega_n t} = \cos \omega_n t + j \sin \omega_n t$$

It can be shown that the sinusoidal functions $\cos \omega_n t$ and $\sin \omega_n t$ are **likewise** eigen functions of linear, time-invariant systems.

The real and imaginary components of eigen function $\exp[j\omega_n t]$ are **also** eigen functions.

Q: *What about the set of **values** $G(\omega_n)$?? Do they have any significance or **importance**??*

A: Absolutely!

Recall the values $G(\omega_n)$ (one for each *n*) depend on the **impulse response** of the system (e.g., circuit) **only**:

$$G(\omega_n) \doteq \int_0^\infty g(t)\, e^{-j\omega_n t}\, dt$$

Thus, the set of values $G(\omega_n)$ completely **characterizes** a linear time-invariant **circuit** over time $0 \le t \le T$.

We call the values $G(\omega_n)$ the **eigen values** of the linear, time-invariant circuit.

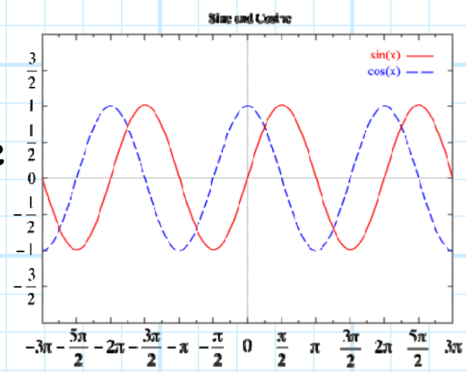
Q: *OK Poindexter, all **eigen** stuff this **might** be interesting if you're a mathematician, but is it at all **useful** to us **electrical engineers**?*

A: It is **unfathomably** useful to us electrical engineers!

Say a linear, time-invariant circuit is excited (only) by a **sinusoidal** source (e.g., $v_s(t) = \cos \omega_o t$). Since the source function is the **eigen function** of the circuit, we will find that at **every** point in the circuit, **both** the current and voltage will have the **same functional form**.

That is, every current and voltage in the circuit will likewise be a **perfect sinusoid** with frequency ω_o!!

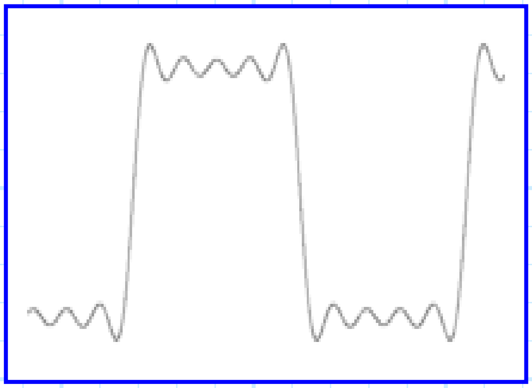
Of course, the **magnitude** of the sinusoidal oscillation will be **different** at different points within the circuit, as will the **relative phase**. But we know that **every** current and voltage in the circuit can be **precisely** expressed as a function of this form:



$$A\cos(\omega_o t + \varphi)$$

Q: *Isn't this pretty **obvious**?*

A: **Why** should it be? Say our source function was instead a **square** wave, or **triangle** wave, or a **sawtooth** wave. We would find that (generally speaking) **nowhere** in the circuit would we find another current or voltage that was a **perfect** square wave (etc.)!



In fact, we would find that not only are the current and voltage functions within the circuit **different** than the source function (e.g. a sawtooth) they are (generally speaking) all different **from each other**.

We find then that a linear circuit will (generally speaking) **distort** any source function—**unless** that function is the **eigen function** (i.e., an sinusoidal function).

Thus, using an **eigen function** as circuit source greatly simplifies our linear circuit analysis problem. **All** we need to accomplish this is to **determine the magnitude** A and **relative phase** φ of the resulting (and otherwise **identical**) sinusoidal function!