The Eigen Values of Linear Circuits

Recall the linear operators that define a **capacitor**:

$$\mathcal{L}_{\mathcal{Y}}^{C}\left[v_C(t)\right] = i_C(t) = C\,\frac{d\,v_C(t)}{d\,t}$$

$$\mathcal{L}_{\mathcal{Z}}^{C}\left[i_C(t)\right] = v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(t')\,dt'$$

We now know that the **eigen function** of these linear, time-invariant operators—like **all** linear, time-invariant operartors—is $\exp\left[j\omega t\right]$.

The question now is, **what** is the **eigen value** of each of these operators? It is this value that **defines** the physical behavior of a given capacitor!

For $v_C(t) = \exp\left[j\omega t\right]$, we find:

$$\begin{aligned} i_C(t) &= \mathcal{L}_{\mathcal{Y}}^{C}\left[v_C(t)\right] \\ &= C\,\frac{d\,e^{j\omega t}}{d\,t} \\ &= \left(j\omega C\right)e^{j\omega t} \end{aligned}$$

Just as we expected, the eigen function $\exp[j\omega t]$ **"survives"** the linear operation **unscathed**—the current function $i(t)$ has **precisely** the same form as the voltage function $v(t)=\exp[j\omega t]$.

The **only** difference between the **current** and **voltage** is the multiplication of the **eigen value**, denoted as $\mathcal{G}_{\mathcal{Y}}^{C}(\omega)$.

$$i(t)=\mathcal{L}_{\mathcal{Y}}^{C}\left[v(t)=e^{j\omega t}\right]=\mathcal{G}_{\mathcal{Y}}^{C}(\omega)\, e^{j\omega t}$$

Since we **just** determined that for this case:

$$i(t)=\left(j\omega C\right) e^{j\omega t}$$

it is **evident** that the eigen value of the linear operation:

$$i(t)=\mathcal{L}_{\mathcal{Y}}^{C}\left[v(t)\right]=C\,\frac{d\,v(t)}{d\,t}$$

is:

$$\mathcal{G}_{\mathcal{Y}}^{C}(\omega)=j\omega C=\omega C\, e^{j\pi/2} \quad !!!$$

So for **example**, if:

$$\begin{aligned} v(t) &= V_m \cos\left(\omega_o t+\varphi\right) \\ &= \text{Re}\left\{\left(V_m\, e^{j\varphi}\right) e^{j\omega_o t}\right\} \end{aligned}$$

we will find that:

$$\mathcal{L}_{\mathcal{Y}}^{C}\left[\left(V_m\, e^{j\varphi}\right)e^{j\omega_o t}\right] = G_{\mathcal{Y}}^{C}(\omega_o)\left(V_m\, e^{j\varphi}\right)e^{j\omega_o t}$$
$$= \left(\omega C\; e^{j\pi/2}\right)\left(V_m\, e^{j\varphi}\right)e^{j\omega_o t}$$
$$= \left(\omega C\, V_m\; e^{j\left(\pi/2+\varphi\right)}\right)e^{j\omega_o t}$$

Therefore:

$$i_C(t) = \operatorname{Re}\left\{\omega C\, V_m e^{j\left(\varphi+\pi/2\right)} e^{j\omega_o t}\right\}$$
$$= \omega C\;\; V_m \cos\left(\omega_o t + \varphi + \pi/2\right)$$
$$= -\omega C\;\; V_m \sin\left(\omega_o t + \varphi\right)$$

Hopefully, this example again emphasizes that these **real-valued** sinusoidal functions can be completely expressed in terms of **complex values**. For example, the complex value:

$$V_C = V_m e^{j\varphi}$$

means that the magnitude of the sinusoidal **voltage** is $|V_C| = V_m$, and its relative phase is $\angle V_C = \varphi$.

The complex value:

$$\mathcal{I}_C = G_{\mathcal{Y}}^{C}(\omega)\, V_C$$
$$= \left(\omega C\; e^{j\pi/2}\right) V_C$$

likewise means that the **magnitude** of the sinusoidal **current** is:

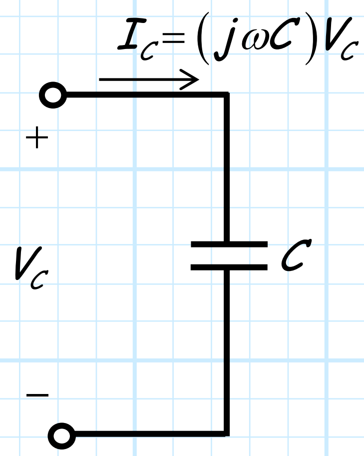
$$\begin{aligned}|I_C| &= |G_y^C(\omega)\,V_C| \\ &= |G_y^C(\omega)|\,|V_C| \\ &= \omega C\,V_m\end{aligned}$$

And the relative **phase** of the sinusoidal **current** is:

$$\begin{aligned}\angle I_C &= \angle G_y^C(\omega) + \angle V_C \\ &= \pi/2 + \varphi\end{aligned}$$

We can thus **summarize** the behavior of a capacitor with the simple **complex equation**:

$$\begin{aligned}I_C &= (j\omega C)\,V_C \\ &= \left(\omega C\, e^{j\pi/2}\right) V_C\end{aligned}$$



Now let's return to the **second** of the two linear operators that describe a capacitor:

$$v_C(t) = \mathcal{L}_z^C\left[i_C(t)\right] = \frac{1}{C}\int_{-\infty}^{t} i_C(t')\,dt'$$

Now, if the capacitor **current** is the eigen function $i_C(t) = \exp[j\omega t]$, we find:

$$\mathcal{L}_{\mathcal{Z}}^{C}\left[e^{j\omega t}\right]=\frac{1}{C}\int_{-\infty}^{t}e^{j\omega t'}\,dt'$$

$$=\left(\frac{1}{j\omega C}\right)e^{j\omega t}$$

where we assume $i(t=-\infty)=0$.

Thus, we can conclude that:

$$\mathcal{L}_{\mathcal{Z}}^{C}\left[e^{j\omega t}\right]=G_{\mathcal{Z}}^{C}(\omega)\,e^{j\omega t}=\left(\frac{1}{j\omega C}\right)e^{j\omega t}$$

Hopefully, it is evident that the **eigen value** of this linear operator is:

$$G_{\mathcal{Z}}^{C}(\omega)=\frac{1}{j\omega C}=\frac{-j}{\omega C}=\frac{1}{\omega C}e^{j\left(3\pi/2\right)}$$

And so:

$$V_C=\left(\frac{1}{j\omega C}\right)I_C$$

Q: *Wait a second! Isn't this essentially the* ***same*** *result as the one derived for operator* $\mathcal{L}_{\mathcal{Y}}^{C}$ *??*

A: It's **precisely** the same! For both operators we find:

$$\frac{V_C}{I_C}=\frac{1}{j\omega C}$$

This should **not** be surprising, as **both** operators $\mathcal{L}_{\mathcal{Y}}^{C}$ and $\mathcal{L}_{\mathcal{Z}}^{C}$ relate the current through and voltage across the **same** device (a capacitor).

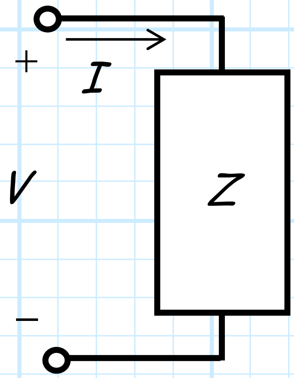
The **ratio** of complex voltage to complex current is of course referred to as the complex device **impedance** Z.

$$Z \doteq \frac{V}{I}$$

An **impedance** can be determined for **any** linear, time-invariant **one-port** network—but **only** for linear, time-invariant one-port networks!

Generally speaking, impedance is a **function of frequency**. In fact, the impedance of a one-port network is simply the **eigen value** $G_{\mathcal{Z}}(\omega)$ of the linear operator $\mathcal{L}_{\mathcal{Z}}$:

$$V = Z\,I$$



$$\mathcal{L}_{\mathcal{Z}}\left[i(t)\right] = v(t)$$

$$Z = G_{\mathcal{Z}}(\omega)$$

Note that impedance is a **complex** value that provides us with **two** things:

1. The **ratio of the magnitudes** of the sinusoidal voltage and current:

$$|Z| = \frac{|V|}{|I|}$$

2. The **difference in phase** between the sinusoidal voltage and current:

$$\angle Z = \angle V - \angle I$$

Q: *What about the linear operator:*

$$\mathcal{L}_{\mathcal{Y}}\left[v(t)\right] = i(t) \quad ??$$

A: Hopefully it is now evident to **you** that:

$$G_{\mathcal{Y}}(\omega) = \frac{1}{G_{\mathcal{Z}}(\omega)} = \frac{1}{Z}$$

The inverse of impedance is **admittance** Y:

$$Y \doteq \frac{1}{Z} = \frac{I}{V}$$

Now, returning to the **other two** linear circuit elements, we find (and **you** can verify) that for resistors:

$$\mathcal{L}_{\mathcal{Y}}^{R}\left[v_R(t)\right] = i_R(t) \quad \Rightarrow G_{\mathcal{Y}}^{R}(\omega) = 1/R$$

$$\mathcal{L}_{\mathcal{Z}}^{R}\left[i_R(t)\right] = v_R(t) \quad \Rightarrow G_{\mathcal{Z}}^{R}(\omega) = R$$

and for inductors:

$$\mathcal{L}_{\mathcal{Y}}^{L}\left[v_L(t)\right] = i_L(t) \quad \Rightarrow \mathcal{G}_{\mathcal{Y}}^{L}(\omega) = \frac{1}{j\omega L}$$

$$\mathcal{L}_{\mathcal{Z}}^{L}\left[i_L(t)\right] = v_L(t) \quad \Rightarrow \mathcal{G}_{\mathcal{Z}}^{L}(\omega) = j\omega L$$

meaning:

$$Z_R = \frac{1}{Y_R} = R = R\, e^{j0} \qquad \text{and} \qquad Z_L = \frac{1}{Y_L} = j\omega L = \omega L\, e^{j\left(\pi/2\right)}$$

Now, note that the relationship

$$Z = \frac{V}{I}$$

forms a **complex "Ohm's Law"** with regard to complex currents and voltages.

Additionally, ICBST (It Can Be Shown That) **Kirchoff's Laws** are likewise valid for complex currents and voltages:

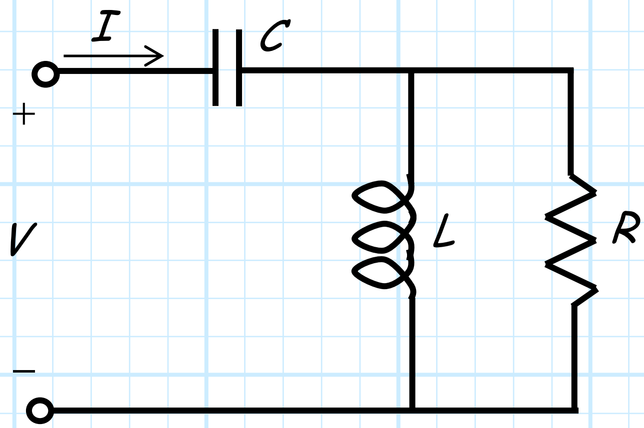
$$\sum_n I_n = 0 \qquad \sum_n V_n = 0$$

where of course the summation represents **complex addition**.

As a result, the impedance (i.e., the eigen value) of **any** one-port device can be determined by simply applying a **basic** knowledge of **linear circuit analysis**!

Returning to the example:

$$Z = \frac{V}{I}$$



And thus using out **basic** circuits knowledge, we find:

$$Z = Z_C + Z_R \| Z_L = 1/_{j\omega C} + R \| j\omega L$$

Thus, the eigen value of the linear operator:

$$\mathcal{L}_{\mathcal{Z}}\left[i(t)\right] = v(t)$$

For **this** one-port network is:

$$G_{\mathcal{Z}}(\omega) = 1/_{j\omega C} + R \| j\omega L$$

Look what we did! We were able to determine $G_{\mathcal{Z}}(\omega)$ **without** explicitly determining impulse response $g_{\mathcal{Z}}(t)$, or having to perform **any** integrations!

Now, if we actually **need** to determine the voltage function $v(t)$ created by some **arbitrary** current function $i(t)$, we integrate:

$$\begin{aligned} v(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} G_{\mathcal{Z}}(\omega)\, I(\omega)\, e^{j\omega t} d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(1/_{j\omega C} + R \| j\omega L \right) I(\omega)\, e^{j\omega t} d\omega \end{aligned}$$

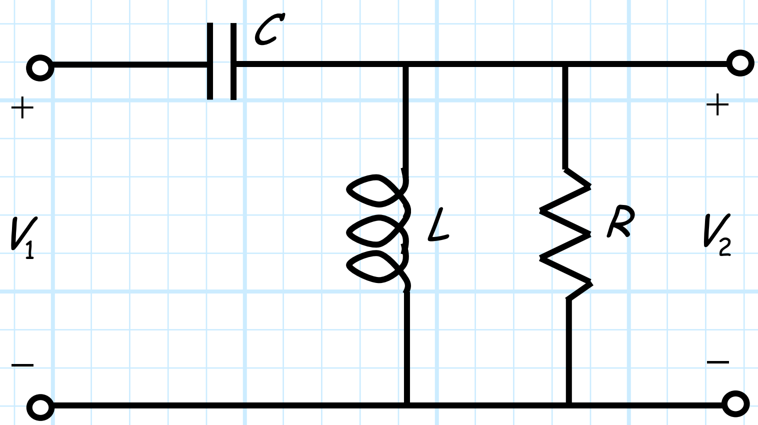
where:

$$I(\omega) = \int_{-\infty}^{+\infty} i(t)\, e^{-j\omega t} dt$$

Otherwise, if our current function is **time-harmonic** (i.e., sinusoidal with frequency ω), we can simply relate complex current I and complex voltage V with the equation:

$$\begin{aligned} V &= Z\, I \\ &= \left(1/_{j\omega C} + R \| j\omega L \right) I \end{aligned}$$

Similarly, for our **two-port** example:



we can likewise determine from **basic** circuit theory the **eigen value** of the linear operator:

$$\mathcal{L}_{21}\left[v_1(t)\right] = v_2(t)$$

is:

$$G_{21}(\omega) = \frac{Z_L \| Z_R}{Z_C + Z_L \| Z_R} = \frac{j\omega L \| R}{\dfrac{1}{j\omega C} + j\omega L \| R}$$

so that:

$$V_2 = G_{21}(\omega)\, V_1$$

or more generally:

$$v_2(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_{21}(\omega)\, V_1(\omega)\, e^{j\omega t}\, d\omega$$

where:

$$V_1(\omega) = \int_{-\infty}^{+\infty} v_1(t)\, e^{-j\omega t}\, dt$$