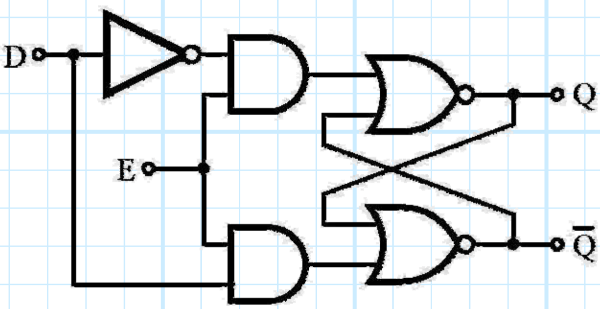
EECS 412 Introduction

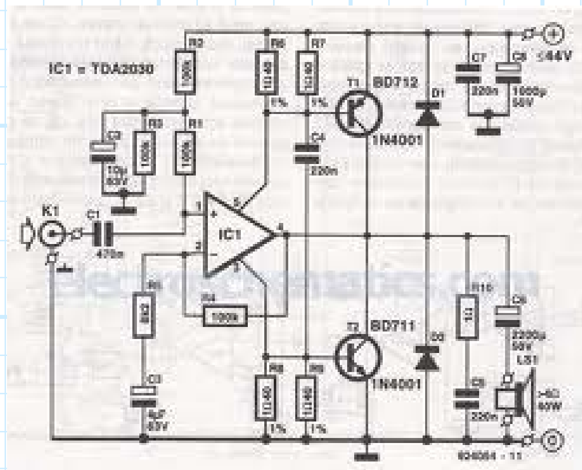
Q: *So what's this class all about? What is its **purpose**?*

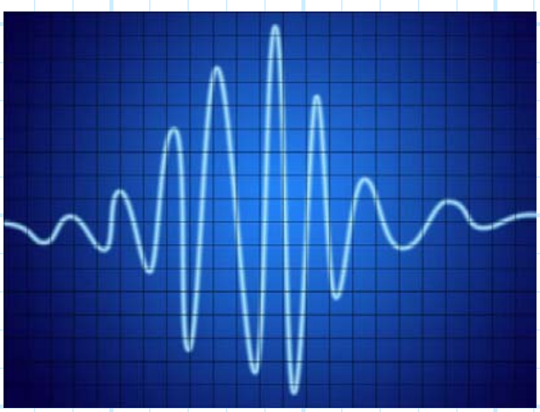
A: In EECS **312** you learned about:



* Electronic **devices** (e.g., transistors and diodes)
* How we use transistors to make **digital** devices (e.g., inverters, gates, flip-flops, and memory).

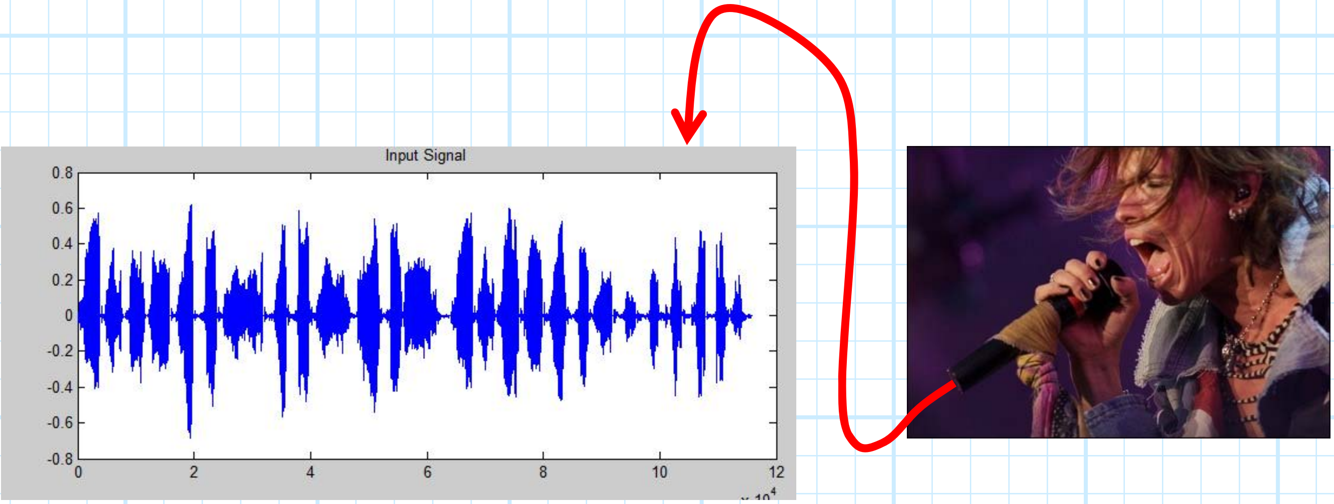
In contrast, EECS 412 will teach you how we use transistors to make **analog** devices (e.g., amplifiers, filters, summers, integrators, etc.).

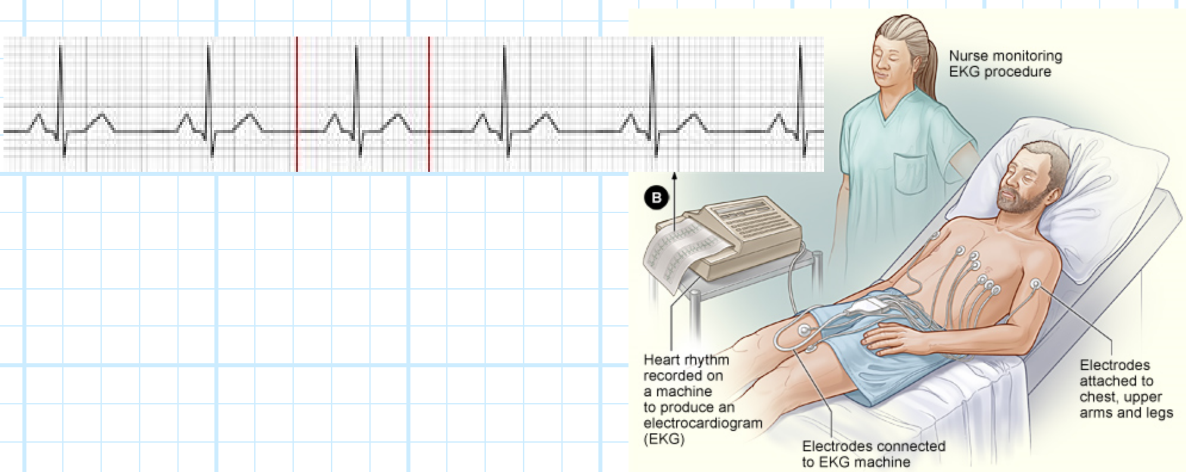




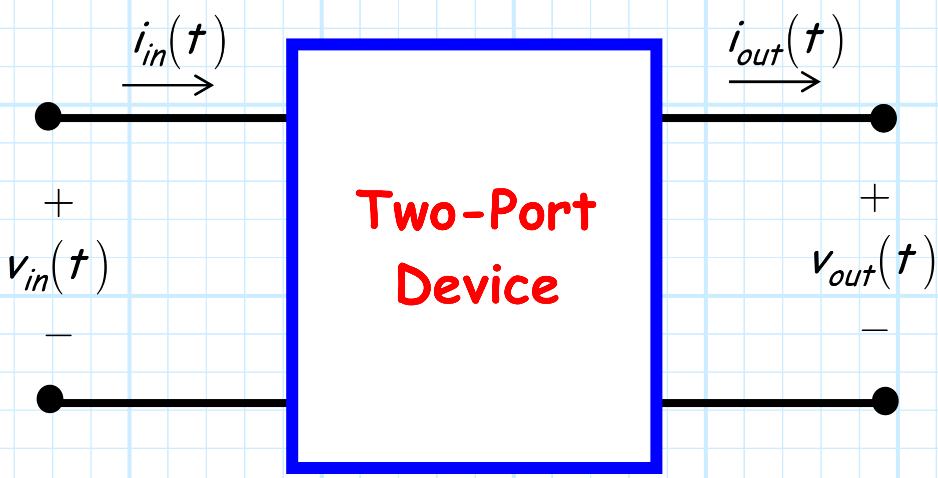
Analog circuits and devices operate on analog signals—usually voltage signals—that represent a **continuous**, time-varying analog of some physical function.

For **example**, the analog voltage signal $v(t)$ can represent an audio pressure wave (i.e., sound), or the beating of a human heart.





Quite often, an analog device has two ports—an **input** port and an **output** port:



A fundamental question in electrical engineering is **determining** the **output** signal $v_{out}(t)$ when the input signal $v_{in}(t)$ is known.

This is frequently a difficult question to answer, but it becomes significantly easier if the two-port device is constructed of **linear**, time-invariant circuit elements!

HO: THE LINEAR, TIME-INVARIANT CIRCUIT

Linear circuit behavior would be not at all useful except for the **unfathomably important** concept of signal expansion via **basis functions**!

HO: SIGNAL EXPANSIONS

Linear systems theory is useful for electrical engineers because most **analog devices and systems are linear** (at least approximately so!).

HO: LINEAR CIRCUIT ELEMENTS

The most powerful tool for analyzing linear systems is its **Eigen function**.

HO: THE EIGEN FUNCTION OF LINEAR SYSTEMS

Complex voltages and currents at times cause much **head scratching**; let's make sure we know what these complex values and functions **physically** mean.

HO: A COMPLEX REPRESENTATION OF SINUSOIDAL FUNCTIONS

Signals may **not** have the explicit form of an Eigen function, **but** our linear systems theory allows us to (relatively) easily analyze this case as well.

HO: ANALYSIS OF CIRCUITS DRIVEN BY ARBITRARY FUNCTIONS

If our linear system is a linear **circuit**, we can apply **basic** circuit analysis to determine all its **Eigen values**!

HO: THE EIGEN SPECTRUM OF LINEAR CIRCUITS

A more general form of the Fourier Transform is the **Laplace** Transform.

HO: THE EIGEN VALUES OF THE LAPLACE TRANSFORM

The numerical value of **frequency** ω has tremendous practical ramifications to us EEs.

HO: FREQUENCY BANDS

A set of **four** Eigen values can completely characterize a two-port linear system.

HO: THE IMPEDANCE AND ADMITTANCE MATRIX

A really important linear (sort of) device is the **amplifier**.

HO: THE AMPLIFIER

The two most important parameter of an amplifier is its **gain** and its **bandwidth**.

HO: AMPLIFIER GAIN AND BANDWIDTH

Amplifier circuits can be quite complex; however, we can use a relatively simple **equivalent circuit** to analyze the result when we connect things to them!

HO: CIRCUIT MODELS FOR AMPLIFIERS

One very useful application of the circuit model is to analyze and characterize types of amplifiers.

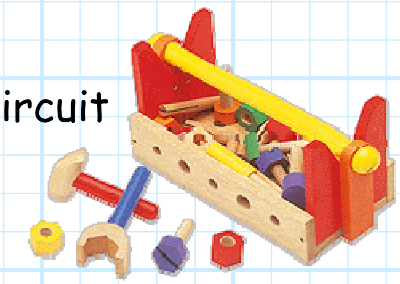
HO: CURRENT AND VOLTAGE AMPLIFIERS

It turns out that amplifiers are only **approximately linear**. It is important that we understand their **non-linear** characteristics and properties.

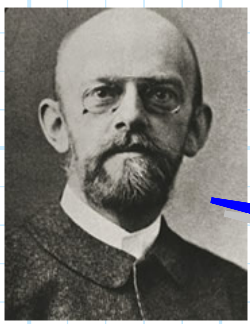
HO: NON-LINEAR BEHAVIOR OF AMPLIFIERS

Linear Circuits

Many analog devices and circuits are **linear** (or approximately so).



Let's make sure that we understand what this term means, as if a circuit is linear, we can apply a large and helpful **mathematical** toolbox!



Mathematicians often speak of **operators**, which is "mathspeak" for any mathematical operation that can be applied to a single **element** (e.g., value, variable, vector, matrix, or function).

...operators, operators, operators!!

For example, a **function** $f(x)$ describes an operation **on** variable x (i.e., $f(x)$ is operator on x). E.G.:

$$f(y) = y^2 - 3 \qquad g(t) = 2t \qquad y(x) = |x|$$

Functions can be operated on

Moreover, we find that functions can likewise be operated **on**!

For example, **integration** and **differentiation** are likewise mathematical operations—operators that operate **on functions**. E.G.,:

$$\int f(y)\,dy \qquad \frac{d\,g(t)}{dt} \qquad \int_{-\infty}^{\infty} |y(x)|\,dx$$

A special and very important class of operators are **linear operators**.

Linear operators are **denoted** as $\mathcal{L}[y]$, where:

* $\mathcal{L}$ symbolically denotes the mathematical **operation**;

* And y denotes the **element** (e.g., function, variable, vector) being **operated on**.

We call this linear superpostion

A **linear** operator is any operator that satisfies the following **two** statements for any and **all** y:

1. $\mathcal{L}[y_1 + y_2] = \mathcal{L}[y_1] + \mathcal{L}[y_2]$

2. $\mathcal{L}[a\,y] = a\,\mathcal{L}[y]$, where a is any **constant**.

From these two statements we can **likewise** conclude that a linear operator has the property:

$$\mathcal{L}[a\,y_1 + b\,y_2] = a\,\mathcal{L}[y_1] + b\,\mathcal{L}[y_2]$$

where both a and b are constants.

Essentially, a linear operator has the property that any weighted sum of solutions is **also** a solution!

An example of a linear function

For **example**, consider the function:

$$\mathcal{L}[t] = g(t) = 2t$$

At $t = 1$:

$$g(t = 1) = 2(1) = 2$$

and at $t = 2$:

$$g(t = 2) = 2(2) = 4$$

Now at $t = 1 + 2 = 3$ we find:

$$\begin{aligned} g(1+2) &= 2(3) \\ &= 6 \\ &= 2 + 4 \\ &= g(1) + g(2) \end{aligned}$$

See, it works like it's suppose to!

More generally, we find that:

$$\begin{aligned} g(t_1+t_2) &= 2(t_1+t_2) \\ &= 2t_1+2t_2 \\ &= g(t_1)+g(t_2) \end{aligned}$$

and

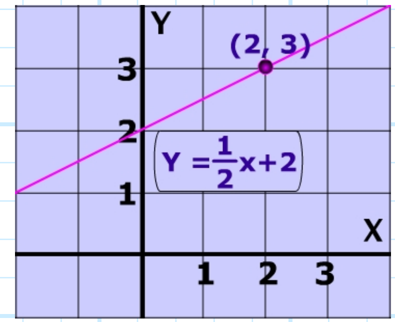
$$\begin{aligned} g(at) &= 2at \\ &= a\,2t \\ &= a\,g(t) \end{aligned}$$

Thus, we conclude that the function $g(t)=2t$ is **indeed** a **linear** function!

Surely this is linear

Now consider **this** function:

$$y(x) = m\,x + b$$



Q: *But that's the equation of a* ***line****! That* ***must*** *be a linear function, right?*

A: I'm **not sure**—let's find out!

We find that:

$$\begin{aligned} y(a\,x) &= m(ax) + b \\ &= a\,mx + b \end{aligned}$$

but:

$$\begin{aligned} a\,y(x) &= a(m\,x + b) \\ &= a\,mx + ab \end{aligned}$$

therefore:

$$y(a\,x) \neq a\,y(x) \quad !!!$$

It's not; and stop calling me Shirley

Likewise:

$$y(x_1 + x_2) = m(x_1 + x_2) + b$$
$$= m x_1 + m x_2 + b$$

but:

$$y(x_1) + y(x_2) = (m x_1 + b) + (m x_2 + b)$$
$$= m x_1 + m x_2 + 2b$$

therefore:

$$y(x_1 + x_2) \neq y(x_1) + y(x_2) \quad !!!$$

➡ The equation of a line is **not** a linear function!

Moreover, **you** can show that the functions:

$$f(y) = y^2 - 3 \qquad\qquad y(x) = |x|$$

are likewise **non-linear**.

The derivative is a linear operator

Remember, linear operators need **not** be **functions**.

Consider the derivative operator, which operates **on** functions.

$$\frac{d\,f(x)}{dx}$$

Note that:

$$\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d\,f(x)}{dx}+\frac{d\,g(x)}{dx}$$

and also:

$$\frac{d}{dx}\left[a\,f(x)\right]=a\frac{d\,f(x)}{dx}$$

We thus can conclude that the **derivative** operation is a **linear** operator **on function** $f(x)$:

$$\frac{d\,f(x)}{dx}=\mathcal{L}\left[f(x)\right]$$

Most operators are not linear

You can likewise show that the **integration** operation is likewise a **linear operator**:

$$\int f(y)\,dy = \mathcal{L}\left[f(y)\right]$$

But, you will find that operations such as:

$$\frac{d\,g^2(t)}{dt} \qquad\qquad \int_{-\infty}^{\infty} |y(x)|\,dx$$

are **not** linear operators (i.e., they are **non-linear** operators).

We find that **most** mathematical operations are in fact **non-linear**!

Linear operators are thus form a small **subset** of all possible mathematical operations.

Linear operators allow for "easy" evaluation

Q: *Yikes! If linear operators are so **rare**, we are we **wasting** our time learning about them??*

A: **Two** reasons!

Reason 1: In electrical engineering, the behavior of most of our fundamental **circuit elements** are described by **linear operators**—linear operations are prevalent in **circuit analysis**!

Reason 2: To our great relief, the two characteristics of linear operators allow us to **perform** these mathematical operations with **relative ease**!

Signal Expansions

Q: ***How** is performing a **linear** operation easier than performing a **non-linear** one??*

A: The "secret" lies is the result:

$$\mathcal{L}\left[a\,y_1 + b\,y_2\right] = a\,\mathcal{L}\left[y_1\right] + b\,\mathcal{L}\left[y_2\right]$$

Note here that the linear operation performed on a relatively **complex** element $a\,y_1 + b\,y_2$ can be determined immediately from the result of operating on the **"simple"** elements y_1 and y_2.

To see how this might work, let's consider some **arbitrary** function of **time** $v(t)$, a function that exists over some **finite** amount of time T (i.e., $v(t) = 0$ for $t < 0$ and $t > T$).

Say we wish to perform some **linear** operation **on this function**:

$$\mathcal{L}\left[v(t)\right] = ??$$

Complex signals as collections of simple elements

Depending on the **difficulty** of the operation $\mathcal{L}$, and/or the **complexity** of the function $v(t)$, directly performing this operation could be very **painful** (i.e., approaching impossible).

Instead, we find that we can often **expand** a very complex and **stressful** function in the following way:

$$v(t) = a_0\,\psi_0(t) + a_1\,\psi_1(t) + a_2\,\psi_2(t) + \cdots = \sum_{n=-\infty}^{\infty} a_n\,\psi_n(t)$$

where the values a_n are **constants** (i.e., coefficients), and the functions $\psi_n(t)$ are known as **basis functions**.

Ms. Nomial's first name is Poly

For example, we could **choose** the basis functions:

$$\psi_n(t) = t^n \quad \text{for} \quad n \geq 0$$

Resulting in a **polynomial** of variable t:

$$v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots = \sum_{n=0}^{\infty} a_n t^n$$

This signal expansion is of course know as the **Taylor Series** expansion.

Choose your basis – but choose wisely

However, there are **many other** useful expansions (i.e., many other useful basis $\psi_n(t)$).

* The key thing is that the basis functions $\psi_n(t)$ are **independent** of the function $v(t)$. That is to say, the basis functions are **selected** by the engineer doing the analysis (i.e., **you**).

* The set of selected basis functions form what's known as a **basis**. With this basis we can **analyze** the function $v(t)$.

* The **result** of this analysis provides the **coefficients** a_n of the signal expansion. Thus, the coefficients **are** directly dependent on the form of function $v(t)$ (as well as the basis used for the analysis). As a result, the set of coefficients $\{a_1, a_2, a_3, \cdots\}$ **completely describe** the function $v(t)$!

It's simpler to operate on each element

Q: *I don't see why this "expansion" of function of $v(t)$ is helpful, it just looks like a lot more **work** to me.*

A: Consider what happens when we wish to perform a **linear** operation on this function:

$$\mathcal{L}\left[v(t)\right] = \mathcal{L}\left[\sum_{n=-\infty}^{\infty} a_n \, \psi_n(t)\right] = \sum_{n=-\infty}^{\infty} a_n \, \mathcal{L}\left[\psi_n(t)\right]$$

Look what happened!

Instead of performing the linear operation on the arbitrary and **difficult** function $v(t)$, we can apply the operation to **each** of the individual basis functions $\psi_n(t)$.

Choose a basis that makes this "easy"

Q: *And that's supposed to be **easier**??*

A: It **depends** on the linear operation and on the basis functions $\psi_n(t)$.

Hopefully, the operation $\mathcal{L}[\psi_n(t)]$ is **simple** and straightforward.

Ideally, the solution to $\mathcal{L}[\psi_n(t)]$ is **already known**!

Q: *Oh yeah, like I'm going to get so **lucky**. I'm sure in all my circuit analysis problems evaluating $\mathcal{L}[\psi_n(t)]$ will be long, frustrating, and **painful**.*

A: Remember, **you** get to choose the **basis** over which the function $v(t)$ is analyzed.

A **smart** engineer will **choose** a basis for which the operations $\mathcal{L}[\psi_n(t)]$ are simple and **straightforward**!

This basis is quite popular

Q: *But I'm **still** confused. **How** do I choose what basis $\psi_n(t)$ to use, and **how** do I analyze the function $v(t)$ to determine the coefficients a_n??*

A: Perhaps an **example** would help. Among the **most popular** basis is this one:

$$\psi_n = \begin{cases} e^{j\left(\frac{2\pi n}{T}\right)t} & 0 \leq t \leq T \\ 0 & t \leq 0, t \geq T \end{cases}$$

and:

$$a_n = \frac{1}{T}\int_0^T v(t)\,\psi_n^*(t)\,dt = \frac{1}{T}\int_0^T v(t)\,e^{-j\left(\frac{2\pi n}{T}\right)t}\,dt$$

So therefore:

$$v(t) = \sum_{n=-\infty}^{\infty} a_n\, e^{j\left(\frac{2\pi n}{T}\right)t} \qquad \text{for } 0 \leq t \leq T$$

The **astute** among you will recognize this signal expansion as the **Fourier Series**!

It has a very important property!

Q: *Yes, just why is Fourier analysis* ***so*** *prevalent?*

A: The answer reveals itself when we apply a **linear operator** to the signal expansion:

$$\mathcal{L}[v(t)] = \mathcal{L}\left[\sum_{n=-\infty}^{\infty} a_n \, e^{-j\left(\frac{2\pi n}{T}\right)t}\right] = \sum_{n=-\infty}^{\infty} a_n \, \mathcal{L}\left[e^{-j\left(\frac{2\pi n}{T}\right)t}\right]$$

Note then that we must **simply** evaluate:

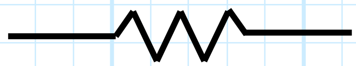
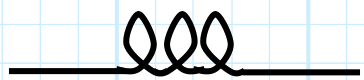
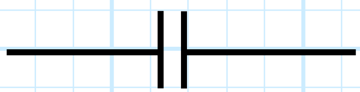
$$\mathcal{L}\left[e^{-j\left(\frac{2\pi n}{T}\right)t}\right]$$

for all *n*.

We will find that **performing** almost any linear operation $\mathcal{L}$ on basis functions of this type to be exceeding **simple** (more on this later)!

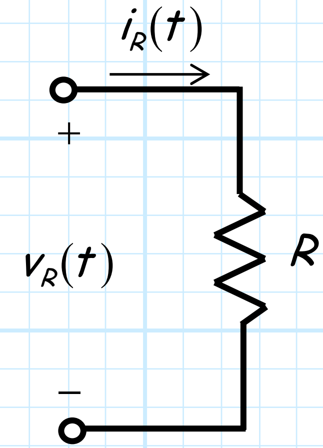
Linear Circuit Elements

Most microwave devices can be described or modeled in terms of the **three** standard circuit elements:

1. Resistance (R) 
2. Inductance (L) 
3. Capacitance (C) 

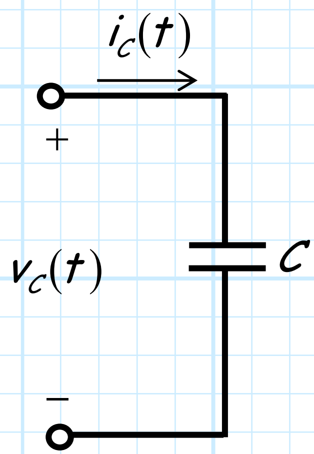
For the purposes of circuit analysis, each of these three elements are **defined** in terms of the **mathematical** relationship between the difference in electric potential $v(t)$ between the two terminals of the device (i.e., the **voltage** across the device), and the **current** $i(t)$ flowing through the device.

We find that for these three circuit elements, the relationship between $v(t)$ and $i(t)$ can be expressed as a linear operator!



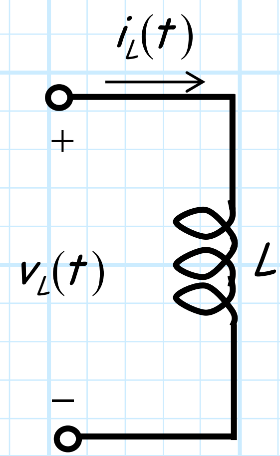
$$\mathcal{L}_{\mathcal{Y}}^{R}\left[v_R(t)\right] = i_R(t) = \frac{v_R(t)}{R}$$

$$\mathcal{L}_{\mathcal{Z}}^{R}\left[i_R(t)\right] = v_R(t) = R\, i_R(t)$$



$$\mathcal{L}_{\mathcal{Y}}^{C}\left[v_C(t)\right] = i_C(t) = C\,\frac{d\,v_C(t)}{d\,t}$$

$$\mathcal{L}_{\mathcal{Z}}^{C}\left[i_C(t)\right] = v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(t')\,dt'$$



$$\mathcal{L}_{\mathcal{Y}}^{L}\left[v_L(t)\right] = i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t')\,dt'$$

$$\mathcal{L}_{\mathcal{Z}}^{L}\left[i_L(t)\right] = v_L(t) = L\,\frac{d\,i_L(t)}{d\,t}$$

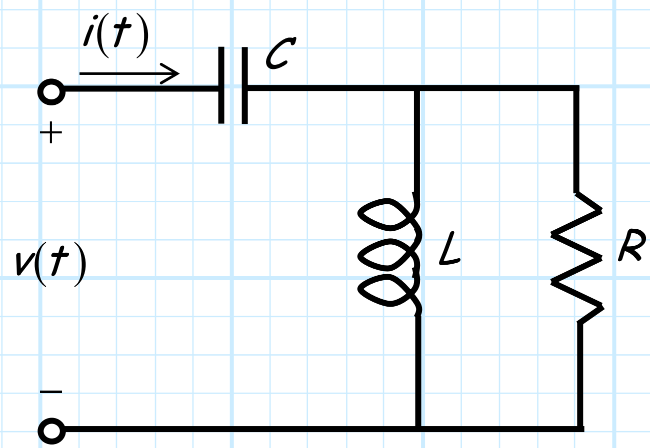
Since the circuit behavior of these devices can be expressed with **linear** operators, these devices are referred to as **linear circuit elements**.

A linear operator describes any relationship

Q: *Well, that's simple enough, but what about an element formed from a **composite** of these fundamental elements?*

*For **example**, for example, how are $v(t)$ and $i(t)$ related in the circuit below??*

$$\mathcal{L}_z\left[i(t)\right] = v(t) = ???$$



A: It turns out that **any** circuit constructed **entirely** with linear circuit elements is **likewise** a linear system (i.e., a linear circuit).

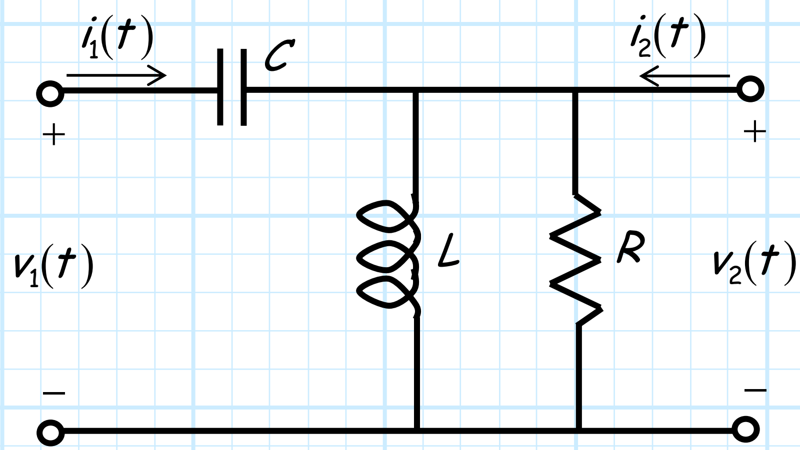
As a result, we know that that there **must** be some linear operator that relates $v(t)$ and $i(t)$ in your example!

$$\mathcal{L}_z\left[i(t)\right] = v(t)$$

This is very useful for multi-port networks

The circuit above provides a good example of a **single-port** (a.k.a. **one-port**) network.

We can of course construct networks with **two or more** ports; an example of a **two-port network** is shown below:



Since this circuit is **linear**, the relationship between **all** voltages and currents can likewise be expressed as **linear operators, e.g.**:

$$\mathcal{L}_{21}\left[v_1(t)\right]=v_2(t) \qquad \mathcal{L}_{Z21}\left[i_1(t)\right]=v_2(t) \qquad \mathcal{L}_{Z22}\left[i_2(t)\right]=v_2(t)$$

The linear operator is a convolution integral

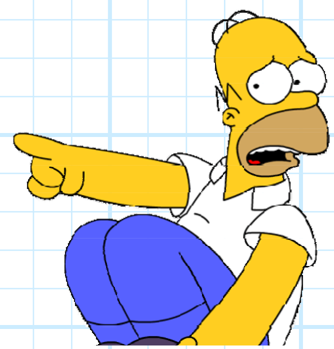
Q: *Yikes! What would these linear operators for this circuit **be**? How can we **determine** them?*

A: It turns out that linear operators for **all** linear circuits can all be expressed in precisely the **same** form!

For example, the linear operators of a single-port network are:

$$v(t) = \mathcal{L}_{\mathcal{Z}}\left[i(t)\right] = \int_{-\infty}^{t} g_{\mathcal{Z}}(t-t')\ i(t')\,dt'$$

$$i(t) = \mathcal{L}_{\mathcal{Y}}\left[v(t)\right] = \int_{-\infty}^{t} g_{\mathcal{Y}}(t-t')\ v(t')\,dt'$$



In other words, the linear operator of linear circuits can always be expressed as a **convolution** integral—a convolution with a **circuit impulse function** $g(t)$.

The impulse response

Q: *But just what **is** this "circuit impulse response"??*

A: An impulse response is simply the **response** of one circuit function (i.e., $i(t)$ or $v(t)$) due to a **specific** stimulus by another.

→ That specific stimulus is the **impulse function** $\delta(t)$.

The impulse function **can** be defined as:

$$\delta(t) = \lim_{\tau \to 0} \frac{1}{\tau} \frac{\sin\left(\frac{\pi t}{\tau}\right)}{\left(\frac{\pi t}{\tau}\right)}$$

Such that is has the following two **properties**:

1. $\delta(t) = 0 \quad \text{for} \quad t \neq 0$

2. $\int_{-\infty}^{\infty} \delta(t)\, dt = 1.0$

We can define all sorts of impulse responses

The impulse responses of the **one-port example** are therefore defined as:

$$g_z(t) \doteq v(t)\big|_{i(t)=\delta(t)}$$

and:

$$g_y(t) \doteq i(t)\big|_{v(t)=\delta(t)}$$

Meaning simply that $g_z(t)$ is equal to the **voltage** function $v(t)$ when the circuit is "thumped" with a **impulse current** (i.e., $i(t)=\delta(t)$), and $g_y(t)$ is equal to the **current** $i(t)$ when the circuit is "thumped" with a **impulse voltage** (i.e., $v(t)=\delta(t)$).

We can make convolution integrals simple!

Similarly, the relationship between the **input** and the **output** of a **two-port** network can be expressed as:

$$v_2(t) = \mathcal{L}_{21}\left[v_1(t)\right] = \int_{-\infty}^{t} g(t-t')\, v_1(t')\, dt'$$

where:

$$g(t) \doteq v_2(t)\Big|_{v_1(t)=\delta(t)}$$

Note that the circuit impulse response must be **causal** (nothing can occur at the output **until** something occurs at the input), so that:

$$g(t) = 0 \quad \text{for} \quad t < 0$$

Q: *Yikes! I recall evaluating convolution integrals to be messy, difficult and* ***stressful****. Surely there is an* ***easier*** *way to describe linear circuits!?!*

A: Nope! The convolution integral is **all** there is.

However, we can use our linear systems theory toolbox to greatly **simplify the evaluation** of a convolution integral!

The Eigen Function of Linear Systems

Recall that that we can express (**expand**) a time-limited signal with a weighted summation of **basis functions:**

$$v(t) = \sum_n a_n \psi_n(t)$$

where $v(t) = 0$ for $t < 0$ and $t > T$.

Say now that we **convolve** this signal with some system **impulse function** $g(t)$:

$$\begin{aligned}\mathcal{L}\left[v(t)\right] &= \int_{-\infty}^{t} g(t-t')\, v(t')\, dt' \\ &= \int_{-\infty}^{t} g(t-t') \sum_n a_n\, \psi_n(t')\, dt' \\ &= \sum_n a_n \int_{-\infty}^{t} g(t-t')\, \psi_n(t')\, dt'\end{aligned}$$

Look what happened!

Convolve with the basis functions – not the signal

Instead of convolving the general function $v(t)$, we now find that we must simply convolve with the set of basis functions $\psi_n(t)$.

Q: *Huh? You say we must* ***"simply"*** *convolve the set of basis functions* $\psi_n(t)$. ***Why*** *would this be any simpler?*

A: Remember, **you** get to **choose** the basis $\psi_n(t)$. If you're **smart**, you'll choose a set that makes the convolution integral **"simple"** to perform!

Q: *But don't I first need to* ***know*** *the explicit form of* $g(t)$ ***before*** *I intelligently choose* $\psi_n(t)$*??*

A: Not necessarily!

Time to use our "special" basis

The **key** here is that the convolution integral:

$$\mathcal{L}\left[\psi_n(t)\right] = \int_{-\infty}^{t} g(t-t')\,\psi_n(t')\,dt'$$

is a **linear, time-invariant** operator.

Because of this, there exists one **basis** with an **astonishing** property!

These **special** basis functions are:

$$\psi_n(t) = \begin{cases} e^{j\omega_n t} & for \quad 0 \le t \le T \\ 0 \quad for & t < 0, t > T \end{cases} \qquad \text{where} \qquad \omega_n = n\left(\frac{2\pi}{T}\right)$$

Prof. Stiles: So darn lame

Now, inserting this function (get ready, here comes the **astonishing** part!) into the convolution integral:

$$\mathcal{L}\left[e^{j\omega_n t}\right] = \int_{-\infty}^{t} g(t-t')\, e^{j\omega_n t'}\, dt'$$

and using the substitution $u = t - t'$, we get:

$$\begin{aligned}\int_{-\infty}^{t} g(t-t')\, e^{j\omega_n t}\, dt' &= \int_{t-(-\infty)}^{t-t} g(u)\, e^{j\omega_n (t-u)}\, (-du) \\ &= e^{j\omega_n t} \int_{+\infty}^{0} g(u)\, e^{-j\omega_n u}\, (-du) \\ &= e^{j\omega_n t} \int_{0}^{\infty} g(u)\, e^{-j\omega_n u}\, du\end{aligned}$$

See! Doesn't **that** astonish!

Q: *I'm only astonished by how **lame** you are. How is this result any **more** "astonishing" than any of the **other** "useful" things you've been telling us?*

Convolution becomes multiplication

A: Note that the integration in this **result** is **not** a convolution—the integral is simply a **value** that depends on n (but **not** time t):

$$G(\omega_n) \doteq \int_0^\infty g(t)\, e^{-j\omega_n t}\, dt$$

As a result, convolution with this "special" set of basis functions can **always** be expressed as:

$$\int_{-\infty}^{t} g(t-t')\, e^{j\omega_n t'}\, dt' = \mathcal{L}\left[e^{j\omega_n t}\right] = G(\omega_n)\, e^{j\omega_n t}$$

The **remarkable** thing about this result is that the linear operation on function $\psi_n(t) = exp[j\omega_n t]$ results in precisely the **same** function of **time** t (save the **complex** multiplier $G(\omega_n)$)! I.E.:

$$\mathcal{L}[\psi_n(t)] = G(\omega_n)\ \psi_n(t)$$

Convolution with $\psi_n(t) = exp[j\omega_n t]$ is accomplished by simply **multiplying** the function by the **complex** number $G(\omega_n)$!

This only works for complex exponentials

Note this is true **regardless** of the impulse response $g(t)$ (the function $g(t)$ affects the **value** of $G(\omega_n)$ **only**)!

Q: *Big deal! Aren't there lots of **other** functions that would satisfy the equation above equation?*

A: Nope. The **only** function where this is true is:

$$\psi_n(t) = e^{j\omega_n t}$$

This function is thus **very** special.

We call this function the **eigen function** of linear, time-invariant systems.

But complex exponentials are two sinusoidal functions

Q: *Are you **sure** that there are no other Eigen functions??*

A: Well, sort of.

Recall from **Euler's equation** that:

$$e^{j\omega_n t} = \cos\omega_n t + j\sin\omega_n t$$

It can be shown that the sinusoidal functions $\cos\omega_n t$ and $\sin\omega_n t$ are **likewise** Eigen functions of linear, time-invariant systems.

The real and imaginary components of Eigen function $exp[j\omega_n t]$ are **also** Eigen functions.

Every linear operator has its Eigen value

Q: *What about the set of* ***values*** $G(\omega_n)$ *?? Do they have any significance or* ***importance****??*

A: Absolutely!

Recall the values $G(\omega_n)$ (one for each n) depend on the **impulse response** of the system (e.g., circuit) **only**:

$$G(\omega_n) \doteq \int_0^{\infty} g(t)\, e^{-j\omega_n t}\, dt$$

Thus, the set of values $G(\omega_n)$ completely **characterizes** a linear time-invariant **circuit** over time $0 \leq t \leq T$.

We call the values $G(\omega_n)$ the Eigen values of the linear, time-invariant circuit.

We're electrical engineers: why should we care?

Q: *OK Poindexter, all **Eigen** stuff this **might** be interesting if you're a mathematician, but is it at all **useful** to us **electrical engineers**?*

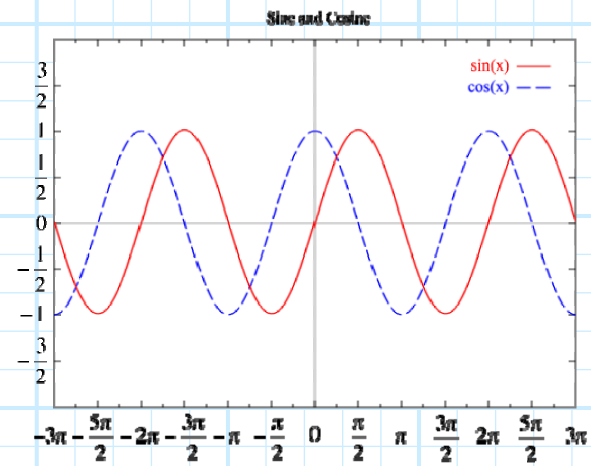
A: It is **unfathomably** useful to us electrical engineers!

Say a linear, time-invariant circuit is excited (only) by a **sinusoidal** source (e.g., $v_s(t) = \cos\omega_o t$).

Since the source function is the **Eigen function** of the circuit, we will find that at **every** point in the circuit, **both** the current and voltage will have the **same functional form**.

That is, every current and voltage in the circuit will likewise be a **perfect sinusoid** with frequency ω_o!!

Haven't you wondered why we always use these?



Of course, the **magnitude** of the sinusoidal oscillation will be **different** at different points within the circuit, as will the **relative phase**.

But we know that **every** current and voltage in the circuit can be **precisely** expressed as a function of this form:

$$A\cos\left(\omega_o t + \varphi\right)$$

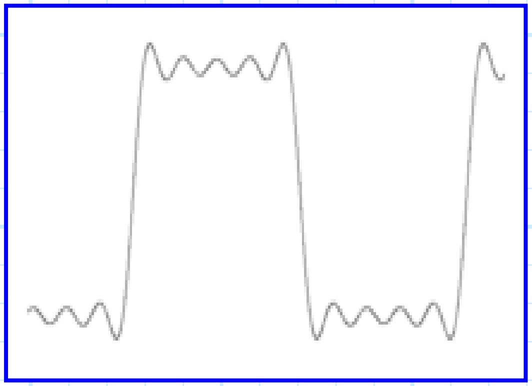
Q: *Isn't this pretty **obvious**?*

A: **Why** should it be?

Say our source function was instead a **square** wave, or **triangle** wave, or a **sawtooth** wave.

We would find that (generally speaking) **nowhere** in the circuit would we find another current or voltage that was a **perfect** square wave (etc.)!

We "just" have to determine magnitude and phase!



In fact, we would find that not only are the current and voltage functions within the circuit **different** than the source function (e.g. a sawtooth) they are (generally speaking) all different **from each other**.

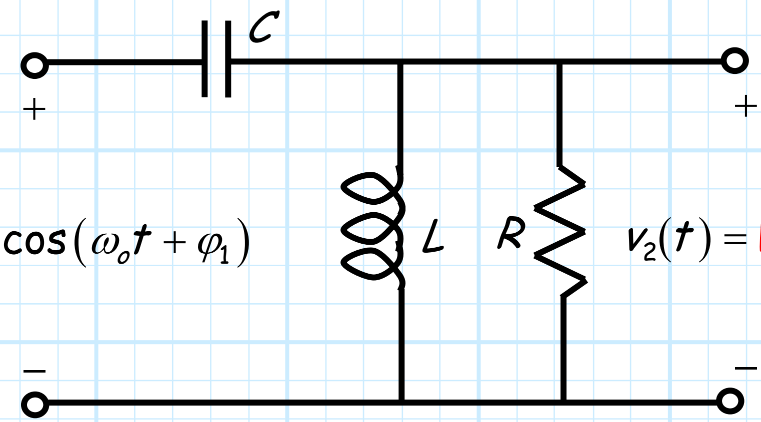
We find then that a linear circuit will (generally speaking) **distort** any source function—**unless** that function is the **Eigen function** (i.e., a sinusoidal function).

Thus, using an **Eigen function** as circuit source greatly simplifies our linear circuit analysis problem.

All we need to accomplish this is to **determine the magnitude** A and **relative phase** φ of the resulting (and otherwise **identical**) sinusoidal function!

A Complex Representation of Sinusoidal Functions

Q: *So, you say (for example) if a linear two-port circuit is driven by a sinusoidal source with arbitrary frequency ω_o, then the output will be identically sinusoidal, only with a different magnitude and relative phase.*



$$v_1(t) = V_{m1}\cos(\omega_o t + \varphi_1) \qquad v_2(t) = V_{m2}\cos(\omega_o t + \varphi_2)$$

How do we determine the unknown magnitude V_{m2} and phase φ_2 of this output?

Eigen values are complex

A: Say the input and output are related by the impulse response $g(t)$:

$$v_2(t) = \mathcal{L}\left[v_1(t)\right] = \int_{-\infty}^{t} g(t-t')\, v_1(t')\, dt'$$

We now know that **if** the input were **instead**:

$$v_1(t) = e^{j\omega_0 t}$$

then:

$$v_2(t) = \mathcal{L}\left[e^{j\omega_0 t}\right] = G(\omega_0)\, e^{j\omega_0 t}$$

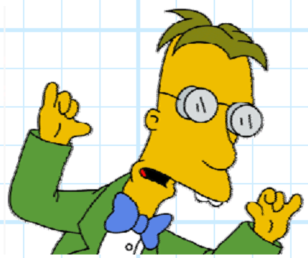
where:

$$G(\omega_0) \doteq \int_{0}^{\infty} g(t)\, e^{-j\omega_0 t}\, dt$$

Thus, we simply multiply the input $v_1(t) = e^{j\omega_0 t}$ by the **complex** eigen value $G(\omega_0)$ to determine the **complex** output $v_2(t)$:

$$v_2(t) = G(\omega_0)\, e^{j\omega_0 t}$$

Complex voltages and currents are your friend!



Q: *You professors drive me crazy with all this math involving **complex** (i.e., real and imaginary) voltage functions. In the lab I can only generate and measure **real-valued** voltages and **real-valued** voltage functions. Voltage is a **real-valued, physical** parameter!*

A: You are quite **correct**.

Voltage **is** a real-valued parameter, expressing electric potential (in Joules) per unit charge (in Coulombs).

Q: *So, all your **complex** formulations and **complex** eigen values and **complex** eigen functions may all be sound **mathematical abstractions**, but aren't they **worthless** to us **electrical engineers** who work in the **"real"** world (pun intended)?*

A: Absolutely not! Complex analysis actually **simplifies** our analysis of real-valued voltages and currents in **linear circuits** (but **only** for linear circuits!).

Remember Euler

The key relationship comes from **Euler's Identity**:

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

Meaning:

$$\text{Re}\{e^{j\omega t}\} = \cos\omega t$$

Now, consider a **complex value** C. We of course **can** write this complex number in terms of it **real** and **imaginary** parts:

$$C = a + j\,b \qquad \therefore \quad a = \text{Re}\{C\} \quad \text{and} \quad b = \text{Im}\{C\}$$

But, we can **also** write it in terms of its **magnitude** $|C|$ and **phase** φ!

$$\boxed{C = |C|\,e^{j\varphi}}$$

where:

$$|C| = C\,C^* = a^2 + b^2 \qquad \varphi = \tan^{-1}\left[{}^{b}\!/\!{}_{a}\right]$$

A complex number has magnitude and phase

Thus, the complex **function** $C\,e^{j\omega_0 t}$ is:

$$
\begin{aligned}
C\,e^{j\omega_0 t} &= |C|\,e^{j\varphi}\,e^{j\omega_0 t} \\
&= |C|\,e^{j\omega_0 t+\varphi} \\
&= |C|\cos(\omega_0 t+\varphi) + j\,|C|\sin(\omega_0 t+\varphi)
\end{aligned}
$$

Therefore we find:

$$|C|\cos(\omega_0 t+\varphi) = \mathrm{Re}\left\{C\,e^{j\omega_0 t}\right\}$$

Now, consider again the **real-valued** voltage function:

$$v_1(t) = V_{m1}\cos(\omega t+\varphi_1)$$

This function is of course **sinusoidal** with a magnitude V_{m1} and phase φ_1.

Using what we have learned above, we can **likewise** express this real function as:

$$v_1(t) = V_{m1}\cos(\omega t+\varphi_1) = \mathrm{Re}\left\{V_1\,e^{j\omega t}\right\}$$

where V_1 is the **complex number**: $V_1 = V_{m1}\,e^{j\varphi_1}$

But what is the output signal?

Q: *I see! A* ***real-valued sinusoid*** *has a magnitude and phase, just like* ***complex number****.*

A ***single*** *complex number (V) can be used to specify* ***both*** *of the fundamental (real-valued) parameters of our sinusoid (V_m, φ).*

What I don't see is how this helps us in our circuit analysis.

After all:

$$v_2(t) \neq G(\omega_o)\ Re\left\{V_1 e^{j\omega_o t}\right\}$$

What then is the ***real-valued*** *output $v_2(t)$ of our two-port network when the input $v_1(t)$ is the* ***real-valued*** *sinusoid:*

$$\begin{aligned} v_1(t) &= V_{m1}\cos(\omega_o t + \varphi_1) \\ &= \text{Re}\left\{V_1 e^{j\omega_o t}\right\} \end{aligned} \quad ???$$

The math will reveal the answer!

A: Let's go back to our **original** convolution integral:

$$v_2(t) = \int_{-\infty}^{t} g(t-t')\, v_1(t')\, dt'$$

If:

$$\begin{aligned} v_1(t) &= V_{m1} \cos(\omega_o t + \varphi_1) \\ &= \text{Re}\left\{ V_1 e^{j\omega_o t} \right\} \end{aligned}$$

then:

$$v_2(t) = \int_{-\infty}^{t} g(t-t')\, Re\left\{ V_1 e^{j\omega_o t'} \right\} dt'$$

Now, since the impulse function $g(t)$ is **real-valued** (this is really important!) it **can** be shown that:

$$\begin{aligned} v_2(t) &= \int_{-\infty}^{t} g(t-t')\, Re\left\{ V_1 e^{j\omega_o t'} \right\} dt' \\ &= Re\left\{ \int_{-\infty}^{t} g(t-t')\, V_1 e^{j\omega_o t'} dt' \right\} \end{aligned}$$

The output signal

Now, applying what we have **previously** learned;

$$v_2(t) = Re\left\{\int_{-\infty}^{t} g(t-t')\, V_1\, e^{j\omega_o t'}\, dt'\right\}$$

$$= Re\left\{V_1 \int_{-\infty}^{t} g(t-t')\; e^{j\omega_o t'}\, dt'\right\}$$

$$= Re\left\{V_1\, G(\omega_0)\, e^{j\omega_o t}\right\}$$

Thus, we **finally** can conclude the real-valued output $v_2(t)$ due to the **real-valued input**:

$$v_1(t) = V_{m1}\cos\left(\omega_o t + \varphi_1\right) = \text{Re}\left\{V_1\, e^{j\omega_o t}\right\}$$

is:

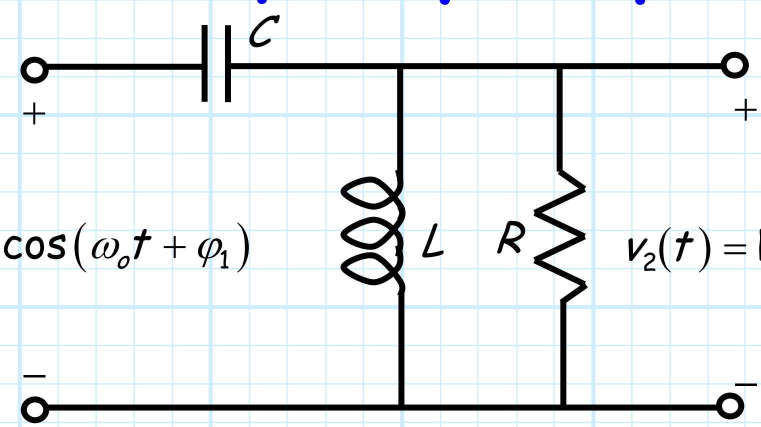
$$v_2(t) = Re\left\{V_2\; e^{j\omega_o t}\right\} = V_{m2}\, cos\left(\omega_o t + \varphi_2\right)$$

where:

$$V_2 = G(\omega_o)\, V_1$$

The **really important** result here is the last one!

The Eigen value of the Linear operator is its "Frequency Response"



$v_1(t) = V_{m1}\cos(\omega_o t + \varphi_1)$

$v_2(t) = \text{Re}\{G(\omega_o)\, V_1\, e^{j\omega_o t}\}$

The magnitude and phase of the **output** sinusoid (expressed as **complex** value V_2) is related to the magnitude and phase of the **input** sinusoid (expressed as **complex** value V_1) by the system **eigen value** $G(\omega_o)$:

$$\frac{V_2}{V_1} = G(\omega_o)$$

Therefore we find that **really** often in electrical engineering, we:

1. Use sinusoidal (i.e., eigen function) sources.

2. Express the voltages and currents created by these sources as complex **values** (i.e., **not** as real functions of time)!

Make sure you know what complex voltages and currents physically represent!

For **example**, we might say " $V_3 = 2.0$", meaning:

$$V_3 = 2.0 = 2.0\, e^{j0} \quad \Rightarrow \quad v_3(t) = \text{Re}\left\{2.0\, e^{j0} e^{j\omega_o t}\right\} = 2.0 \cos \omega_o t$$

Or "$I_L = -3.0$", meaning:

$$I_L = -2.0 = 3.0\, e^{j\pi} \quad \Rightarrow \quad i_L(t) = \text{Re}\left\{3.0\, e^{j\pi} e^{j\omega_o t}\right\} = 3.0 \cos\left(\omega_o t + \pi\right)$$

Or "$V_s = j$", meaning:

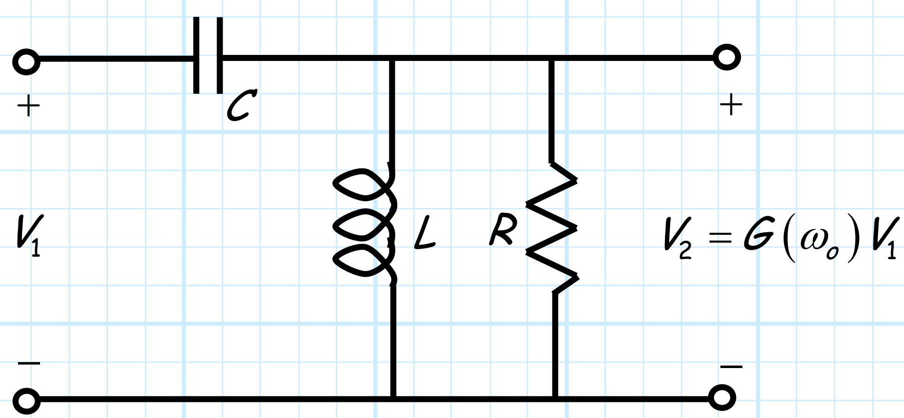
$$V_s = j = 1.0\, e^{j\left(\pi/2\right)} \quad \Rightarrow \quad v_s(t) = \text{Re}\left\{1.0\, e^{j\left(\pi/2\right)} e^{j\omega_o t}\right\} = 1.0 \cos\left(\omega_o t + \pi/2\right)$$

Summarizing

* Remember, if a linear circuit is excited by a sinusoid (e.g., **eigen function** $\exp\left[j\omega_0 t\right]$), then the **only** unknowns are the magnitude and phase of the sinusoidal **currents** and **voltages** associated with **each element** of the circuit.

* These unknowns are **completely** described by complex values, as complex values **likewise** have a magnitude and phase.

* We can always **"recover"** the **real-valued** voltage or current function by multiplying the complex value by $\exp\left[j\omega_0 t\right]$ and then taking the real part, but typically we don't—after all, **no** new or unknown information is revealed by this operation!



Analysis of Circuits Driven by Arbitrary Functions

Q: *What happens if a linear circuit is excited by some function that is **not** an "eigen function"? Isn't limiting our analysis to sinusoids **too restrictive**?*

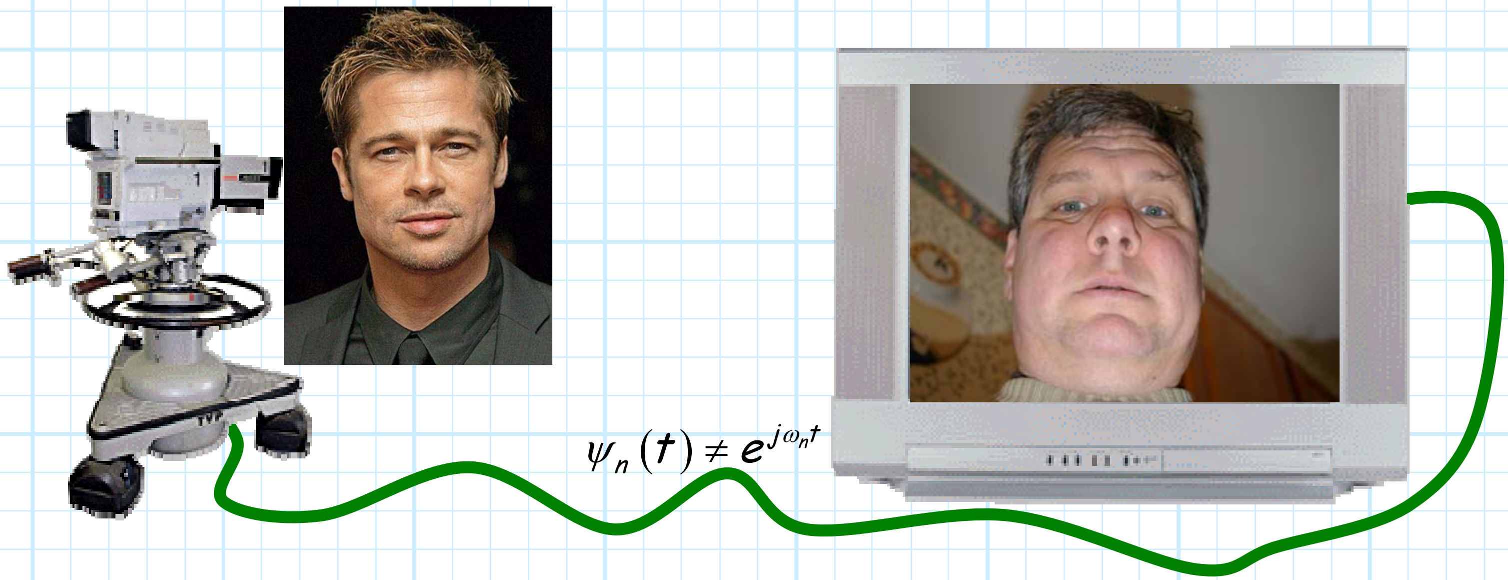
A: Not as restrictive as you might think.

Because sinusoidal functions are the eigen-functions of linear, time-invariant systems, they have become **fundamental** to much of our electrical engineering infrastructure—particularly with regard to **communications**.

For example, every radio and TV station is assigned its **very own eigen function** (i.e., its own frequency ω)!

Eigen functions: without them communication would be impossible

It is **very** important that we use eigen functions for electromagnetic communication, otherwise the **received** signal might look **grotesquely** different from the one that was **transmitted**!



With sinusoidal functions (being eigen functions and all), we **know** that receive function will have **precisely** the same form as the one transmitted (albeit quite a bit **smaller**).

Thus, our assumption that a linear circuit is excited by a sinusoidal function is often a very **accurate** and **practical** one!

What if the signal is not sinusoidal?

Q: *Still, we often find a circuit that is **not** driven by a sinusoidal source. How would we analyze **this** circuit?*

A: Recall the property of **linear operators**:

$$\mathcal{L}\left[a\,y_1 + b\,y_2\right] = a\,\mathcal{L}\left[y_1\right] + b\,\mathcal{L}\left[y_2\right]$$

We now know that we can **expand** the function:

$$v(t) = a_0\,\psi_0(t) + a_1\,\psi_1(t) + a_2\,\psi_2(t) + \cdots = \sum_{n=-\infty}^{\infty} a_n\,\psi_n(t)$$

and we found that:

$$\mathcal{L}\left[v(t)\right] = \mathcal{L}\left[\sum_{n=-\infty}^{\infty} a_n\,\psi_n(t)\right] = \sum_{n=-\infty}^{\infty} a_n\,\mathcal{L}\left[\psi_n(t)\right]$$

Let's choose Eigen functions as our basis

We found that any linear operation $\mathcal{L}[\psi_n(t)]$ is greatly simplified **if** we choose as our basis function the **eigen function** of linear systems:

$$\psi_n(t) = \begin{cases} e^{j\omega_n t} & for \quad 0 \leq t \leq T \\ 0 & for \quad t < 0, t > T \end{cases} \qquad \text{where} \qquad \omega_n = n\left(\frac{2\pi}{T}\right)$$

so that:

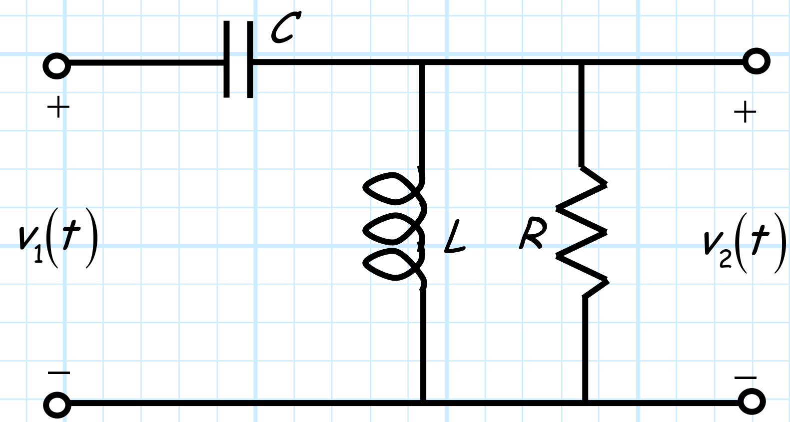
$$\mathcal{L}[\psi_n(t)] = G(\omega_n)\, e^{j\omega_n t}$$

And so:

$$\mathcal{L}\left[v(t)\right] = \mathcal{L}\left[\sum_{n=-\infty}^{\infty} a_n\, e^{j\omega_n t}\right] = \sum_{n=-\infty}^{\infty} a_n\, \mathcal{L}\left[e^{j\omega_n t}\right] = \sum_{n=-\infty}^{\infty} a_n\, G(\omega_n)\, e^{j\omega_n t}$$

Just follow these steps…

Thus, for the example:



We follow these **analysis steps**:

1. Expand the input function $v_1(t)$ using the basis functions $\psi_n(t) = exp[j\omega_n t]$:

$$v_1(t) = V_{01}\, e^{j\omega_0 t} + V_{11}\, e^{j\omega_1 t} + V_{21}\, e^{j\omega_2 t} + \cdots = \sum_{n=-\infty}^{\infty} V_{n1}\, e^{j\omega_n t}$$

where:

$$V_{n1} = \frac{1}{T}\int_0^T v_1(t)\, e^{-j\omega_n t}\, dt$$

…and the output is determined

2. Evaluate the **eigen values** of the linear system:

$$G(\omega_n) = \int_0^\infty g(t)\, e^{-j\omega_n t}\, dt$$

3. Perform the **linear operation** (the convolution integral) that relates $v_2(t)$ to $v_1(t)$:

$$
\begin{aligned}
v_2(t) &= \mathcal{L}\left[v_1(t)\right] \\
&= \mathcal{L}\left[\sum_{n=-\infty}^{\infty} V_{n1}\, e^{j\omega_n t}\right] \\
&= \sum_{n=-\infty}^{\infty} V_{n1}\, \mathcal{L}\left[e^{j\omega_n t}\right] \\
&= \sum_{n=-\infty}^{\infty} V_{n1}\, G(\omega_n)\, e^{j\omega_n t}
\end{aligned}
$$

A Summary

Summarizing:

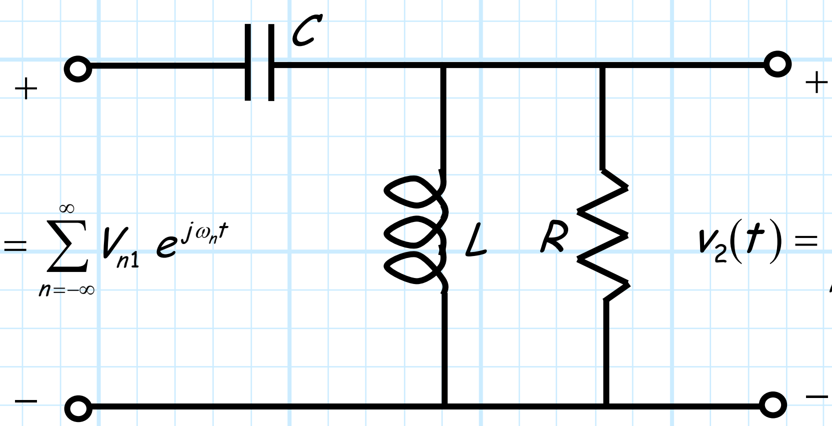
$$v_2(t) = \sum_{n=-\infty}^{\infty} V_{n2}\, e^{j\omega_n t}$$

where:

$$V_{n2} = G(\omega_n)\, V_{n1}$$

and:

$$V_{n1} = \frac{1}{T}\int_0^T v_1(t)\, e^{-j\omega_n t}\, dt \qquad G(\omega_n) = \int_0^\infty g(t)\, e^{-j\omega_n t}\, dt$$



$$v_1(t) = \sum_{n=-\infty}^{\infty} V_{n1}\, e^{j\omega_n t}$$

$$v_2(t) = \sum_{n=-\infty}^{\infty} G_1(\omega_n)\, V_{n1}\, e^{j\omega_n t}$$

As stated earlier, the signal expansion used here is the **Fourier Series**.

The Fourier Transform

Say that the **timewidth** T of the signal $v_1(t)$ becomes **infinite**. In this case we find our analysis becomes:

$$v_2(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} V_2(\omega)\, e^{j\omega t} d\omega$$

where:

$$V_2(\omega) = G(\omega)\, V_1(\omega)$$

and:

$$V_1(\omega) = \int_{-\infty}^{+\infty} v_1(t)\, e^{-j\omega t} dt \qquad G(\omega) = \int_{-\infty}^{+\infty} g(t)\, e^{-j\omega t}\, dt$$

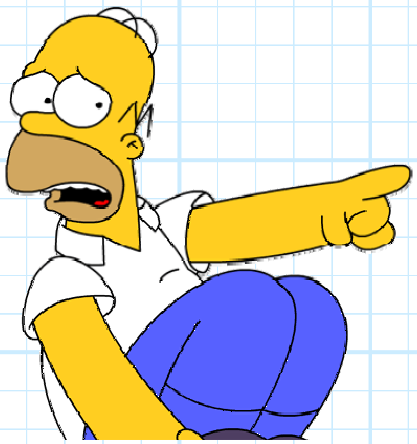
The signal expansion in this case is the **Fourier Transform.**

We find that as $T \rightarrow \infty$ the number of **discrete** system eigen values $G(\omega_n)$ become so numerous that they form a **continuum**—$G(\omega)$ is a **continuous** function of frequency ω.

We thus call the function $G(\omega)$ the **eigen spectrum** or **frequency response** of the circuit.

This still looks very difficult!

Q: *You* ***claim*** *that all this fancy mathematics (e.g., eigen functions and eigen values) make analysis of linear systems and circuits much* ***easier****, yet to apply these techniques, we must* ***determine*** *the eigen values or eigen spectrum:*



$$G(\omega_n) = \int_0^{\infty} g(t)\, e^{-j\omega_n t}\, dt \qquad G(\omega) = \int_{-\infty}^{+\infty} g(t)\, e^{-j\omega t}\, dt$$

Neither *of these operations look* ***at all*** *easy.*

And in addition to performing the integration, we must ***somehow*** *determine the* ***impulse function*** $g(t)$ *of the linear system as well !*

Just how are we supposed to do ***that****?*

It's not nearly as difficult as it appears!

A: An insightful question!

Determining the impulse response $g(t)$ and then the frequency response $G(\omega)$ **does** appear to be **exceedingly** difficult—and for many linear systems it indeed **is**!

However, much to our great **relief**, we can determine the eigen spectrum $G(\omega)$ of linear circuits **without** having to perform a difficult integration.

In fact, we **don't** even need to know the impulse response $g(t)$!

The Eigen Values of Linear Circuits

Recall the linear operators that define a **capacitor**:

$$\mathcal{L}_{\mathcal{Y}}^{C}\left[v_C(t)\right] = i_C(t) = C\,\frac{d\,v_C(t)}{d\,t}$$

$$\mathcal{L}_{\mathcal{Z}}^{C}\left[i_C(t)\right] = v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(t')\,dt'$$

We now know that the **Eigen function** of these linear, time-invariant operators—like **all** linear, time-invariant operators—is $\exp\left[j\omega t\right]$.

The question now is: **what** is the **Eigen value** of each of these operators?

It is this value that **defines** the physical behavior of a given capacitor!

The operator is linear

For $v_c(t) = \exp[j\omega t]$, we find:

$$
\begin{aligned}
i_c(t) &= \mathcal{L}_{\mathcal{Y}}^{c}\left[v_c(t)\right] \\
&= C\,\frac{d\,e^{j\omega t}}{d\,t} \\
&= (j\omega C)\,e^{j\omega t}
\end{aligned}
$$

Just as we expected, the Eigen function $\exp[j\omega t]$ **"survives"** the linear operation **unscathed**—the current function $i(t)$ has **precisely** the same form as the voltage function $v(t) = \exp[j\omega t]$.

The **only** difference between the **current** and **voltage** is the multiplication of the **Eigen value**, denoted as $G_{\mathcal{Y}}^{c}(\omega)$.

$$
i_c(t) = \mathcal{L}_{\mathcal{Y}}^{c}\left[v(t) = e^{j\omega t}\right] = G_{\mathcal{Y}}^{c}(\omega)\,e^{j\omega t}
$$

The Eigen value of a capacitor

Since we **just** determined that for this case:

$$i_C(t) = (j\omega C)\, e^{j\omega t}$$

it is **evident** that the Eigen value of the linear operation:

$$i(t) = \mathcal{L}_{\mathcal{Y}}^{C}[v(t)] = C\,\frac{d\,v(t)}{d\,t}$$

is:

$$G_{\mathcal{Y}}^{C}(\omega) = j\omega\, C = \omega\, C\, e^{j\pi/2} \quad !!!$$

Let's now consider real-valued functions

So for **example**, if:

$$v(t) = V_m \cos(\omega_o t + \varphi) = \text{Re}\left\{\left(V_m e^{j\varphi}\right) e^{j\omega_o t}\right\}$$

we will find that:

$$\begin{aligned}\mathcal{L}_{\mathcal{Y}}^{C}\left[\left(V_m e^{j\varphi}\right) e^{j\omega_o t}\right] &= G_{\mathcal{Y}}^{C}(\omega_o)\left(V_m e^{j\varphi}\right) e^{j\omega_o t} \\ &= \left(\omega C \, e^{j\pi/2}\right)\left(V_m e^{j\varphi}\right) e^{j\omega_o t} \\ &= \left(\omega C \, V_m \, e^{j\left(\pi/2+\varphi\right)}\right) e^{j\omega_o t}\end{aligned}$$

Therefore:

$$\begin{aligned}i_C(t) &= \text{Re}\left\{\omega C \, V_m e^{j\left(\varphi+\pi/2\right)} e^{j\omega_o t}\right\} \\ &= \omega C \; V_m \cos\left(\omega_o t + \varphi + \pi/2\right) \\ &= -\omega C \; V_m \sin\left(\omega_o t + \varphi\right)\end{aligned}$$

Remember what the complex value means

Hopefully, this example again emphasizes that these **real-valued** sinusoidal functions can be completely expressed in terms of **complex values**.

For **example**, the complex value:

$$V_c = V_m e^{j\varphi}$$

means that the magnitude of the sinusoidal **voltage** is $|V_c| = V_m$, and its relative phase is $\angle V_c = \varphi$. The complex value:

$$I_c = G_y^c(\omega)\, V_c = \left(\omega C\, e^{j\pi/2}\right) V_c$$

likewise means that the **magnitude** of the sinusoidal **current** is:

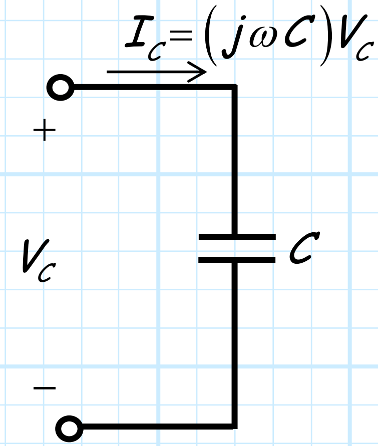
$$|I_c| = \left|G_y^c(\omega)\, V_c\right| = \left|G_y^c(\omega)\right| |V_c| = \omega C\, V_m$$

And the relative **phase** of the sinusoidal **current** is:

$$\angle I_c = \angle G_y^c(\omega) + \angle V_c = \pi/2 + \varphi$$

Now find the voltage from the current

We can thus **summarize** the behavior of a capacitor with the simple **complex equation**:



$$\begin{aligned} I_C &= (j\omega C)\, V_C \\ &= \left(\omega C\, e^{j\pi/2}\right) V_C \end{aligned}$$

Now let's return to the **second** of the two linear operators that describe a capacitor:

$$v_C(t) = \mathcal{L}_Z^C\left[i_C(t)\right] = \frac{1}{C}\int_{-\infty}^{t} i_C(t')\, dt'$$

Now, if the capacitor **current** is the Eigen function $i_C(t) = \exp[j\omega t]$, we find:

$$\mathcal{L}_Z^C\left[e^{j\omega t}\right] = \frac{1}{C}\int_{-\infty}^{t} e^{j\omega t'}\, dt' = \left(\frac{1}{j\omega C}\right) e^{j\omega t}$$

where we assume $i(t = -\infty) = 0$.

The Eigen value of this linear operator

Thus, we can conclude that:

$$\mathcal{L}_{\mathcal{Z}}^{C}\left[e^{j\omega t}\right] = G_{\mathcal{Z}}^{C}(\omega)\, e^{j\omega t} = \left(\frac{1}{j\omega C}\right) e^{j\omega t}$$

Hopefully, it is evident that the **Eigen value** of this linear operator is:

$$G_{\mathcal{Z}}^{C}(\omega) = \frac{1}{j\omega C} = \frac{-j}{\omega C} = \frac{1}{\omega C} e^{j\left(3\pi/2\right)}$$

And so:

$$V_C = \left(\frac{1}{j\omega C}\right) I_C$$

Impedance is simply an Eigen value!

Q: *Wait a second! Isn't this essentially the **same** result as the one derived for operator $\mathcal{L}_y^C$??*

A: It's **precisely** the same! For both operators we find:

$$\frac{V_C}{I_C} = \frac{1}{j\omega C}$$

This should **not** be surprising, as **both** operators $\mathcal{L}_y^C$ and $\mathcal{L}_z^C$ relate the current through and voltage across the **same** device (a capacitor).

The **ratio** of complex voltage to complex current is of course referred to as the complex device **impedance** Z.

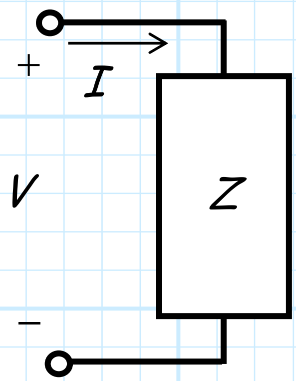
$$Z \doteq \frac{V}{I}$$

An **impedance** can be determined for **any** linear, time-invariant **one-port** network—but **only** for linear, time-invariant one-port networks!

Know what impedance tells you!

Generally speaking, impedance is a **function of frequency**. In fact, the impedance of a one-port network is simply the **Eigen value** $\mathcal{G}_z(\omega)$ of the linear operator $\mathcal{L}_z$:

$$V = Z\, I$$



$$\mathcal{L}_z\left[i(t)\right] = v(t)$$

$$Z = \mathcal{G}_z(\omega)$$

Note that impedance is a **complex** value that provides us with **two** things:

1. The **ratio of the magnitudes** of the sinusoidal voltage and current:

$$|Z| = \frac{|V|}{|I|}$$

2. The **difference in phase** between the sinusoidal voltage and current:

$$\angle Z = \angle V - \angle I$$

Admittance

Q: *What about the linear operator:*

$$\mathcal{L}_y\left[v(t)\right] = i(t) \quad ??$$

A: Hopefully it is now evident to **you** that:

$$G_y(\omega) = \frac{1}{G_z(\omega)} = \frac{1}{Z}$$

The inverse of impedance is **admittance** Y:

$$Y \doteq \frac{1}{Z} = \frac{I}{V}$$

Inductors and resistors

Now, returning to the **other two** linear circuit elements, we find (and you can verify) that for resistors:

$$\mathcal{L}_{\mathcal{Y}}^{R}\left[v_R(t)\right]=i_R(t) \quad \Rightarrow G_{\mathcal{Y}}^{R}(\omega)=1/R$$

$$\mathcal{L}_{\mathcal{Z}}^{R}\left[i_R(t)\right]=v_R(t) \quad \Rightarrow G_{\mathcal{Z}}^{R}(\omega)=R$$

and for inductors:

$$\mathcal{L}_{\mathcal{Y}}^{L}\left[v_L(t)\right]=i_L(t) \quad \Rightarrow G_{\mathcal{Y}}^{L}(\omega)=\frac{1}{j\omega L}$$

$$\mathcal{L}_{\mathcal{Z}}^{L}\left[i_L(t)\right]=v_L(t) \quad \Rightarrow G_{\mathcal{Z}}^{L}(\omega)=j\omega L$$

meaning:

$$Z_R=\frac{1}{Y_R}=R=R\,e^{j0} \qquad \text{and} \qquad Z_L=\frac{1}{Y_L}=j\omega L=\omega L\,e^{j\left(\pi/2\right)}$$

All the rules of circuit theory apply to complex currents and voltages too

Now, note that the relationship

$$Z = \frac{V}{I}$$

forms a **complex "Ohm's Law"** with regard to complex currents and voltages.

Additionally, ICBST (It Can Be Shown That) **Kirchoff's Laws** are likewise valid for complex currents and voltages:

$$\sum_n I_n = 0 \qquad\qquad \sum_n V_n = 0$$

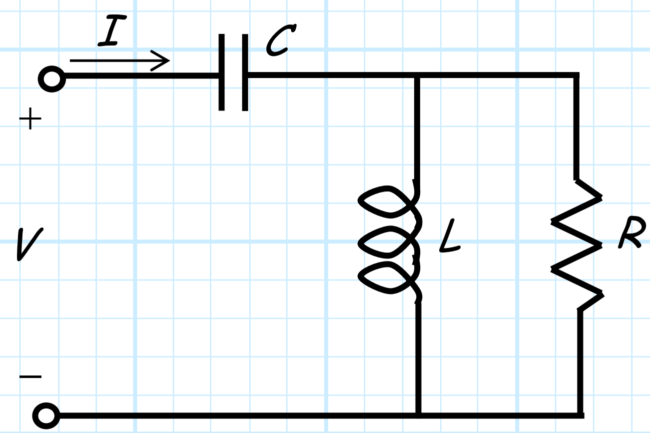
where of course the summation represents **complex addition**.

As a result, the impedance (i.e., the Eigen value) of **any** one-port device can be determined by simply applying a **basic** knowledge of **linear circuit analysis**!

We can determine Eigen values without knowing the impulse response!

Returning to the example:

$$Z = \frac{V}{I}$$



And thus using out **basic** circuits knowledge, we find:

$$Z = Z_C + Z_R \| Z_L = {}^1\!/_{j\omega C} + R \| j\omega L$$

Thus, the Eigen value of the linear operator:

$$\mathcal{L}_Z\left[i(t)\right] = v(t)$$

For **this** one-port network is:

$$G_Z(\omega) = {}^1\!/_{j\omega C} + R \| j\omega L$$

No need for convolution!

Look what we did! We were able to determine $G_z(\omega)$ **without** explicitly determining impulse response $g_z(t)$, or having to perform **any** integrations!

Now, if we actually **need** to determine the voltage function $v(t)$ created by some **arbitrary** current function $i(t)$, we integrate:

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G_z(\omega)\, I(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\tfrac{1}{j\omega C} + R \| j\omega L \right) I(\omega)\, e^{j\omega t}\, d\omega$$

where:

$$I(\omega) = \int_{-\infty}^{+\infty} i(t)\, e^{-j\omega t}\, dt$$

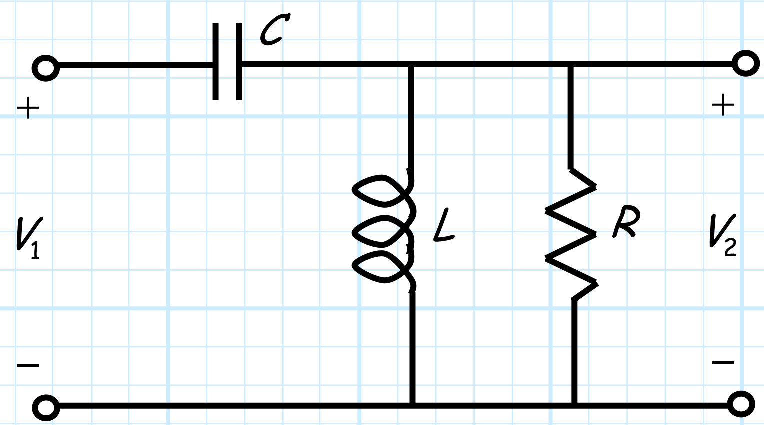
Otherwise, if our current function is **time-harmonic** (i.e., sinusoidal with frequency ω), we can simply relate complex current I and complex voltage V with the equation:

$$V = Z\, I$$

$$= \left(\tfrac{1}{j\omega C} + R \| j\omega L \right) I$$

See how easy this is?

Similarly, for our **two-port** example, we can likewise determine from **basic** circuit theory the **Eigen value** of linear operator:



$$\mathcal{L}_{21}\left[v_1(t)\right]=v_2(t)$$

is:

$$G_{21}(\omega)=\frac{Z_L \| Z_R}{Z_C+Z_L \| Z_R}=\frac{j\omega L \| R}{\dfrac{1}{j\omega C}+j\omega L \| R}$$

so that:

$$V_2=G_{21}(\omega)\,V_1$$

or more generally:

$$v_2(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} G_{21}(\omega)\,V_1(\omega)\,e^{j\omega t}\,d\omega$$

where:

$$V_1(\omega)=\int_{-\infty}^{+\infty} v_1(t)\,e^{-j\omega t}\,dt$$

Eigen Values of the Laplace Transform

Well, I fibbed **a little** when I stated that the Eigen function of linear, time-invariant systems (circuits) is:

$$\mathcal{L}\left\{e^{j\omega t}\right\} = G(\omega)\, e^{j\omega t}$$

Instead, the more **general** Eigen function is:

$$\mathcal{L}\left\{e^{st}\right\} = G(s)\, e^{st}$$

Where s is a **complex** (i.e., real and imaginary) frequency of the form:

$$s = \sigma + j\,\omega$$

such that:

$$e^{st} = e^{(\sigma + j\omega)\,t} = e^{\sigma t} e^{j\omega t}$$

Note then, **if** $\sigma = 0$, the Eigen function e^{st} becomes the previously described Eigen function $e^{j\omega t}$!

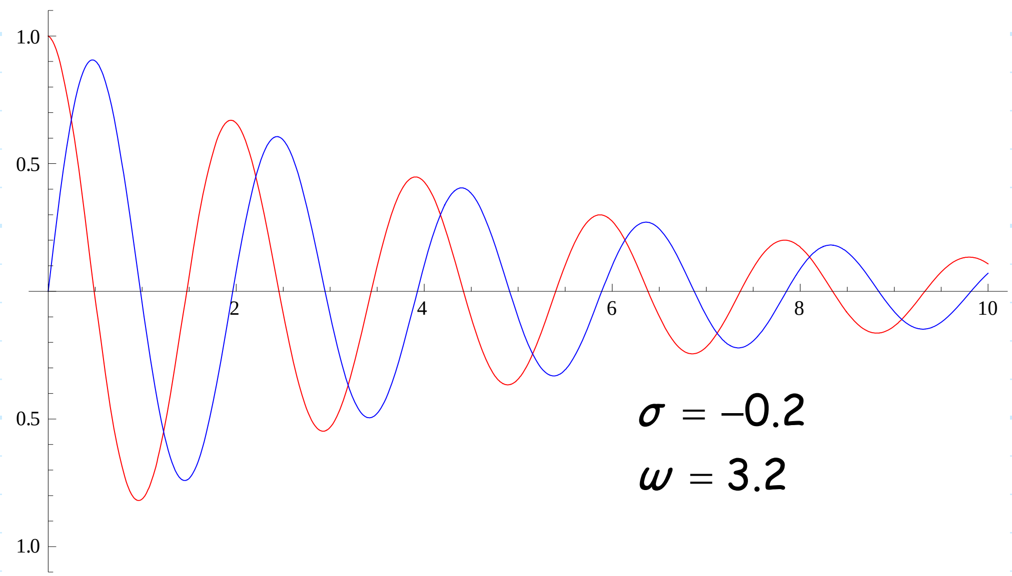
What does this function mean?

Q: *Yikes! I understand e^{st} even **less** than I understood $e^{j\omega t}$! What does this function mean?*

A: Remember, the function e^{st} is a **complex** function—it is actually an expression of **two real-value** functions.

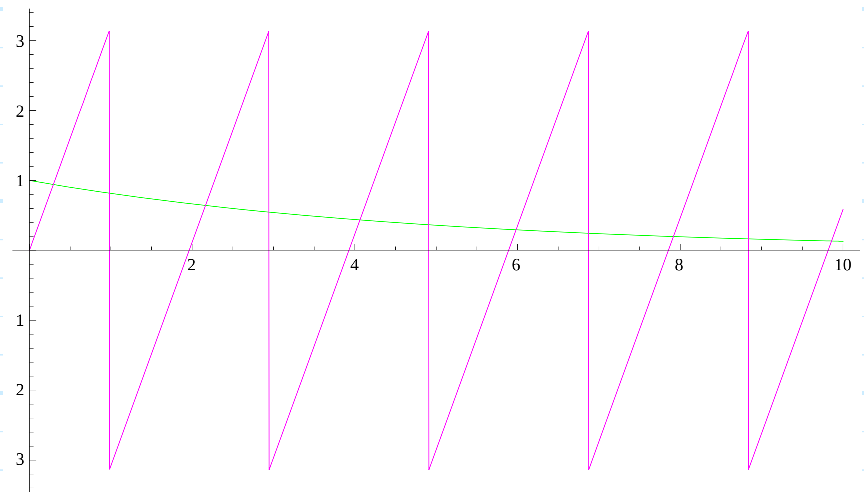
These two real-valued functions could be its **real** and **imaginary** components:

$$
\begin{aligned}
e^{st} &= e^{\sigma t} e^{+j\omega t} \\
&= e^{\sigma t}\left(\cos \omega t + j \sin \omega t\right) \\
&= e^{\sigma t} \cos \omega t + j\, e^{\sigma t} \sin \omega t
\end{aligned}
$$



Magnitude and phase

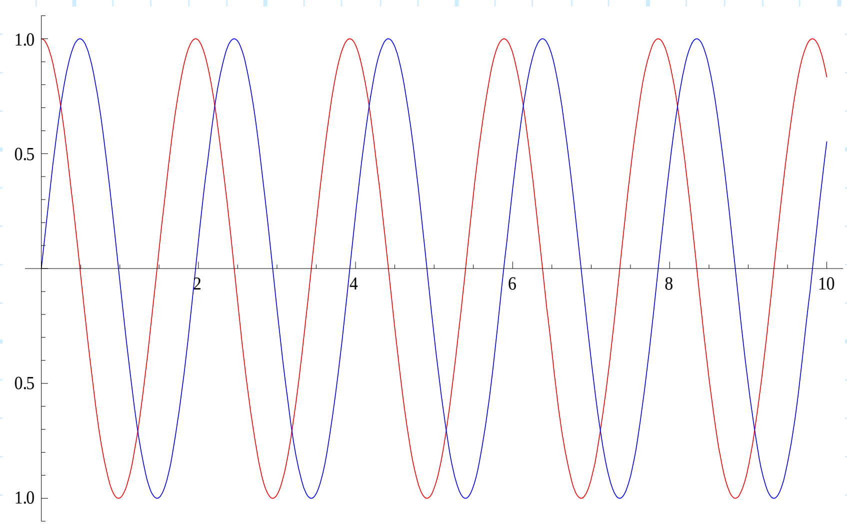
Or, the two real-valued functions could alternatively be the complex values **magnitude** and **phase**:



$\sigma = -0.2$

$\omega = 3.2$

If $\sigma = 0$, then $e^{st} = e^{+j\omega t}$, and we're **back** to the time-harmonic Eigen function:



$\sigma = 0.0$

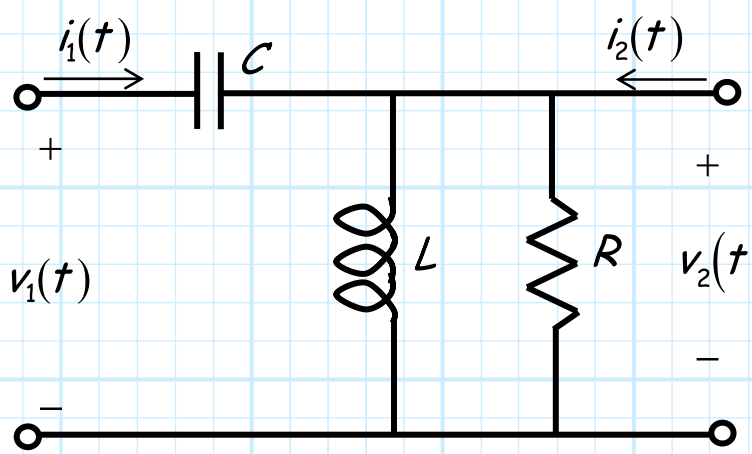
$\omega = 3.2$

Can we use this as a basis?

Q: *What about **basis functions**? Can we use these Eigen function to **expand** a signal?*

A: Sure! Instead of the Fourier Transform, the result of expanding a signal with basis function e^{st} is the **Laplace Transform**.

For example, **again** consider the following linear circuit:



$$v_2(t) = \mathcal{L}\{v_1(t)\} = \int_{-\infty}^{\infty} g(t-t')\, v_1(t')\, dt'$$

A summary

Using the Laplace transform, we can determine the **output** voltage $v_2(t)$ by:

1. Expand the **input** signal $v_1(t)$ using the basis function e^{st}:

$$V_1(s) = \int_0^{+\infty} v_1(t)\, e^{-st}\, dt \quad \text{(or use a \textbf{look-up table!})}$$

2. Determine the **Eigen value** of the linear operator relating $v_1(t)$ to $v_2(t)$:

$$v_2(t) = \mathcal{L}\{v_1(t)\} = \int_{-\infty}^{\infty} g(t-t')\, v_1(t')\, dt'$$

$$\Rightarrow \qquad V_2(s) = G(s)V_1(s)$$

where:

$$G(s) = \int_{-\infty}^{+\infty} g(t)\, e^{-st}\, dt$$

3. Determine $v_2(t)$ from the **inverse** Laplace transform of $V_2(s)$ (**definitely** use a look-up table!).

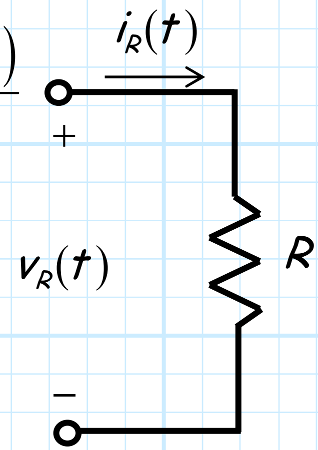
The Eigen values of circuit elements

Q: *But how do we **determine** $G(s)$?*

A: It's just pretty darn **simple**!

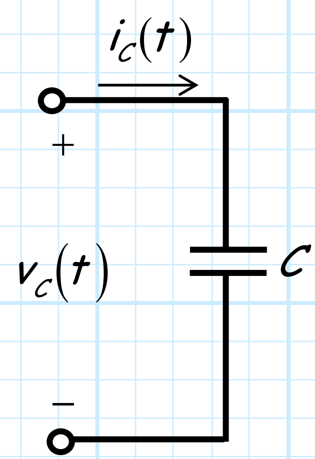
Again, we determine the Eigen value of each linear operator of our **three** linear circuit elements—only **this** time we use the Eigen function e^{st}!

$$i_R(t) = \mathcal{L}_{\mathcal{Y}}^{R}\left[v_R(t)\right] = \frac{v_R(t)}{R}$$



$$\mathcal{L}_{\mathcal{Y}}^{R}\left[e^{st}\right] = \frac{e^{st}}{R}$$

$$I_R(s) = \frac{V_R(s)}{R}$$



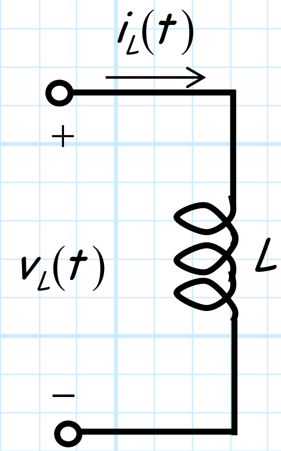
$$i_C(t) = \mathcal{L}_{\mathcal{Y}}^{C}\left[v_C(t)\right] = C\,\frac{d\,v_C(t)}{dt}$$

$$\mathcal{L}_{\mathcal{Y}}^{C}\left[e^{st}\right] = C\,\frac{d\,e^{st}}{dt} = sC\,e^{st}$$

$$I_C(s) = sC\,V_C(s)$$

Just apply your circuits knowledge!

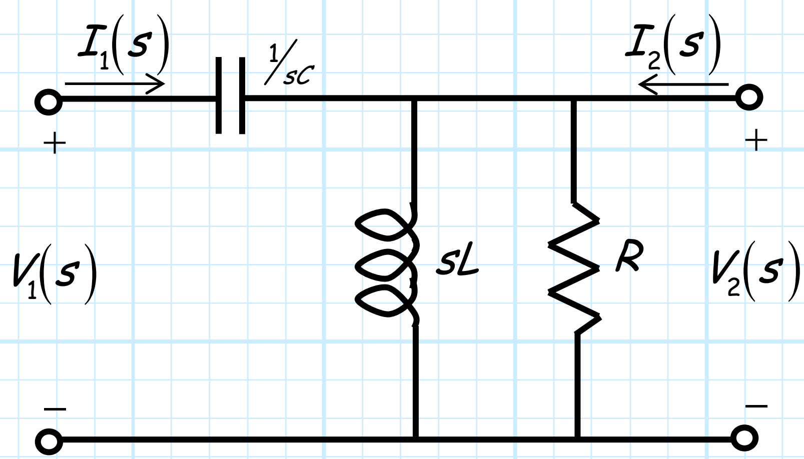
$$\mathcal{L}_{\mathcal{Y}}^{L}\left[v_L(t)\right] = i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t')\, dt'$$



$$\mathcal{L}_{\mathcal{Y}}^{L}\left[e^{st}\right] = \frac{1}{L}\int_{-\infty}^{t} e^{st}\, dt' = \frac{1}{sL}e^{st}$$

$$I_L(s) = \frac{V_L(s)}{s\,L}$$

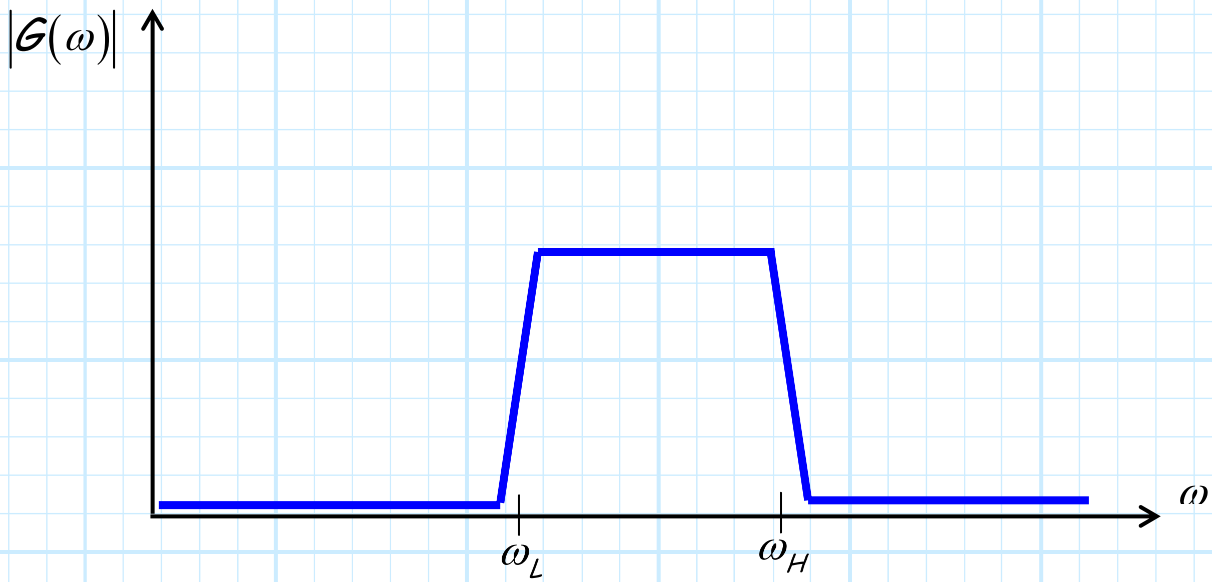
As a result we can determine the Eigen value $G(s)$ of a linear circuit by applying our **circuit theory**:



$$\frac{V_2(s)}{V_1(s)} = G_{21}(s) = \frac{s\,L \| R}{\dfrac{1}{sC} + sL \| R}$$

Frequency Bands

The Eigen value $G(\omega)$ of a linear operator is of course dependent on **frequency** ω—the numeric **value** of $G(\omega)$ depends on the frequency ω of the basis function $e^{j\omega t}$.



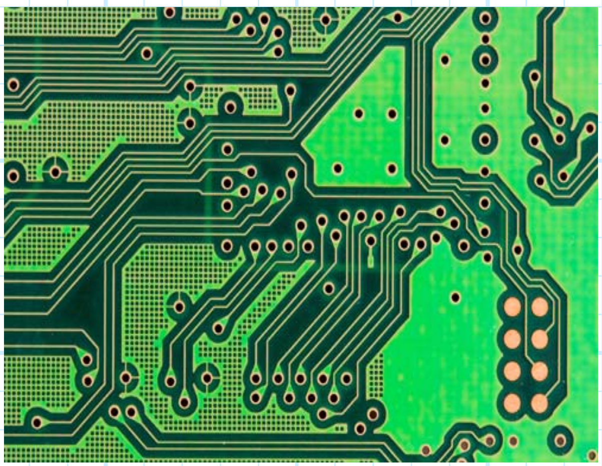
Frequency Response

The frequency ω has units of **radians/second**; it can likewise be expressed as:

$$\omega = 2\pi f$$

where f is the sinusoidal frequency in cycles/second (i.e., **Hertz**).

As a result, the function $G(\omega)$ is also known as the **frequency response** of a linear operator (e.g. a linear circuit).



The numeric value of the signal frequency f has significant **practical** ramifications to us electrical engineers, beyond that of simply determining the numeric value $G(\omega)$.

These practical ramifications include the **packaging**, **manufacturing**, and **interconnection** of electrical and electronic devices.

The problem is that every real circuit is **awash** in inductance and capacitance!

Those darn parasitics!

Q: *If this is such a problem, shouldn't we just **avoid** using capacitors and inductors?*

A: Well, capacitors and inductors are **particular useful** to us EE's.

But, even **without** capacitors and inductors, we find that our circuits are **still** awash in capacitance and inductance!

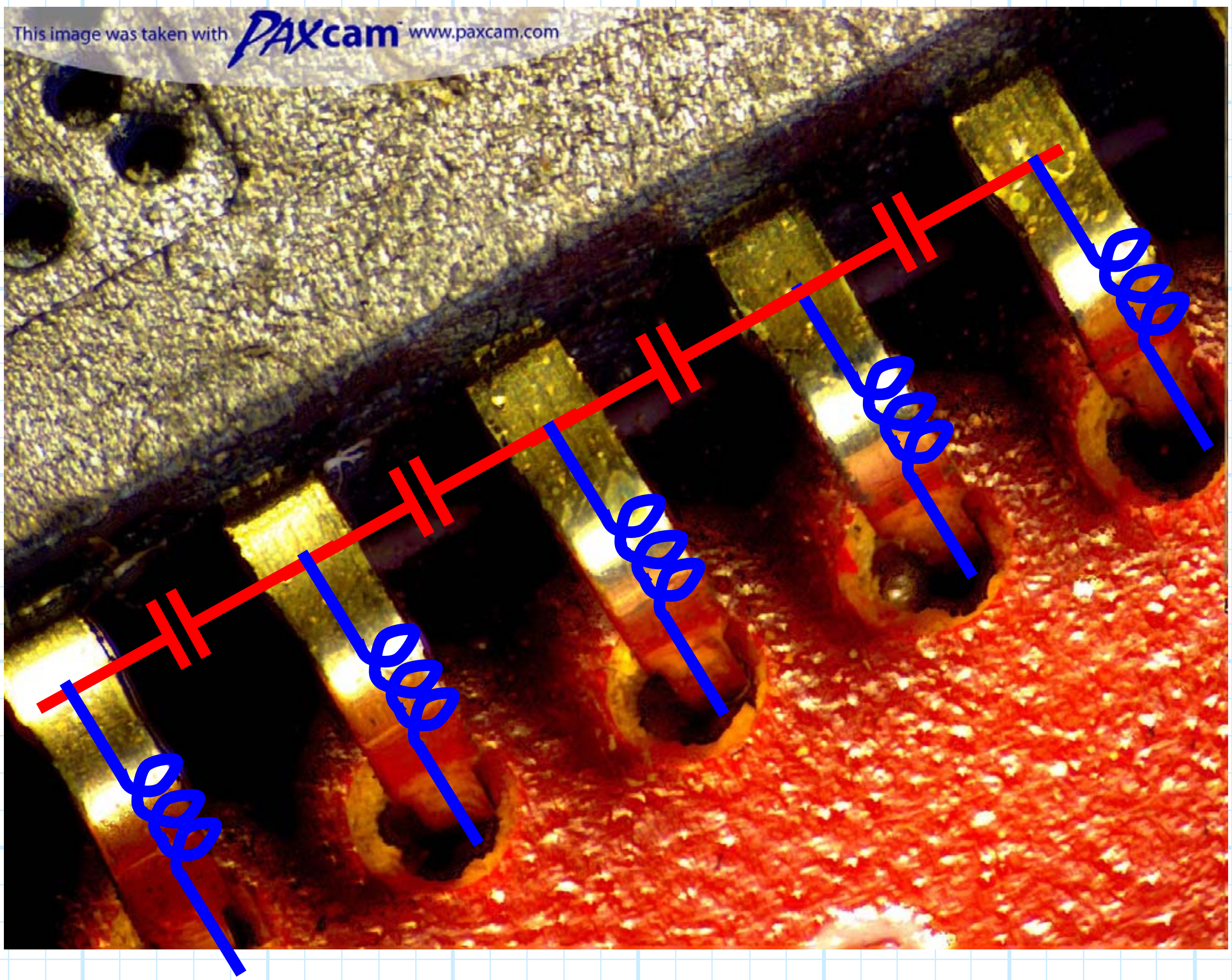
Q: *???*

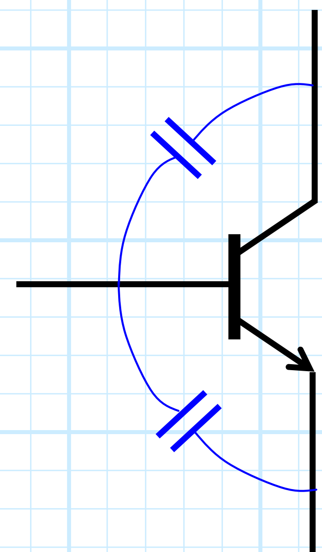
A: Every circuit that we construct will have a inherent set of **parasitic** inductance and capacitance.

Parasitic inductance and capacitance is associated with elements **other** than capacitors and inductors!

Every wire an inductor

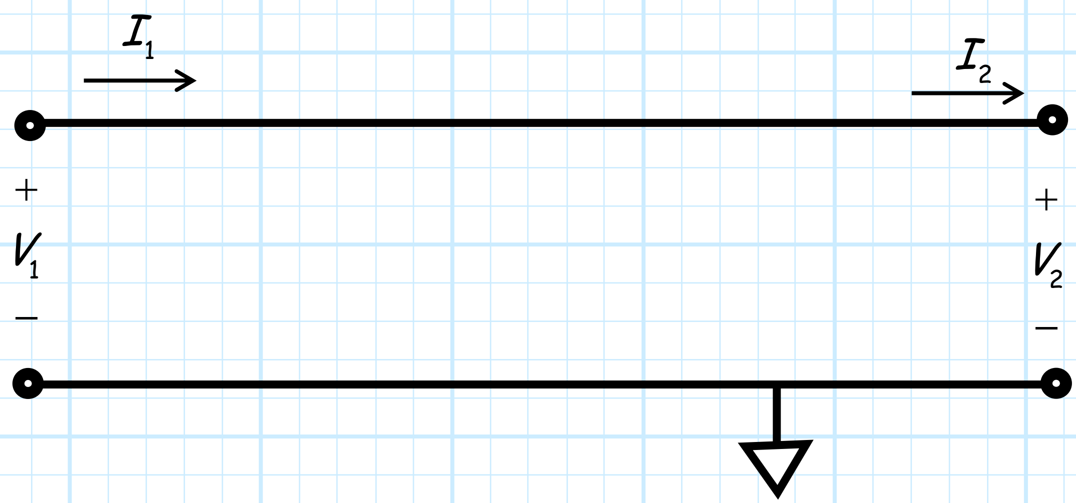
For example, every **wire** and lead has a small inductance associated with it:





Seems simple enough…

Consider then a "wire" above a ground plane:



From **KVL** and **KCL**, we "know" that:

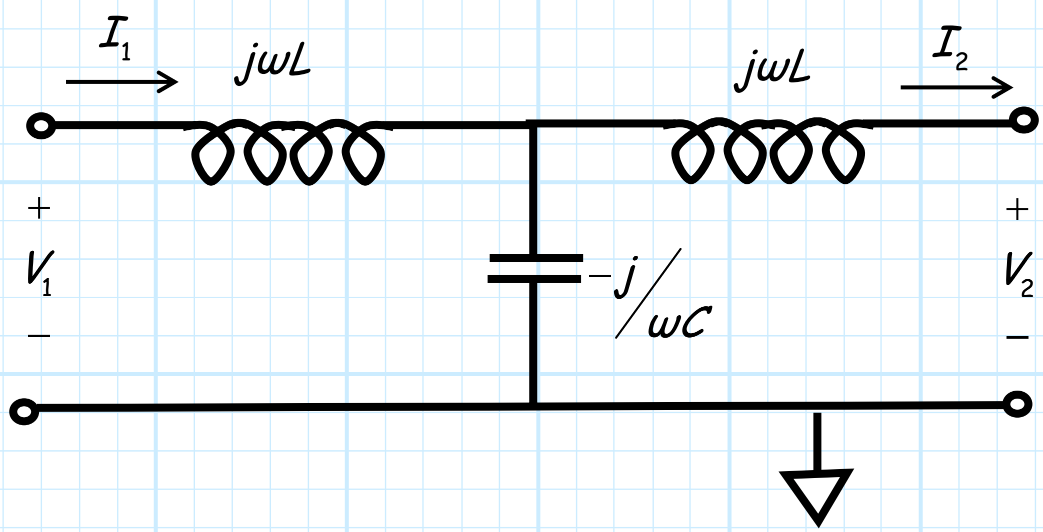
$$V_1 = V_2 \qquad\qquad I_1 = I_2$$

Thus, the linear operator (for example) relating voltage V_1 to voltage V_2 has an **Eigen value** equal to **1.0** for **all frequencies**:

$$\frac{V_2}{V_1} = G(\omega) = 1.0$$

...but its harder than you thought!

But, the unfortunate reality is that the "wire" exhibits **inductance**, and likewise a **capacitance** between it and the ground plane



We now see that the in fact the currents and voltage must be **dissimilar**:

$$V_1 \neq V_2 \qquad\qquad I_1 \neq I_2$$

And so the Eigen value of the linear operator is **not** equal to 1.0!

$$\frac{V_2}{V_1} = G(\omega) \neq 1.0$$

The parasitics are small

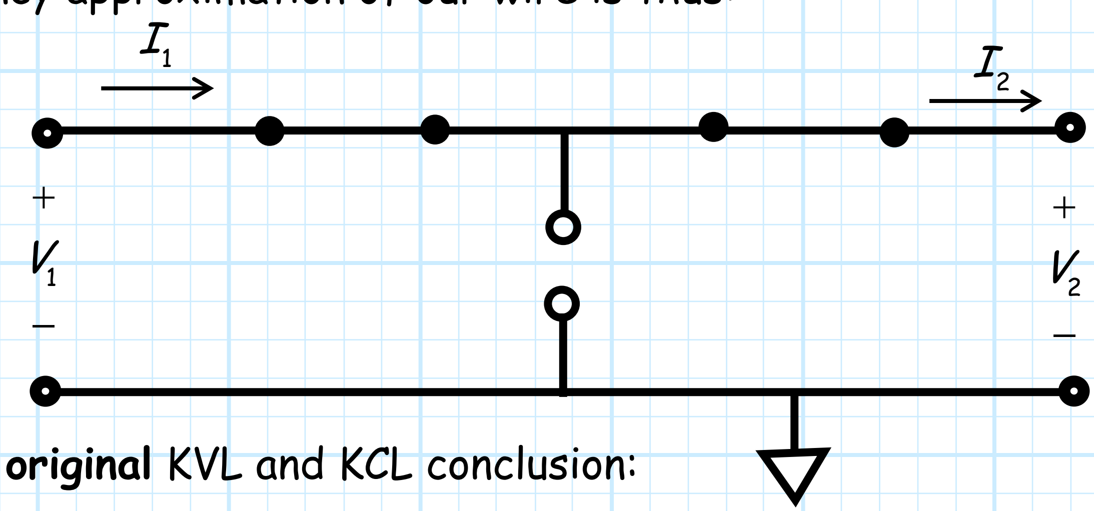
Now, these parasitic values of L and C are likely to be **very small**, so that if the frequency is "low" the **inductive impedance** is quite **small**:

$$|j\omega L| \ll 1 \qquad \text{(almost a \textbf{short} circuit!)}$$

And, the **capacitive impedance** (if the frequency is low) is quite **large**:

$$\left|{}^{-j}\!/_{\omega C}\right| \ll 1 \qquad \text{(almost an \textbf{open} circuit!)}$$

Thus, a low-frequency approximation of our wire is thus:



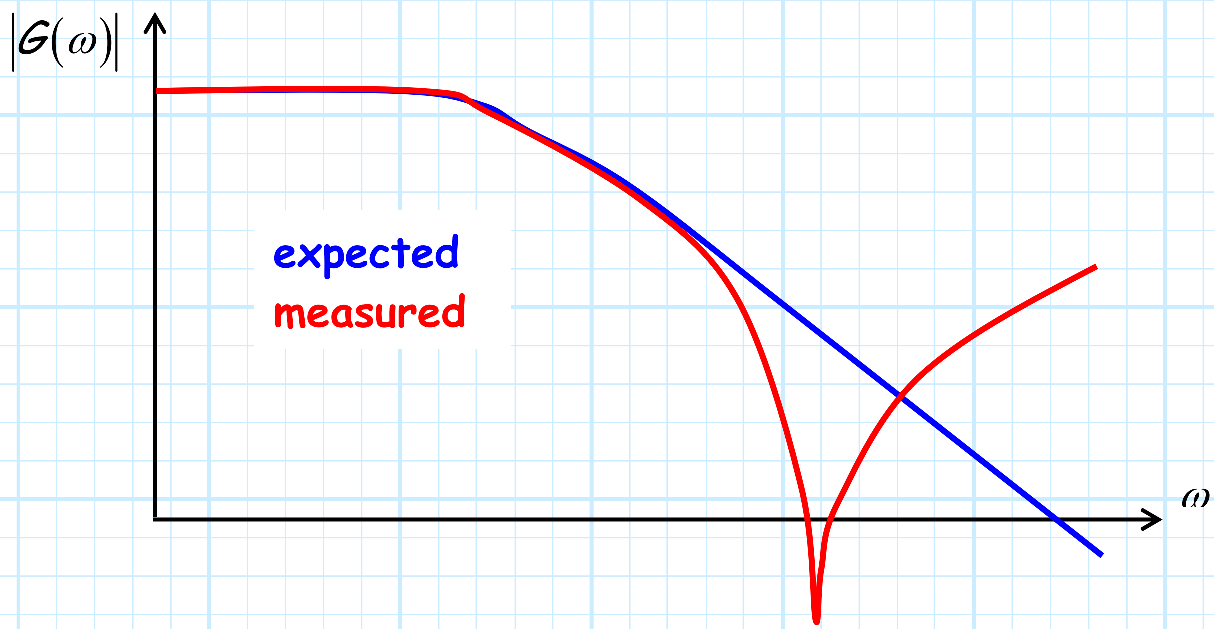
Which leads to our **original** KVL and KCL conclusion:

$$V_1 = V_2 \qquad\qquad I_1 = I_2$$

Parasitics are a problem at "high" frequencies

Thus, as our signal frequency increases, the we often find that the "frequency response" $G(\omega)$ will in reality be **different** from that **predicted** by our circuit model—**unless** explicit parasitics are considered in that model.

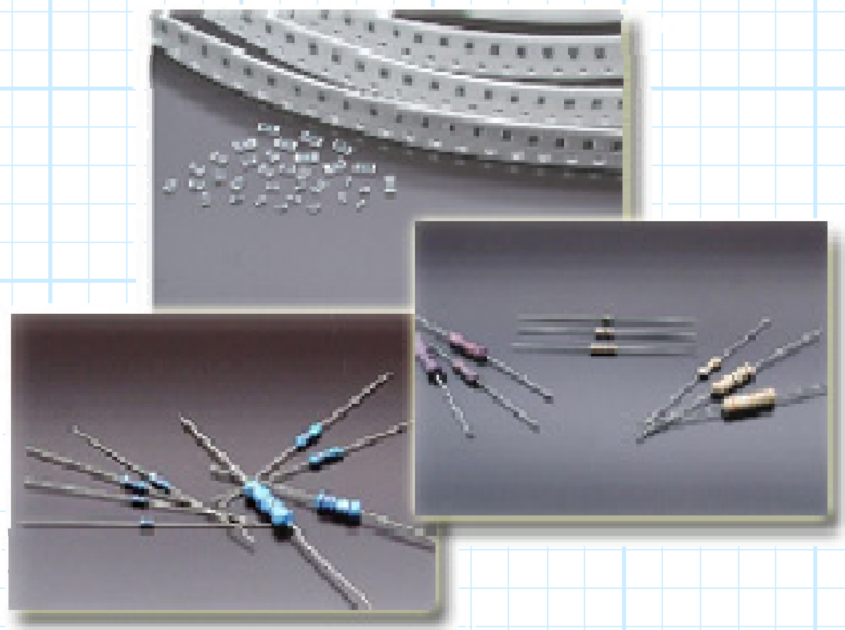
As a result, the response $G(\omega)$ may **vary** from our **expectations** as the signal frequency increases!



Frequency Bands

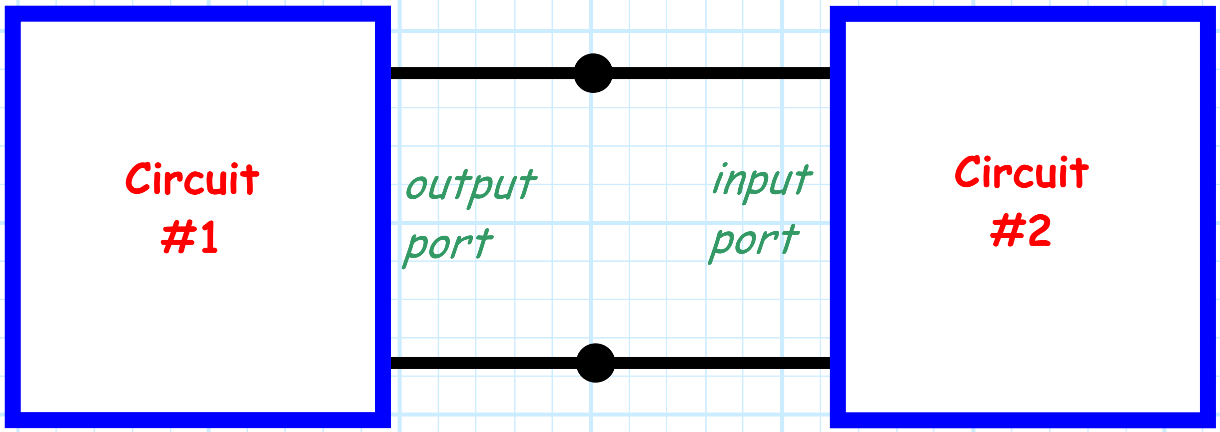
For frequencies in the kilohertz (**audio** band) of megahertz (**video** band), parasitics are generally not a problem.

However, as we move into the 100's of megahertz, or gigahertz (**RF** and **microwave** bands), the effects of parasitic inductance and capacitance are not only significant—they're **unavoidable**!



Impedance and Admittance Parameters

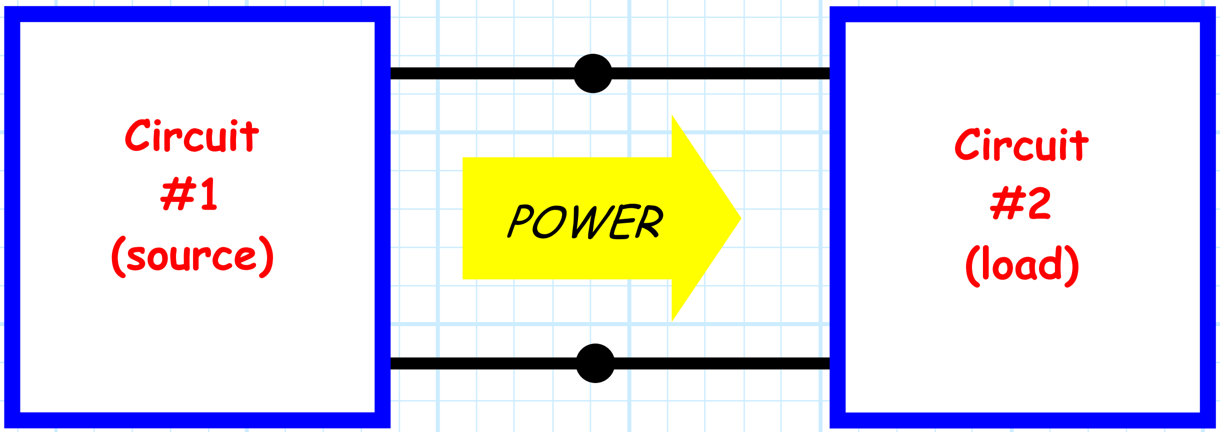
Say we wish to connect the **output** of one circuit to the **input** of another .



The terms "input" and "output" tells us that we wish for signal energy to flow **from** the output circuit **to** the input circuit.

Energy flows from source to load

In this case, the first circuit is the **source**, and the second circuit is the **load**.

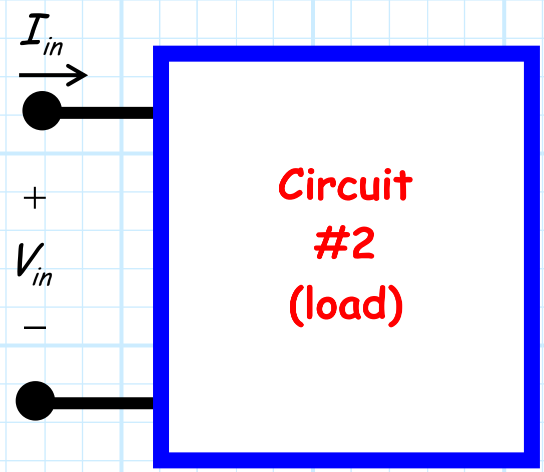


Each of these two circuits may be quite complex, but we can always simply this problem by using **equivalent circuits**.

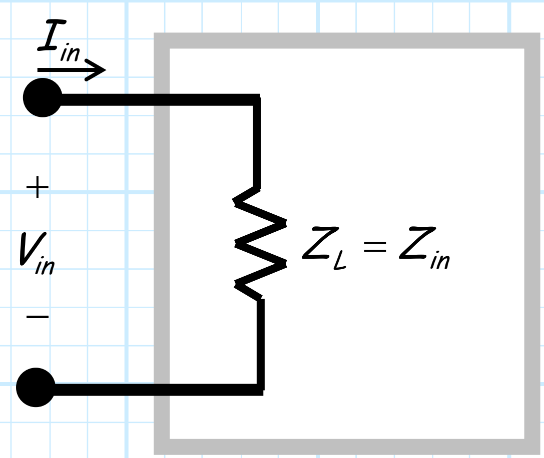
Load is the input impedance

For example, if we assume time-harmonic signals (i.e., eigen functions!), the load can be modeled as a simple lumped **impedance** , with a **complex** value equal to the input impedance of the circuit.

$$Z_{in} = \frac{V_{in}}{I_{in}}$$

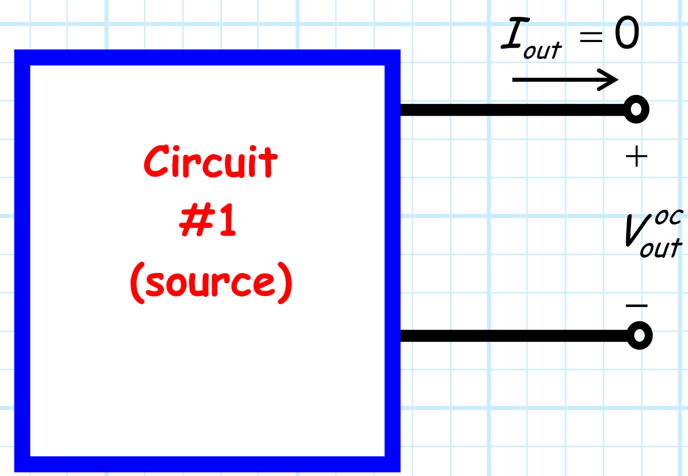


$$\boxed{V_{in} = Z_{in} I_{in}}$$

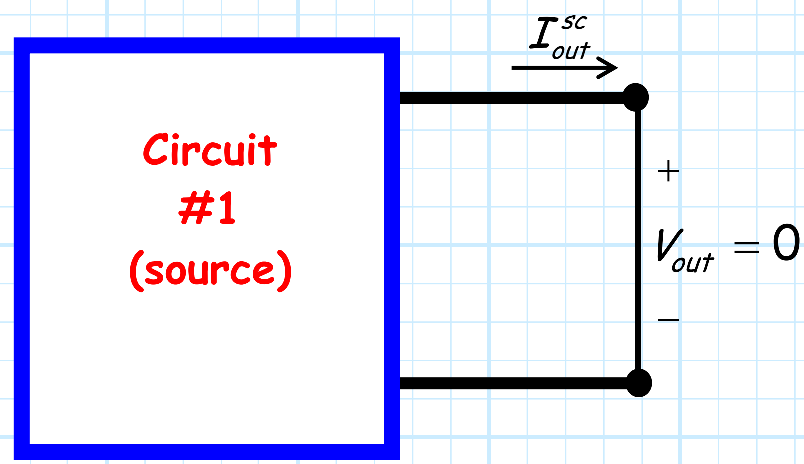


Equivalent Circuits

The source circuit can likewise be modeled using either a Thevenin's or Norton's equivalent.



This equivalent circuit can be determined by first evaluating (or measuring) the **open-circuit** output **voltage** V_{out}^{oc} :

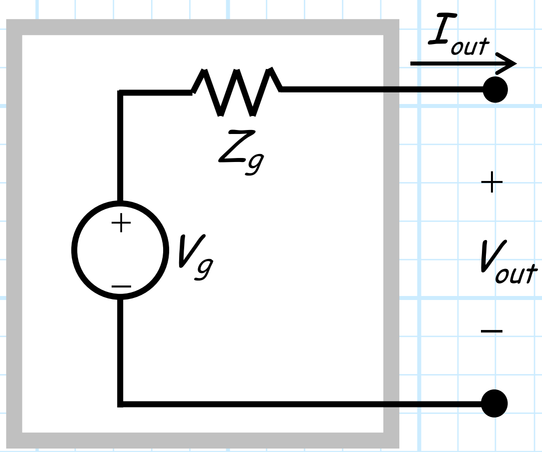


And likewise evaluating (or measuring) the **short-circuit** output **current** I_{out}^{sc} :

Thevenin's

From these two values (V_{out}^{oc} and I_{out}^{sc}) we can determine the **Thevenin's equivalent source**:

$$V_g = V_{out}^{oc} \qquad\qquad Z_g = \frac{V_{out}^{oc}}{I_{out}^{sc}}$$



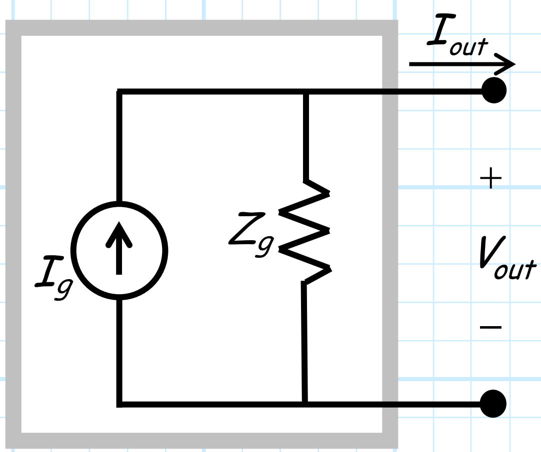
$$V_{out} = V_g - Z_g\, I_{out}$$

$$I_{out} = \frac{V_g - V_{out}}{Z_g}$$

Norton's

Or, we could use a **Norton's** equivalent circuit:

$$I_g = I_{out}^{sc} \qquad\qquad Z_g = \frac{V_{out}^{oc}}{I_{out}^{sc}}$$

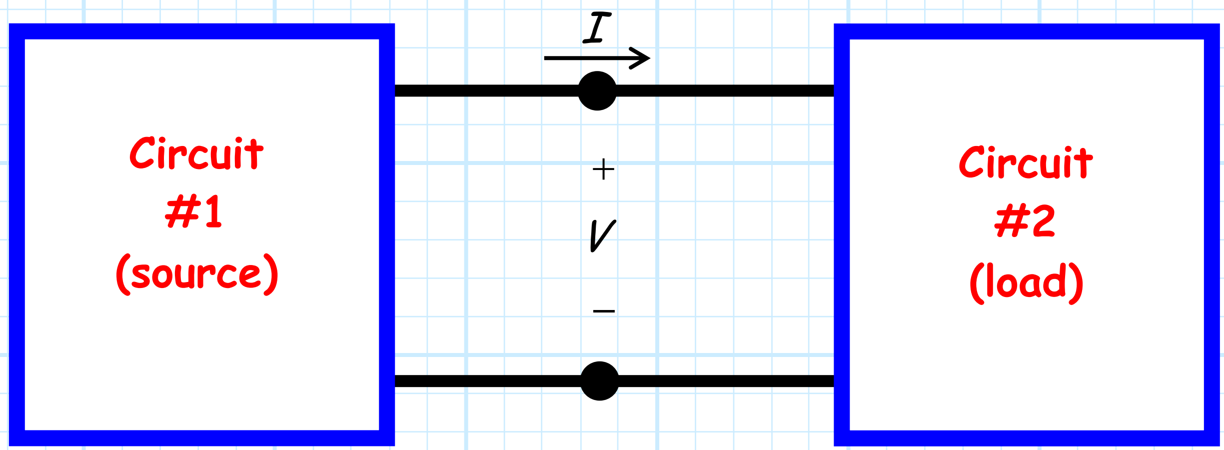


$$I_{out} = I_g - V_{out} / Z_g$$

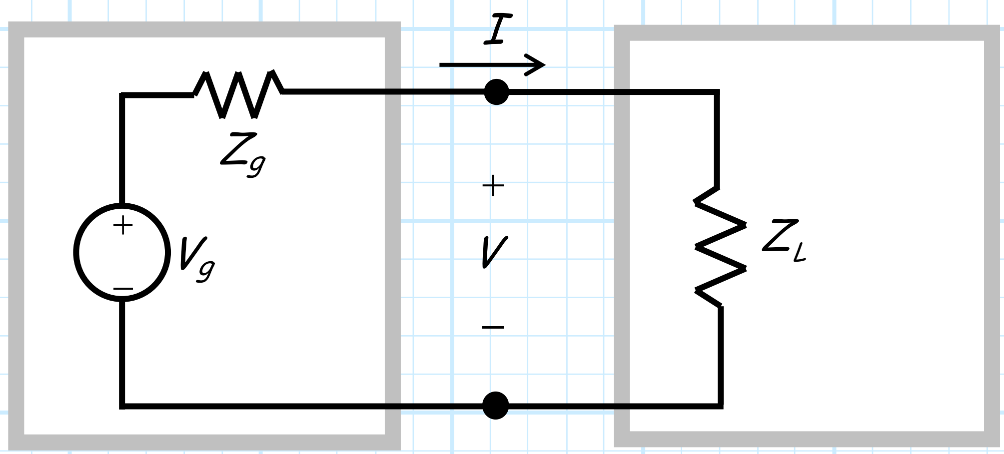
$$V_{out} = \left(I_g - I_{out} \right) Z_g$$

Circuit Model

Thus, the entire circuit:



Can be modeled with equivalent circuits as:



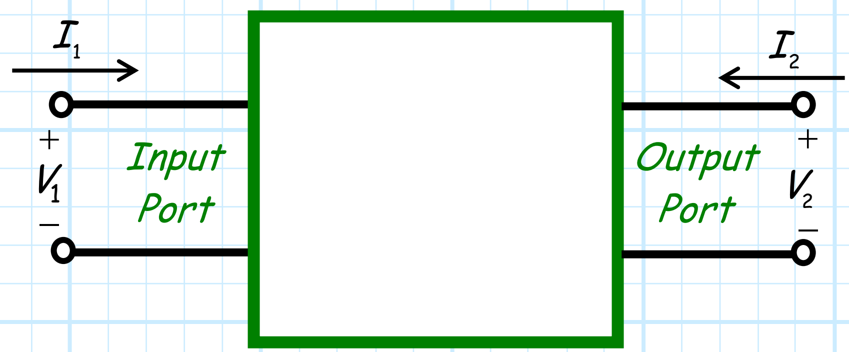
Please note again that we have assumed a **time harmonic** source, such that all the values in the circuit above (V_g, Z_g, I, V, Z_L) are **complex** (i.e., they have a **magnitude** and **phase**).

Two-Port circuits

Q: *But, circuits like filters and amplifiers are two-port devices, they have both an input and an output. How do we characterize a two-port device?*

A: Indeed, many important components are **two-port** circuits.

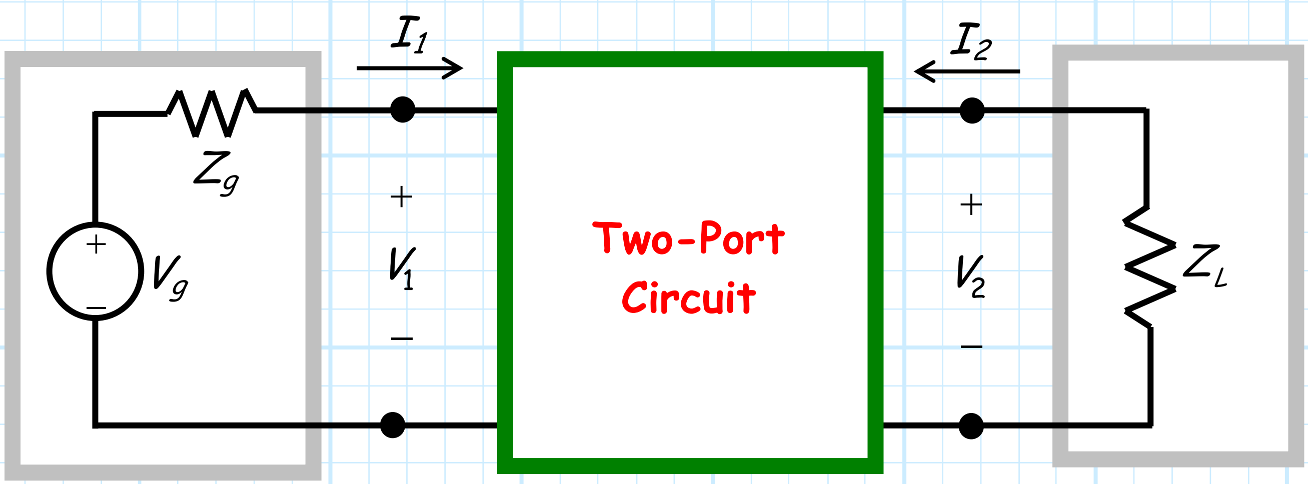
For these devices, the signal power **enters** one port (i.e., the input) and **exits** the other (the output).



Between source and load

These two-port circuits typically do something to **alter** the signal as it passes from input to output (e.g., filters it, amplifies it, attenuates it).

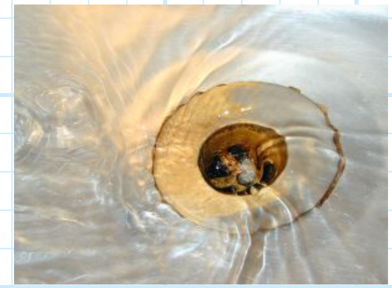
We can thus assume that a **source** is connected to the **input** port, and that a **load** is connected to the **output** port.



How to characterize?

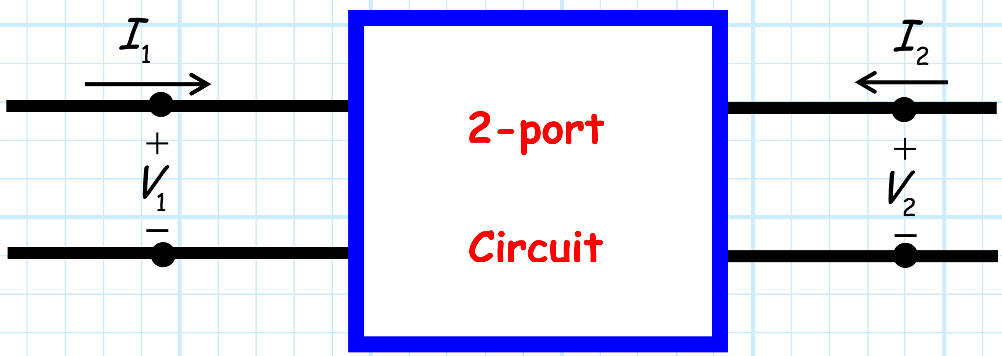
Again, the source circuit may be **quite complex**, consisting of many components. However, at least one of these components must be a **source** of energy.

Likewise, the load circuit might be **quite complex**, consisting of many components. However, at least one of these components must be a **sink** of energy.



Q: *But what about the **two-port circuit** in the middle? How do we characterize it?*

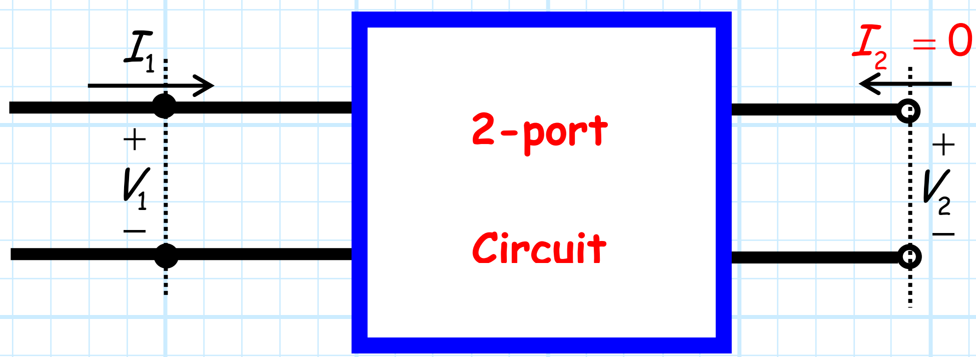
A: A linear two-port circuit is fully characterized by just four **impedance parameters**!



Do this little experiment

Note that inside the "blue box" there could be anything from a very **simple** linear circuit to a very large and **complex** linear system.

Now, say there exists a non-zero current at input **port 1** (i.e., $I_1 \neq 0$), while the current at **port 2** is known to be **zero** (i.e., $I_2 = 0$).



Say we measure/determine the **current** at port **1** (i.e., determine I_1), and we then measure/determine the **voltage** at the port **2** plane (i.e., determine V_2).

Impedance parameters

The complex ratio between V_2 and I_1 is know as the **trans-impedance parameter** Z_{21}:

$$Z_{21}(\omega) = \frac{V_2(\omega)}{I_1(\omega)}$$

Note this trans-impedance parameter is the Eigen value of the linear operator relating current $i_1(t)$ to voltage $v_2(t)$:

$$v_2(t) = \mathcal{L}\{i_1(t)\} \quad \rightarrow \quad V_2(\omega) = G_{21}(\omega)\, I_1(\omega)$$

Thus:

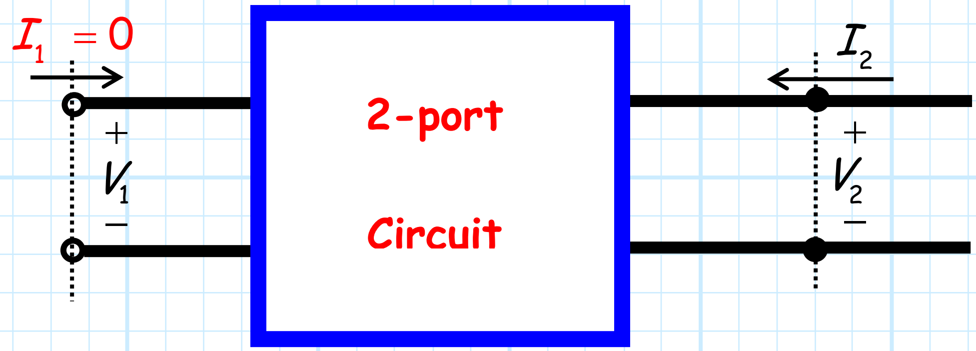
$$G_{21}(\omega) = Z_{21}(\omega)$$

Likewise, the complex ratio between V_1 and I_1 is the **trans-impedance parameter** Z_{11}:

$$Z_{11}(\omega) = \frac{V_1(\omega)}{I_1(\omega)}$$

A second experiment

Now consider the opposite situation, where there exists a non-zero current at **port 2** (i.e., $I_2 \neq 0$), while the current at port 1 is known to be **zero** (i.e., $I_2 = 0$).



The result is two more impedance parameters:

$$Z_{12}(\omega) = \frac{V_1(\omega)}{I_2(\omega)} \qquad\qquad Z_{22}(\omega) = \frac{V_2(\omega)}{I_2(\omega)}$$

Thus, more **generally**, the ratio of the current into port n and the voltage at port m is:

$$Z_{mn} = \frac{V_m}{I_n} \qquad \text{(given that } I_k = 0 \text{ for } k \neq n \text{)}$$

Open circuits enforce I=0

Q: *But how do we ensure that* ***one*** *port current is zero ?*

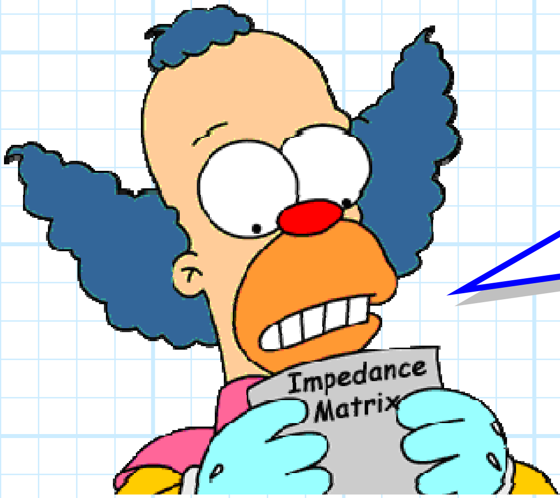
A: Place an **open** circuit **at** that port!

Placing an **open** at a port (and it must be **at** the port!) **enforces** the condition that $I = 0$.

Now, we can thus **equivalently** state the definition of trans-impedance as:

$$Z_{mn} = \frac{V_m}{I_n} \qquad \text{(given that port } k \neq n \text{ is \textbf{open-circuited})}$$

What's the point?



Q: *As impossible as it sounds, this handout is even more* ***pointless*** *than all your previous efforts.* ***Why*** *are we studying this? After all, what is the likelihood that a device will have an* ***open*** *circuit on one of its ports?!*

A: OK, say that **neither** port is **open-circuited**, such that we have currents **simultaneously** on **both** of the two ports of our device.

Since the device is **linear**, the voltage at **one** port is due to **both** port currents.

This voltage is simply the coherent **sum** of the voltage at that port due to **each** of the two currents!

Specifically, the voltage at each port can is:

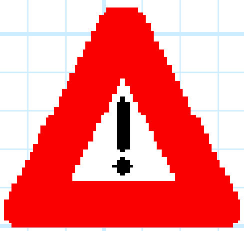
$$V_1 = Z_{11}\, I_1 + Z_{12}\, I_2$$

$$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$$

They're a function of frequency!

Thus, these four impedance parameters **completely characterizes** a linear, *2*-port device.

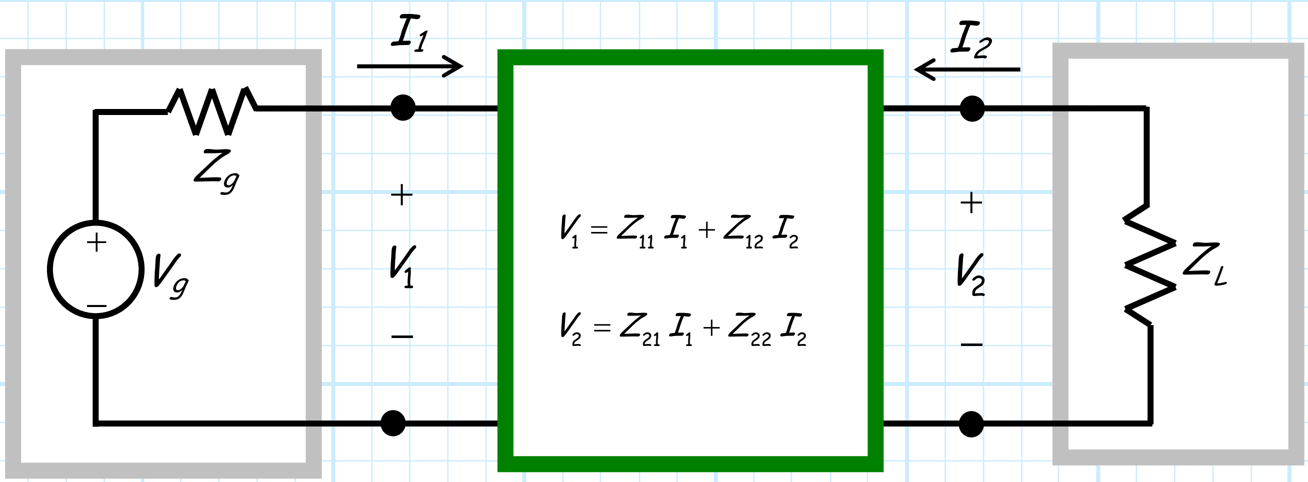
Effectively, these impedance parameters describes a 2-port device the way that Z_L describes a single-port device (e.g., a load)!



But **beware**! The values of the impedance matrix for a particular device or circuit, just like Z_L, are **frequency dependent**!

A complete equivalent circuit

Now, we can use our **equivalent circuits** to model this system:



Note in this circuit there are **4 unknown values**—two voltages (V_1 and V_2), and two currents (I_1 and I_2).

→ Our job is to **determine** these 4 unknown values!

Let's do some algebra!

Let's begin by looking at the source, we can determine from **KVL** that:

$$V_g - Z_g I_1 = V_1$$

And so with a bit of **algebra**:

$$I_1 = \frac{V_g - V_1}{Z_g}$$ (←look, **Ohm's Law**!)

Now let's look at our two-port circuit. If we know the impedance matrix (i.e., all **four trans-impedance** parameters), then:

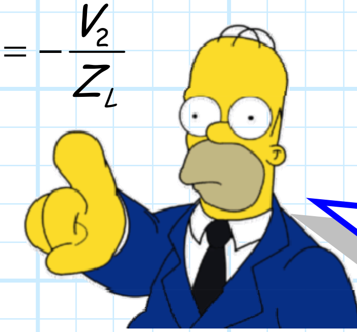
$$V_1 = Z_{11}\, I_1 + Z_{12}\, I_2$$

$$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$$

Watch the minus sign!

Finally, for the load:

$$I_2 = -\frac{V_2}{Z_L}$$



Q: *Are you sure this is correct? I don't recall there being a **minus sign** in Ohm's Law.*

A: Be **very careful** with the notation.

Current I_2 is defined as positive when it is flowing into the two port circuit. This is the notation required for the impedance matrix.

Thus, positive current I_2 is flowing out of the load impedance—the opposite convention to Ohm's Law.

This is why the **minus sign is required**.

A very good thing

Now let's **take stock** of our results. Notice that we have compiled **four** independent equations, involving our **four** unknown values:

$$I_1 = \frac{V_g - V_1}{Z_g}$$

$$I_2 = -\frac{V_2}{Z_L}$$

$$V_1 = Z_{11}\, I_1 + Z_{12}\, I_2$$

$$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$$

Q: *Four equations and four unknowns! That sounds like a **very good thing**!*

A: It **is**! We can apply a bit of **algebra** and solve for the unknown currents and voltages:

$$I_1 = V_g \frac{Z_{22} + Z_L}{(Z_{11} + Z_g)(Z_{22} + Z_L) - Z_{12} Z_{21}} \qquad I_2 = -V_g \frac{Z_{21}}{(Z_{11} + Z_g)(Z_{22} + Z_L) - Z_{12} Z_{21}}$$

$$V_1 = V_g \frac{Z_{11}(Z_{22} + Z_L) - Z_{12} Z_{21}}{(Z_{11} + Z_g)(Z_{22} + Z_L) - Z_{12} Z_{21}} \qquad V_2 = V_g \frac{Z_L\, Z_{21}}{(Z_{11} + Z_g)(Z_{22} + Z_L) - Z_{12} Z_{21}}$$

Admittance Parameters

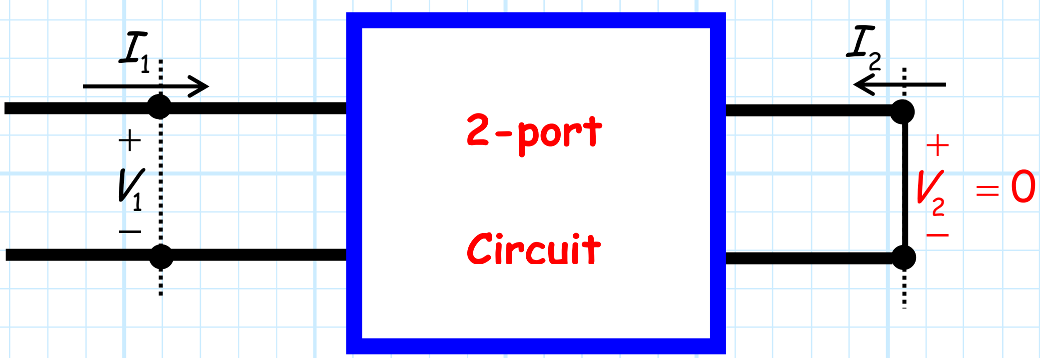
Q: *Are impedance parameters the **only** way to characterize a 2-port linear circuit?*

A: Hardly! Another method uses **admittance parameters**.

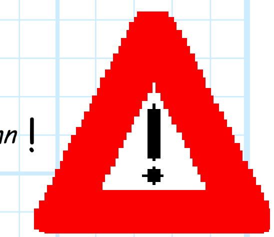
The elements of the Admittance Matrix are the **trans-admittance** parameters Y_{mn}, defined as:

$$Y_{mn} = \frac{I_m}{V_n} \qquad \text{(given that } V_k = 0 \text{ for } k \neq n)$$

Note here that the **voltage** at one port **must** be equal to **zero**. We can ensure that by simply placing a **short circuit** at the zero-voltage port!



Note that $Y_{mn} \neq 1/Z_{mn}$!



Short circuits enforce V=0

Now, we can **equivalently** state the definition of trans-admittance as:

$$Y_{mn} = \frac{V_m}{I_n} \qquad \text{(given that all ports } k \neq n \text{ are } \textbf{short - circuited}\text{)}$$

Just as with the trans-impedance values, we can use the trans-admittance values to evaluate general circuit problems, where **none** of the ports have zero voltage.

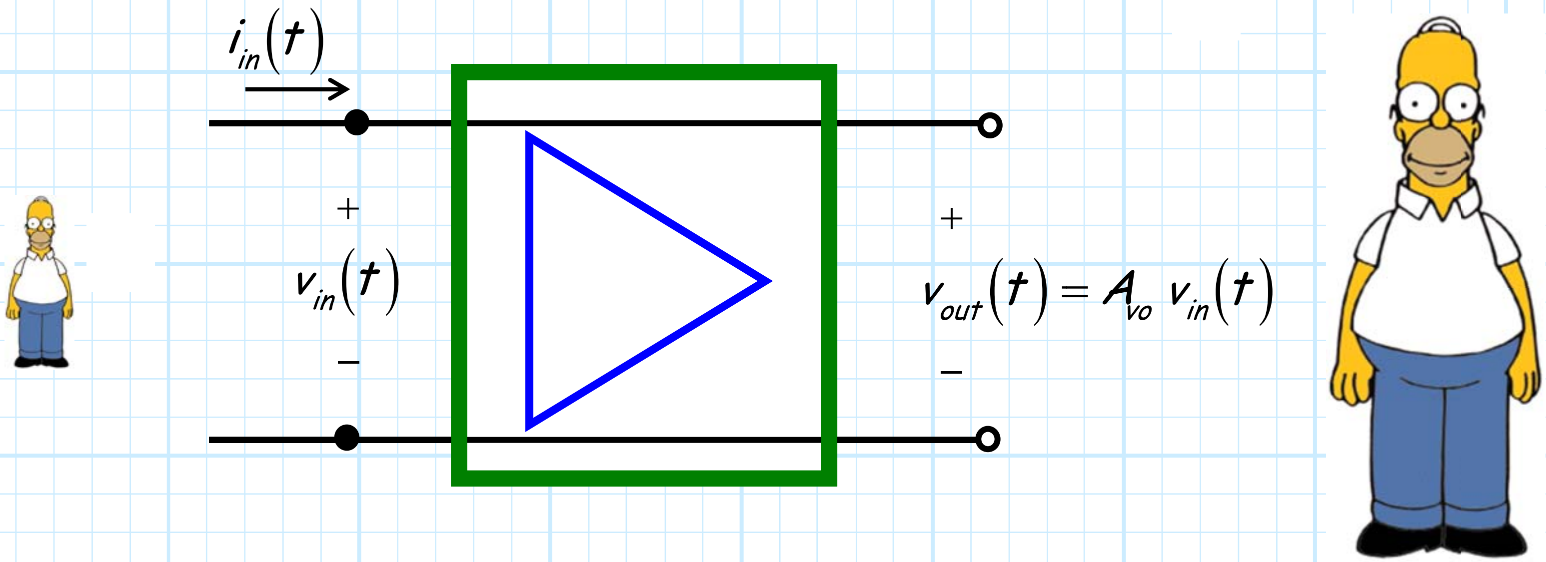
Since the device is **linear**, the current at any **one** port due to **all** the port currents is simply the coherent **sum** of the currents at that port due to **each** of the port voltages!

$$I_1 = Y_{11}\, V_1 + Y_{12}\, V_2$$

$$I_2 = Y_{21}\, V_1 + Y_{22}\, V_2$$

Amplifiers

An **ideal** amplifier is a two-port circuit that takes an input signal $v_{in}(t)$ and reproduces it **exactly** at its output, only with a **larger** magnitude!



The real value A_{vo} is the **open-circuit voltage gain** of this ideal amplifier, and has a magnitude much larger than unity ($A_{vo} \gg 1$).

We actually *can* find $g(t)$!

Now, let's express this result using our knowledge of **linear circuit theory**!

Recall, the output $v_{out}(t)$ of a linear device can be determined by **convolving** its input $v_{in}(t)$ with the device **impulse response** $g(t)$:

$$v_{out}(t) = \int_{-\infty}^{t} g(t-t')\, v_{in}(t')\, dt'$$

Q: *Yikes! What is the impulse response of this ideal amp? How can we determine it?*

A: It's actually quite **simple**!

Remember, the impulse **response** of linear circuit is just the output that results when the **input** is an impulse function $\delta(t)$.

Every function an Eigen function

Since the output of an ideal amplifier is just the input multiplied by A_{vo}, we conclude if $v_{in}(t) = \delta(t)$:

$$g(t) = v_{out}(t) = A_{vo}\,\delta(t)$$

Thus:

$$\begin{aligned} v_{out}(t) &= \int_{-\infty}^{t} g(t-t')\, v_{in}(t')\, dt' \\ &= \int_{-\infty}^{t} A_{vo}\,\delta(t-t')\, v_{in}(t')\, dt' \\ &= A_{vo}\int_{-\infty}^{t} \delta(t-t')\, v_{in}(t')\, dt' \\ &= A_{vo}\, v_{in}(t) \end{aligned}$$

→ **Any** and **every** function $v_{in}(t)$ is an **Eigen function** of an **ideal** amplifier!!

And now the Eigen value

Now, we can determine the **Eigen value** of this linear operator relating input to output:

$$v_{out}(t) = \mathcal{L}\{v_{in}(t)\}$$

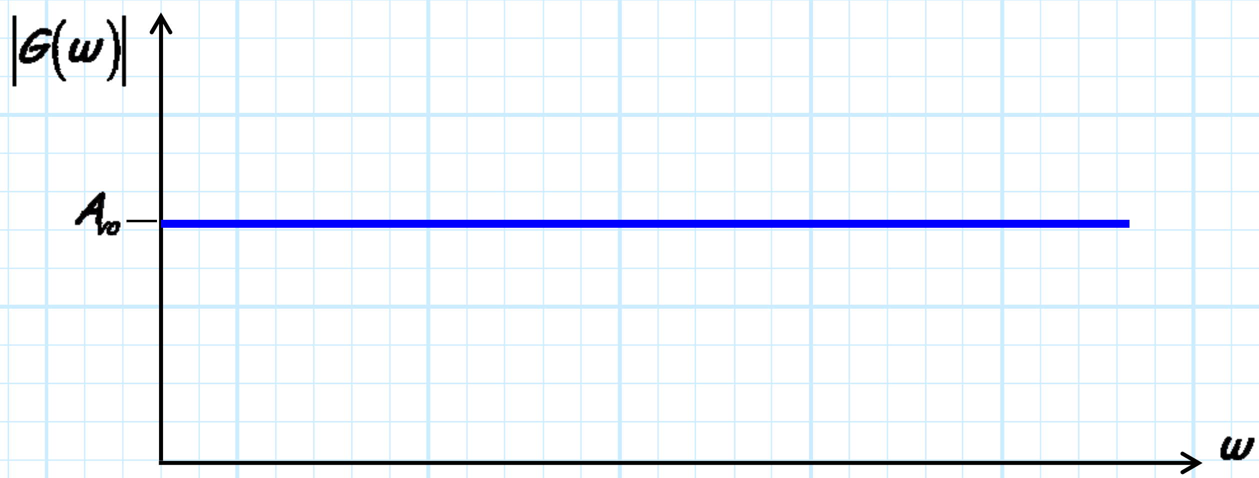
Recall this Eigen value is found from the **Fourier transform** of the impulse response:

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} A_{vo}\, \delta(t)\, e^{-j\omega t}\, dt \\ &= A_{vo} + j0 \\ &= A_{vo} e^{j0} \end{aligned}$$

This result, although simple, has an interesting interpretation…

DC to daylight

...it means that the amplifier exhibits gain of A_{vo} for sinusoidal signals of **any** and **all** frequencies!



BUT, there is one **big** problem with an ideal amplifier:

➡ They are **impossible** to build!!

Real amplifier have finite bandwidths

The **ideal** amplifier has a frequency response of $|G(\omega)| = A_{vo}$.

Note this means that the amplifier gain is A_{vo} for **all** frequencies $0 < \omega < \infty$ (D.C. to daylight!).

The **bandwidth** of the **ideal** amplifier is therefore **infinite**!

* Since every electronic device will exhibit **some** amount of inductance, capacitance, and resistance, every device will have a **finite** bandwidth.

* In other words, there will be frequencies ω where the device does **not work**!

* From the standpoint of an amplifier, "not working" means $|G(\omega)| \ll A_{vo}$ (i.e., **low gain**).

→ Amplifiers therefore have **finite** bandwidths.

Amplifier bandwidth

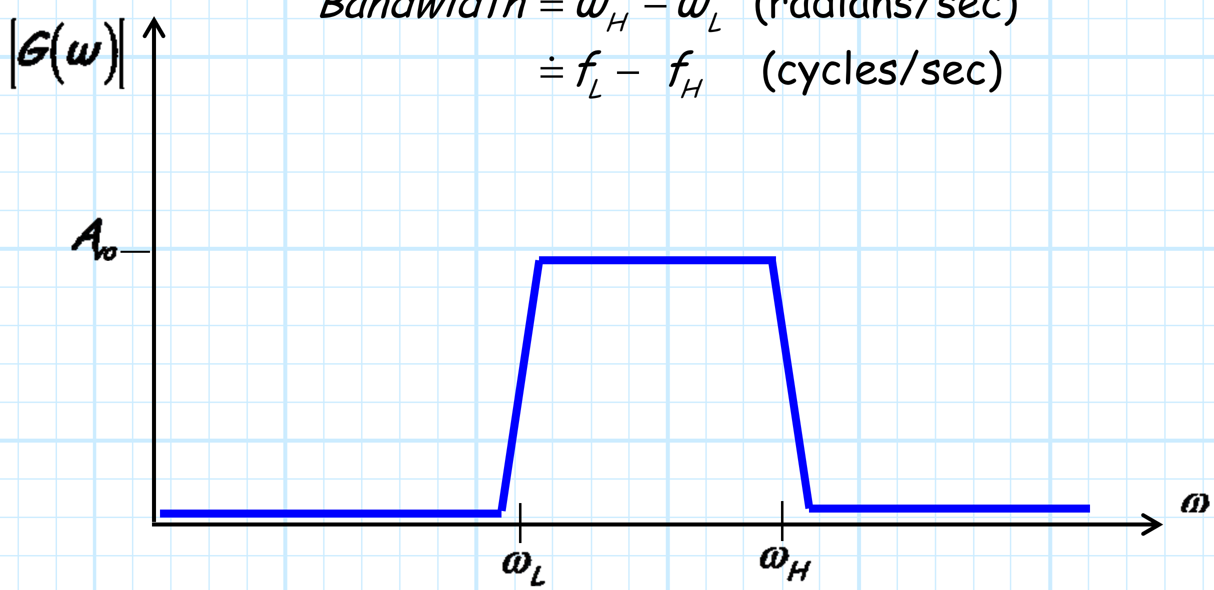
There is a range of frequencies ω between ω_L and ω_H where the gain will (approximately) be A_{vo}.

For frequencies outside this range, the gain will typically be small (i.e. $|G(\omega)| \ll A_{vo}$):

$$|G(\omega)| = \begin{cases} \approx A_{vo} & \omega_L < \omega < \omega_H \\ \ll A_{vo} & \omega < \omega_L, \omega > \omega_H \end{cases}$$

The **width** of this frequency range is called the amplifier **bandwidth**:

$$\begin{aligned} Bandwidth &\doteq \omega_H - \omega_L \quad \text{(radians/sec)} \\ &\doteq f_L - f_H \quad \text{(cycles/sec)} \end{aligned}$$



Wideband is desirable

One result of a **finite bandwidth** is that the amplifier impulse response is **not** an impulse function !

$$h(t) = \int_{-\infty}^{\infty} H(\omega)\, e^{+j\omega t} dt \neq A_{vo}\, \delta(t)$$

therefore **generally** speaking:

$$v_{out}(t) \neq A_{vo}\, v_{in}(t) \;\; !!$$

However, if an input signal **spectrum** $V_{in}(\omega)$ lies completely **within** the amplifier bandwidth, then we find that will (approximately) behave like an **ideal** amplifier:

$$v_{out}(t) \cong A_{vo}\, v_{in}(t) \quad \text{if } \; V_{in}(\omega) \text{ is within the amplifier bandwidth}$$

As a result, **maximizing** the bandwidth of an amplifier is a typically and important **design goal**!

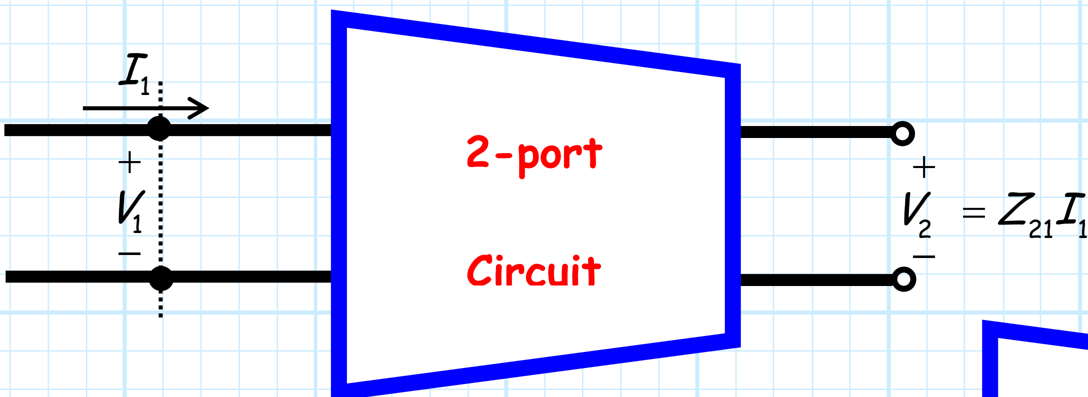
Amplifier Gain

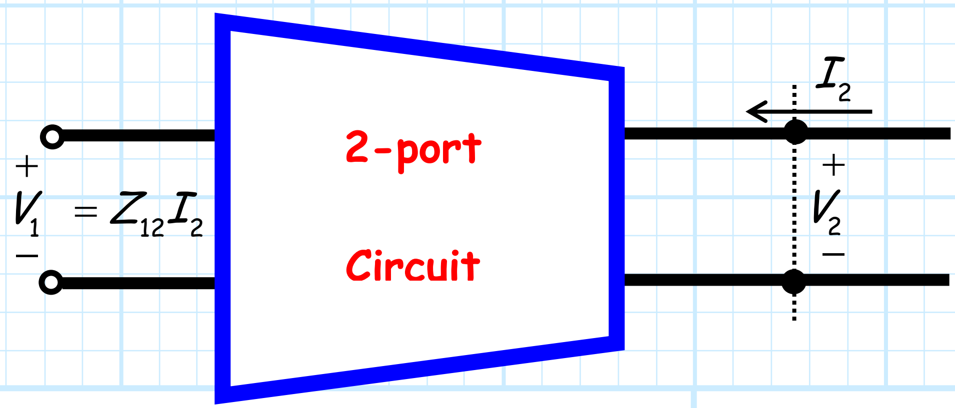
One interesting characteristic of an amplifier is that it is a **unilateral** device—it makes a big difference **which end** you use as the **input**!

Most passive linear circuits (e.g., using only R, L and C) are **reciprocal**. With respect to a 2-port device, **reciprocity** means:

$$Z_{12}(\omega) = Z_{21}(\omega) \qquad \text{and} \qquad Y_{12}(\omega) = Y_{21}(\omega)$$

For example, consider these two **open-circuit voltage** measurements:





Most linear circuits are reciprocal…

If this linear two-port circuit is also **reciprocal**, then when the two currents I_1 and I_2 are equal, so too will be the resulting **open-circuit** voltages V_1 and V_2!

Thus, a **reciprocal** 2-port circuit will have the property:

$$V_1 = V_2 \quad \text{when} \quad I_1 = I_2$$

Note this would likewise mean that:

$$\frac{V_2}{I_1} = \frac{V_1}{I_2}$$

And since (because of the **open-circuits**!):

$$V_2 = Z_{21} I_1 \qquad \text{and} \qquad V_1 = Z_{12} I_2$$

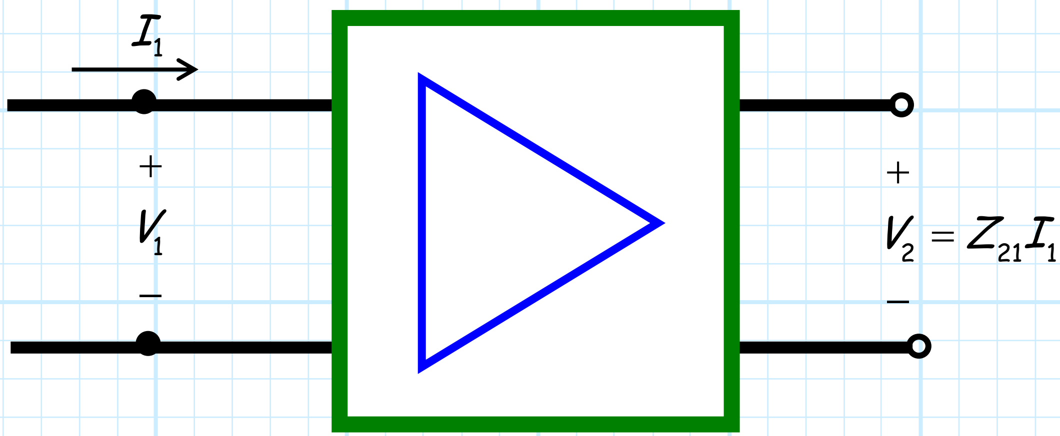
We can conclude from this "experiment" that these trans-impedance parameters of a **reciprocal** 2-port device are **equal**:

$$Z_{12}(\omega) = Z_{21}(\omega)$$

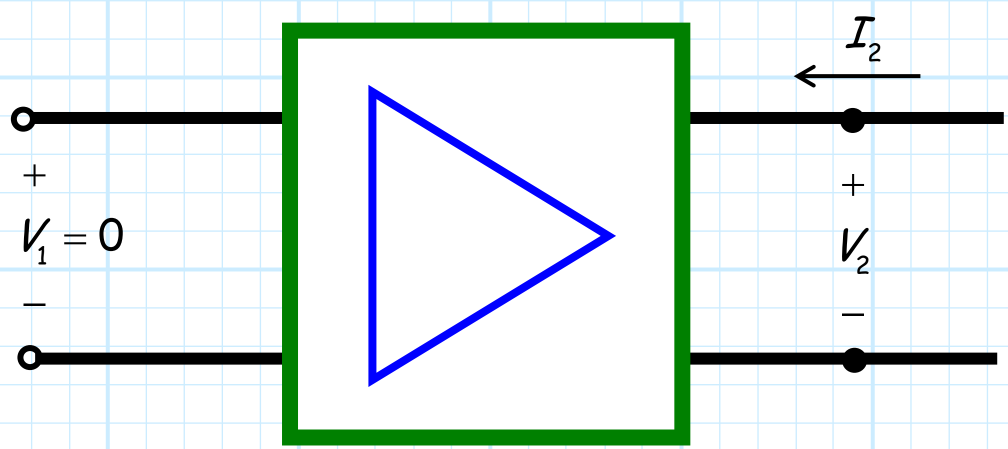
…but amplifiers are not!

Contrast this with an **amplifier**.

A **current** on the input port will indeed produce a **voltage** on an open-circuited output:



However, **amplifiers are not reciprocal**. Placing the same current at the **output** will **not** create the equal voltage on the input—in fact, it will produce **no voltage at all**!



Amps are unilateral: an input and output

Since for this **open-circuited** input port we know that:

$$Z_{12} = \frac{V_1}{I_2},$$

the fact that voltage produced at the input port is zero ($V_1 = 0$) means the trans-impedance parameter Z_{12} is likewise **zero** (or nearly so) for unilateral amplifiers:

$$Z_{12}(\omega) = 0 \qquad \text{(for amplifiers)}$$

Thus, the two equations describing an amplifier (a two-port device) **simplify nicely**.

Here's the simplification

Beginning with:

$$V_1 = Z_{11}\, I_1 + Z_{12}\, I_2$$

$$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$$

Now since $Z_{12} = 0$, we find:

$$V_1 = Z_{11}\, I_1$$

$$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$$

Q: *Gee; I'm sort of* ***unimpressed*** *by this simplification—I was hoping the result would be a little more—simple.*

A: Actually, the two equations above represent a **tremendous** simplification—it completely **decouples** the input port from the output, and it allows us to assign very real **physical interpretations** to the remaining impedance parameters!

To see all these benefits (**try** to remain calm), we will now make a few changes in the **notation**.

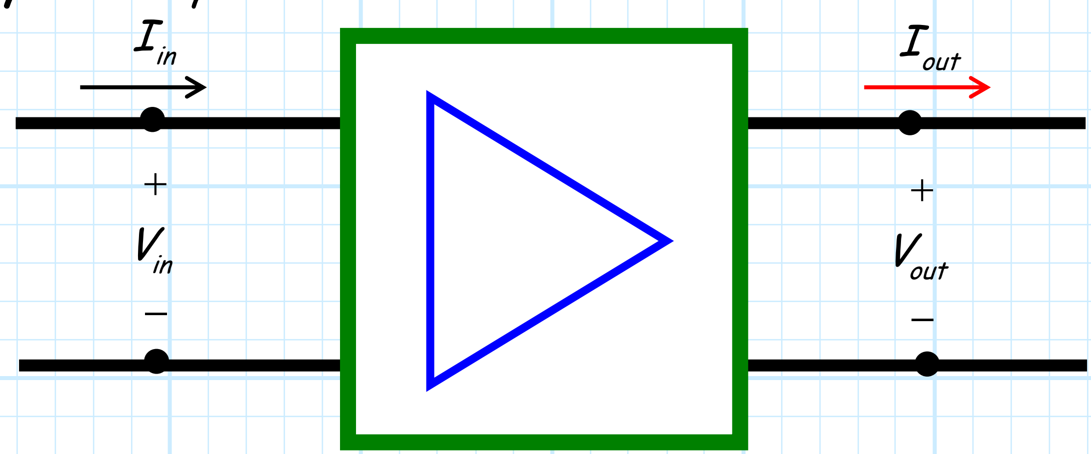
A slight change in notation

First we explicitly denote voltage V_1 as V_{in}, and likewise V_2 as V_{out} (the same with currents I).

Additionally, we change the current **definition** at the output port, **reversing** the direction of positive current as flowing **outward** from the output port. Thus:

$$I_{out} = -I_2$$

And so, a **tidy** summary:



$$V_{in} = Z_{11} I_{in}$$

$$V_{out} = Z_{21} I_{in} - Z_{22} I_{out}$$

The input is independent of the output!

From this summary, it is evident that the relationship between the **input** current and **input** voltage is determined by impedance parameter Z_{11}—and Z_{11} **only**:

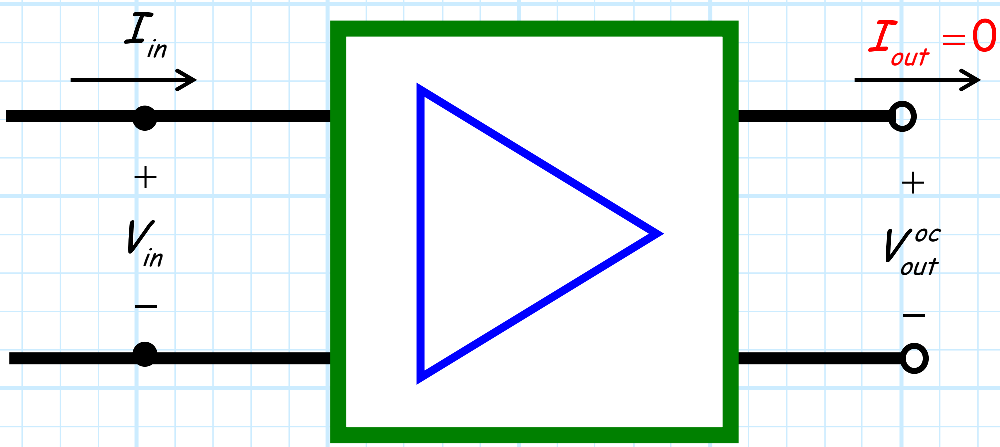
$$Z_{11} = \frac{V_{in}}{I_{in}}$$

Thus, the impedance parameter Z_{11} is known as the **input impedance** Z_{in} of an (unilateral!) amplifier:

$$Z_{in}(\omega) \doteq \frac{V_{in}(\omega)}{I_{in}(\omega)} = Z_{11}(\omega)$$

The open-circuit output voltage

Now, consider the case where the output port of the amplifier is **open-circuited** ($I_{out} = 0$):



The (**open-circuit**) output voltage is therefore simply:

$$\begin{aligned} V_{out} &= Z_{21} I_{in} - Z_{22} I_{out} \\ &= Z_{21} I_{in} - Z_{22}(0) \\ &= Z_{21} I_{in} \end{aligned}$$

The **open-circuit output voltage** is thus proportional to the **input current**.

Open-circuit trans-impedance

The proportionality constant is the impedance parameter Z_{21}—a value otherwise known as the **open-circuit trans-impedance** Z_m:

$$Z_m(\omega) \doteq \frac{V_{out}^{oc}(\omega)}{I_{in}(\omega)} = Z_{21}(\omega)$$

Thus, an (unilateral!) amplifier can be described as:

$$V_{in} = Z_{in}\, I_{in}$$

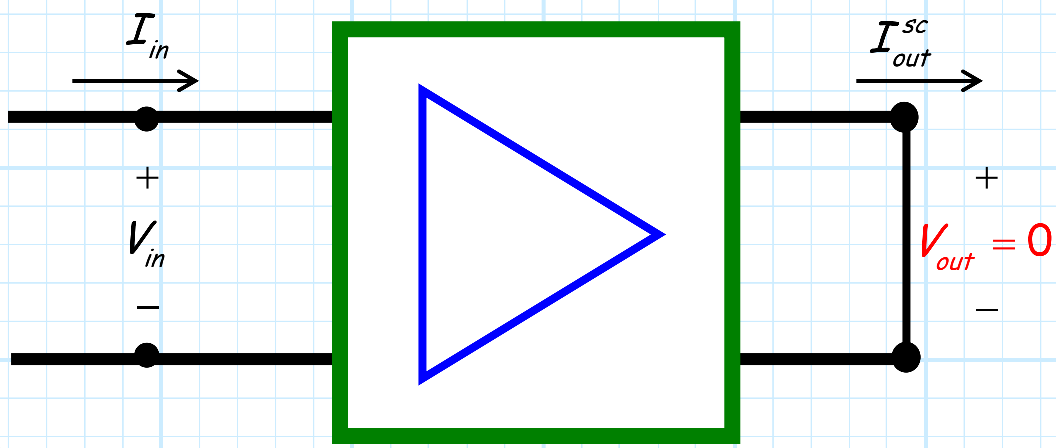
$$V_{out} = Z_m\, I_{in} - Z_{22}\, I_{out}$$

Short-circuit output current

Q: *What about impedance parameter Z_{22}; does it have any **physical** meaning?*

A: It sure does!

Consider now the result of **short-circuiting** the amplifier **output** ($\therefore V_{out} = 0$) :



Since $V_{out} = 0$:

$$V_{out} = 0 = Z_m \, I_{in} - Z_{22} \, I_{out}^{sc}$$

we can quickly determine the **short-circuit output current**:

$$I_{out}^{sc} = \frac{Z_m \, I_{in}}{Z_{22}}$$

The output impedance

Q: *I'm **not** seeing the significance of this result!?*

A: Let's rearrange to determine Z_{22}:

$$Z_{22} = \frac{Z_m \, I_{in}}{I_{out}^{sc}}$$

Note the **numerator—**it is the **open-circuit voltage** $V_{out}^{oc} = Z_m \, I_{in}$, and so:

$$Z_{22} = \frac{Z_m \, I_{in}}{I_{out}^{sc}} = \frac{V_{out}^{oc}}{I_{out}^{sc}}$$

Of course, **you** remember that the ratio of the **open-circuit voltage** to **short-circuit current** is the **output impedance** of a source:

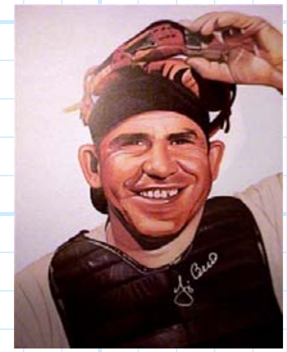
$$Z_{out} \doteq \frac{V_{out}^{oc}}{I_{out}^{sc}} = Z_{22}$$

These equations look familiar!

Thus, the **output impedance** of an (unilateral) amplifier is the impedance parameter Z_{22}, and so:

$$V_{in} = Z_{in}\, I_{in}$$

$$V_{out} = Z_m\, I_{in} - Z_{out}\, I_{out}$$



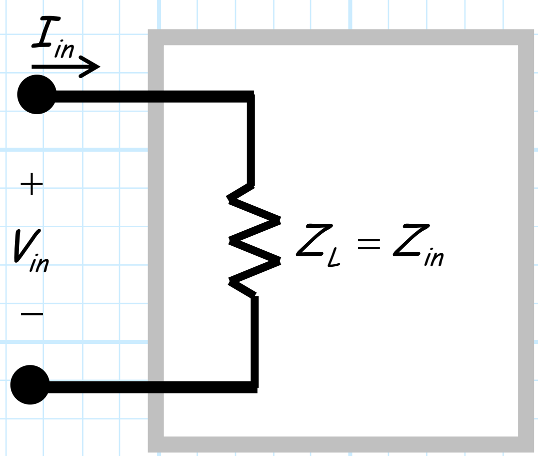
Q: *It's déjà vu all over again; haven't we seen equations like this **before**?*

A: Yes! Recall the first (i.e., input) equation:

$$V_{in} = Z_{in}\, I_{in}$$

is that of a simple **load impedance**:

$$\boxed{V_{in} = Z_{in} I_{in}}$$



Looks like a Thevenin's source

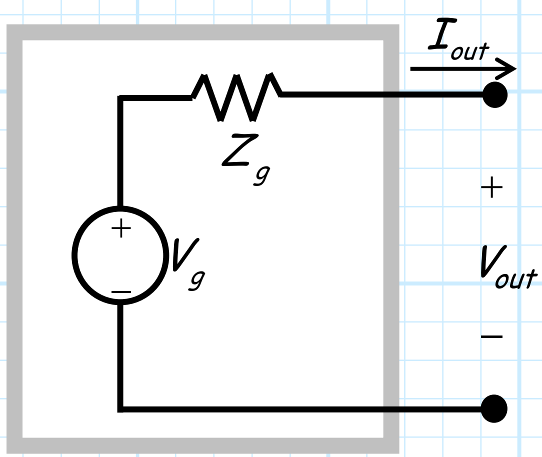
And the **second** (i.e., output) amplifier equation:

$$V_{out} = Z_m I_{in} - Z_{out} I_{out}$$

is of the form of a **Thevenin's source**:

where:

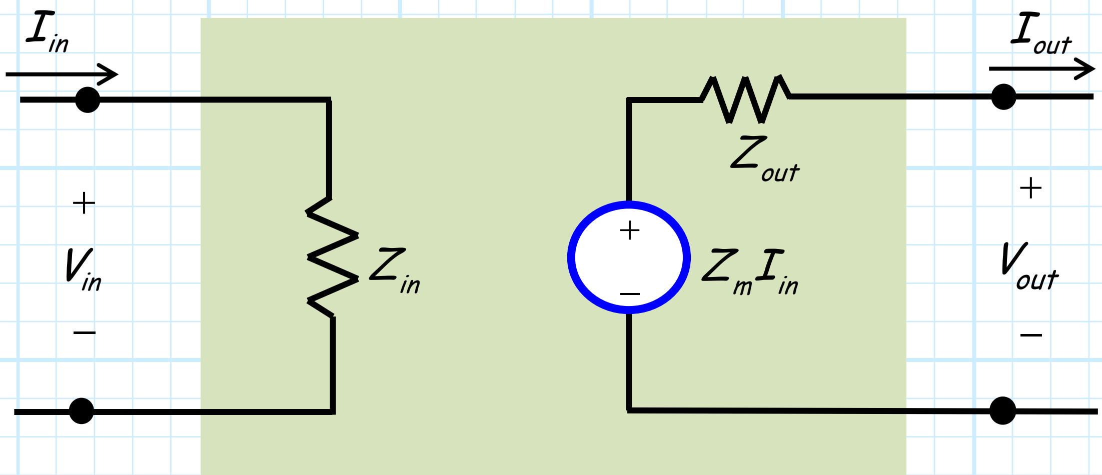
$$V_g = Z_m I_{in} \quad \text{and} \quad Z_g = Z_{out}$$



$$\boxed{V_{out} = V_g - Z_g I_{out}}$$

An equivalent circuit model

We can combine these two observations to form an **equivalent circuit model** of an (unilateral) amplifier:



$$V_{in} = Z_{in} I_{in}$$

$$V_{out} = Z_m I_{in} - Z_{out} I_{out}$$

Note in this model, the output of the amp is a **dependent** Thevenin's source—dependent on the **input current**!

Let's make the model more useful

Q: *So, do we **always** use **this** equivalent circuit to model an amplifier?*

A: Um, actually **no**.

The **truth** is that we EE's rarely use this equivalent circuit (not that there's anything wrong with it!).

Instead, the equivalent circuit we use involves a **slight modification** of the model above.

Relate the input voltage to output voltage

To see this modification, we insert the **first** (i.e., input) equation, expressed as:

$$I_{in} = \frac{V_{in}}{Z_{in}}$$

into the **second** (i.e., output) equation:

$$\begin{aligned} V_{out} &= Z_m \, I_{in} - Z_{out} \, I_{out} \\ &= \left(\frac{Z_m}{Z_{in}} \right) V_{in} - Z_{out} \, I_{out} \end{aligned}$$

Thus, the **open-circuit output voltage** can alternatively be expressed in terms of the **input voltage**!

$$V_{out}^{oc} = \left(\frac{Z_m}{Z_{in}} \right) V_{in}$$

Note the ratio Z_m / Z_{in} is unitless (a coefficient!).

Open-circuit voltage gain

This coefficient is known as the **open-circuit voltage gain** A_{vo} of an amplifier:

$$A_{vo}(\omega) \doteq \frac{V_{out}^{oc}}{V_{in}} = \frac{Z_m}{Z_{in}} = \frac{Z_{21}}{Z_{11}}$$

The **open-circuit voltage gain** $A_{vo}(\omega)$ is perhaps the most **important** of all amplifier parameters.

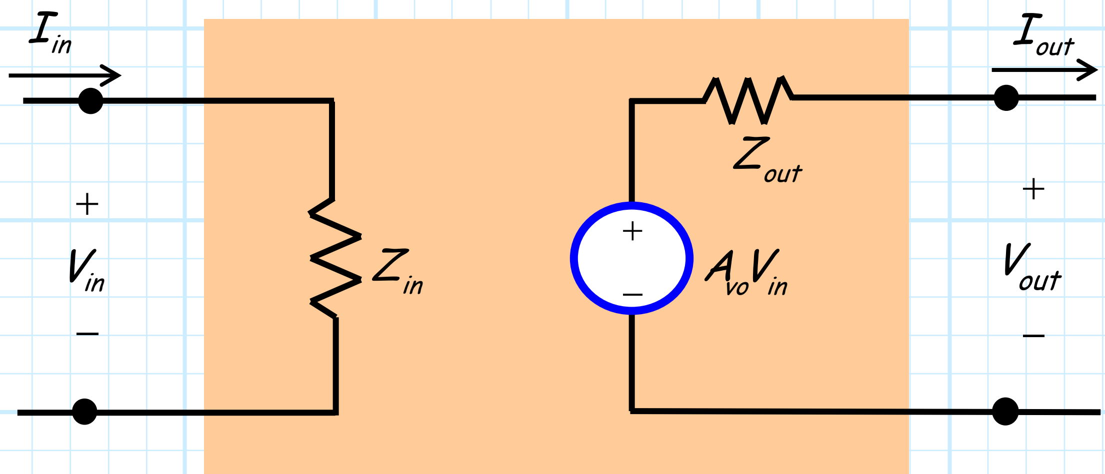
To see why, consider the amplifier equations in terms of this voltage gain:

$$V_{in} = Z_{in} I_{in}$$

$$V_{out} = A_{vo} V_{in} - Z_{out} I_{out}$$

A more "useful" equivalent circuit

The **equivalent circuit** described by these equations is:



In this circuit model, the output Thevenin's source is again dependent—but now it's dependent on the input **voltage**!

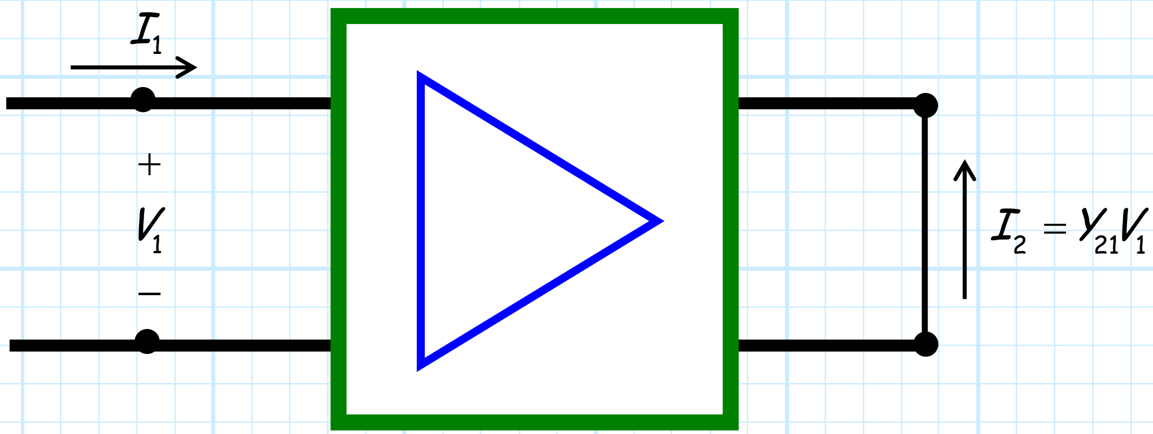
Thus, in this model, the input **voltage** and output **voltage** are more **directly related**.

Now let's consider admittance parameters

Q: *Are these the **only** two was to model a unilateral amplifier?*

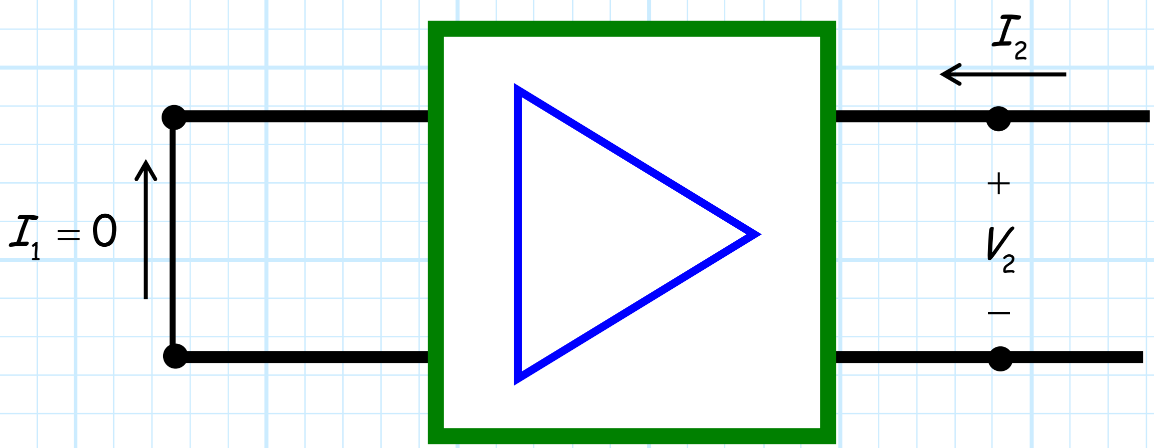
A: Hardly! Consider now **admittance parameters**.

A **voltage** on the **input** port of an amplifier will indeed produce a **short-circuit output current**:



The unilateral amplifier

However, since amplifiers are **not** reciprocal, placing the **same** voltage at the output will **not** create the equal current at the input—in fact, it will produce **no current at all**!



This again shows that amplifiers are **unilateral** devices, and so we find that the trans-admittance parameter Y_{12} is **zero**:

$$Y_{12}(\omega) = 0 \qquad \text{(for amplifiers)}$$

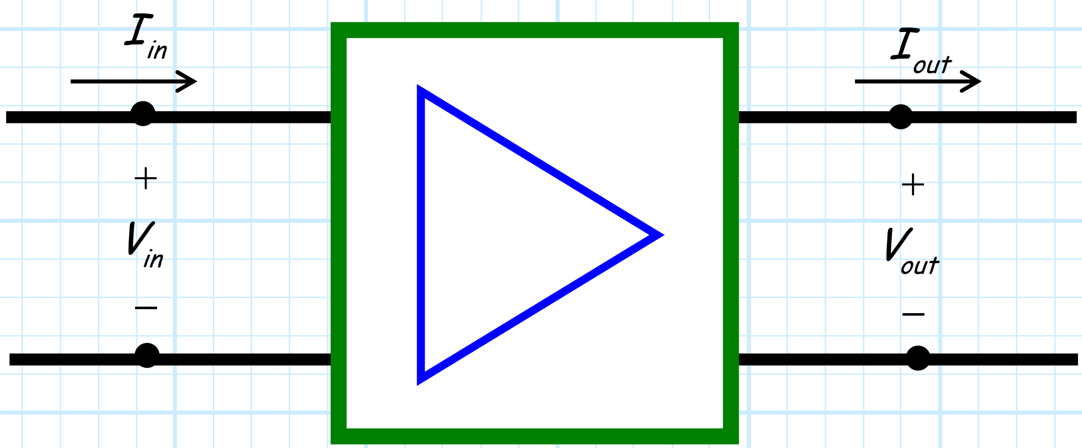
In terms of our new notation

Thus, the two equations using **admittance parameters** simplify to:

$$I_1 = Y_{11} V_1$$

$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

with the same definitions of input and output current/voltage used previously:



$$I_{in} = Y_{11} V_{in}$$

$$I_{out} = -Y_{21} V_{in} - Y_{22} V_{out}$$

Input admittance

As with impedance parameters, it is apparent from this result that the **input** port is **independent** from the **output**.

Specifically, an **input admittance** can be defined as:

$$\mathcal{Y}_{in}(\omega) \doteq \frac{\mathcal{I}_{in}(\omega)}{\mathcal{V}_{in}(\omega)} = \mathcal{Y}_{11}(\omega)$$

Note that the **input admittance** of an amplifier is simply the **inverse** of the **input impedance**:

$$\mathcal{Y}_{in}(\omega) = \frac{\mathcal{I}_{in}(\omega)}{\mathcal{V}_{in}(\omega)} = \frac{1}{\mathcal{Z}_{in}(\omega)}$$

And from this we can conclude that for a **unilateral** amplifier (but **only** because it's unilateral!):

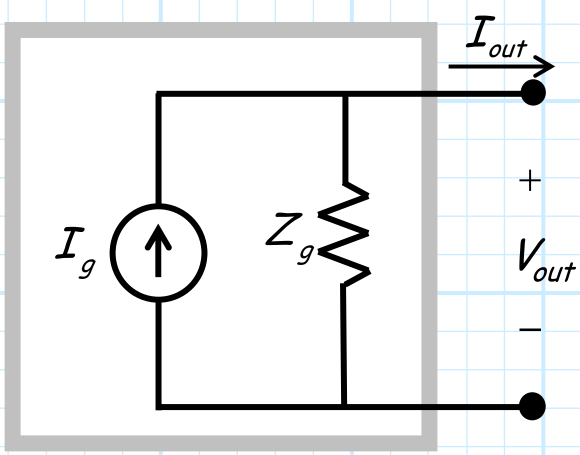
$$\mathcal{Y}_{11} = \frac{1}{\mathcal{Z}_{11}}$$

Looks like a Norton's source!

Likewise, the **second** amplifier equation:

$$I_{out} = -Y_{21} V_{in} - Y_{22} V_{out}$$

is of the form of a **Norton's** source:



$$\boxed{I_{out} = I_g - V_{out} / Z_g}$$

where:

$$I_g = -Y_{21} V_{in} \qquad \text{and} \qquad Z_g = \frac{1}{Y_{22}}$$

Short-circuit trans-admittance

More specifically, we can define a **short-circuit trans-admittance**:

$$\mathcal{Y}_m \doteq -\mathcal{Y}_{21}$$

and an **output impedance**:

$$Z_{out} \doteq \frac{1}{\mathcal{Y}_{22}}$$

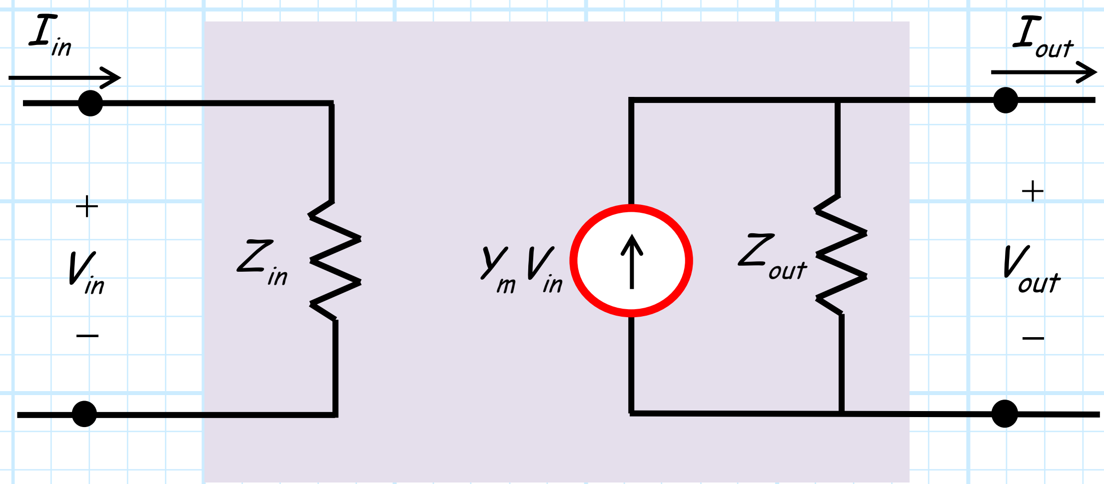
so that the amplifier equations are **now**:

$$I_{in} = V_{in} / Z_{in}$$

$$I_{out} = \mathcal{Y}_m V_{in} - V_{out} / Z_{out}$$

Yet another equivalent circuit model

The equivalent circuit described by **these** equations is:



Note in this model, the output of the amp is a **dependent** Norton's source—dependent on the **input voltage**.

However, this particular amplifier model is likewise **seldom** used.

Short-circuit current gain

Instead, we again insert the **input** equation:

$$V_{in} = \frac{I_{in}}{Y_{11}}$$

Into the **output** equation:

$$\begin{aligned} I_{out} &= -Y_{21}\, V_{in} - V_{out} / Z_{out} \\ &= -\left(\frac{Y_{21}}{Y_{11}}\right) I_{in} - V_{out} / Z_{out} \end{aligned}$$

Note the ratio $-Y_{21}/Y_{11}$ is unitless (a coefficient!).

This coefficient is known as the **short-circuit current gain** A_{is} of an amplifier:

$$A_{is}(\omega) \doteq \frac{I_{out}^{sc}}{I_{in}} = \frac{Y_m}{Y_{11}} = -\frac{Y_{21}}{Y_{11}}$$

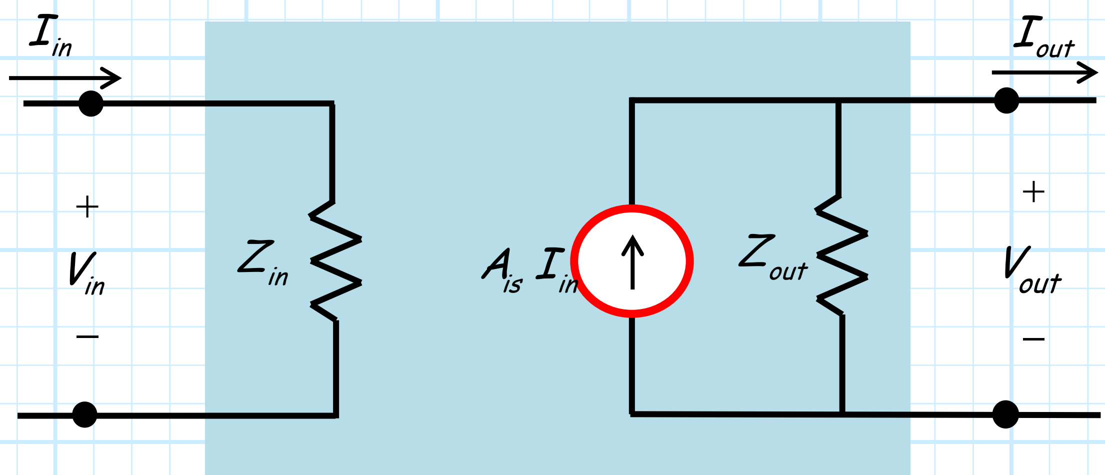
A useful equivalent circuit model

Thus, we can also express the amplifier port equations as:

$$I_{in} = V_{in} / Z_{in}$$

$$I_{out} = A_{is}\, I_{in} - V_{out} / Z_{out}$$

So, the equivalent circuit described by **these** equations is the **last** of **four** we shall consider:

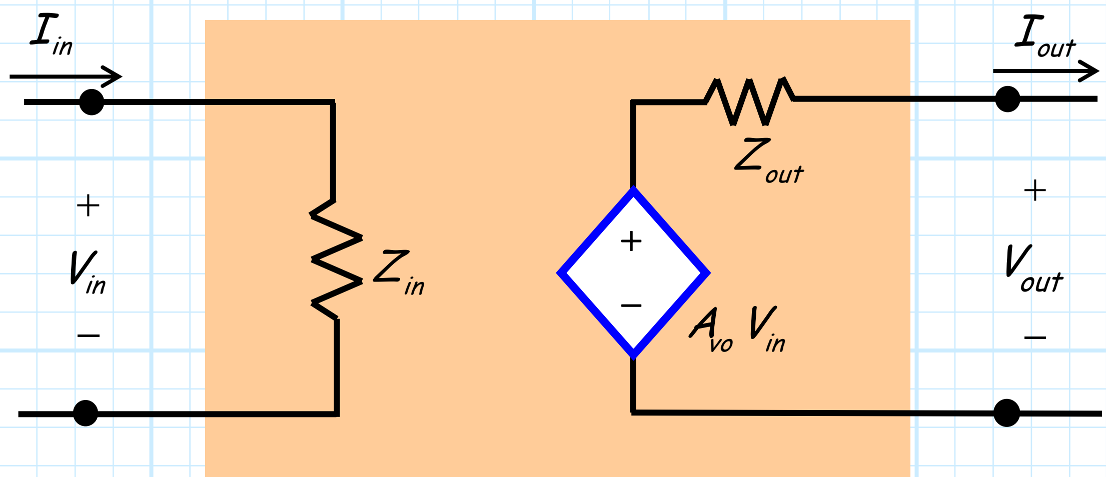


In this circuit model, the output Norton's source is again dependent—but now it's dependent on the input **current**!

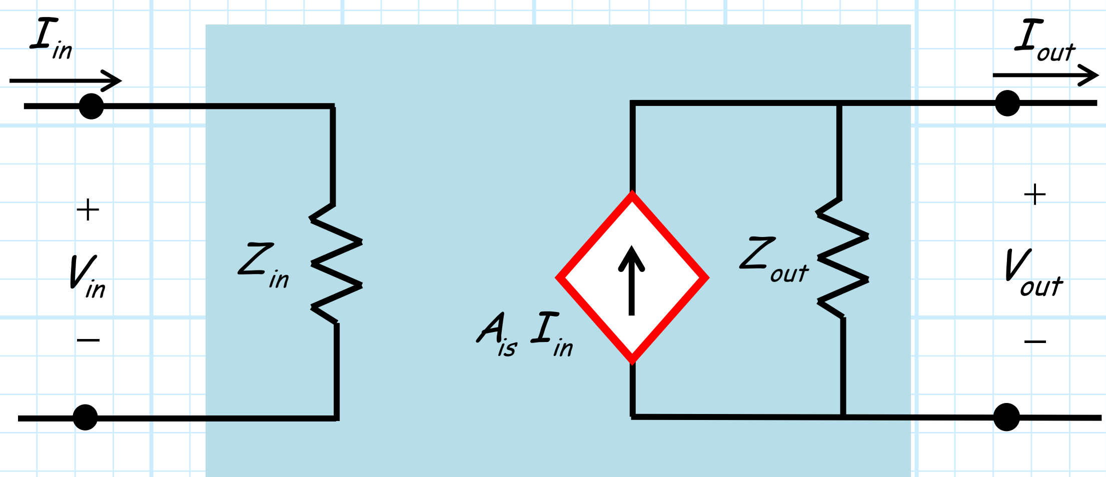
Thus, in this model, the input **current** and output **current** are more directly related.

Circuit Models for Amplifiers

The two most important amplifier circuit models explicitly use the **open-circuit voltage gain** A_{vo}:



And the **short-circuit current gain** A_{is}:



Just three values describe all!

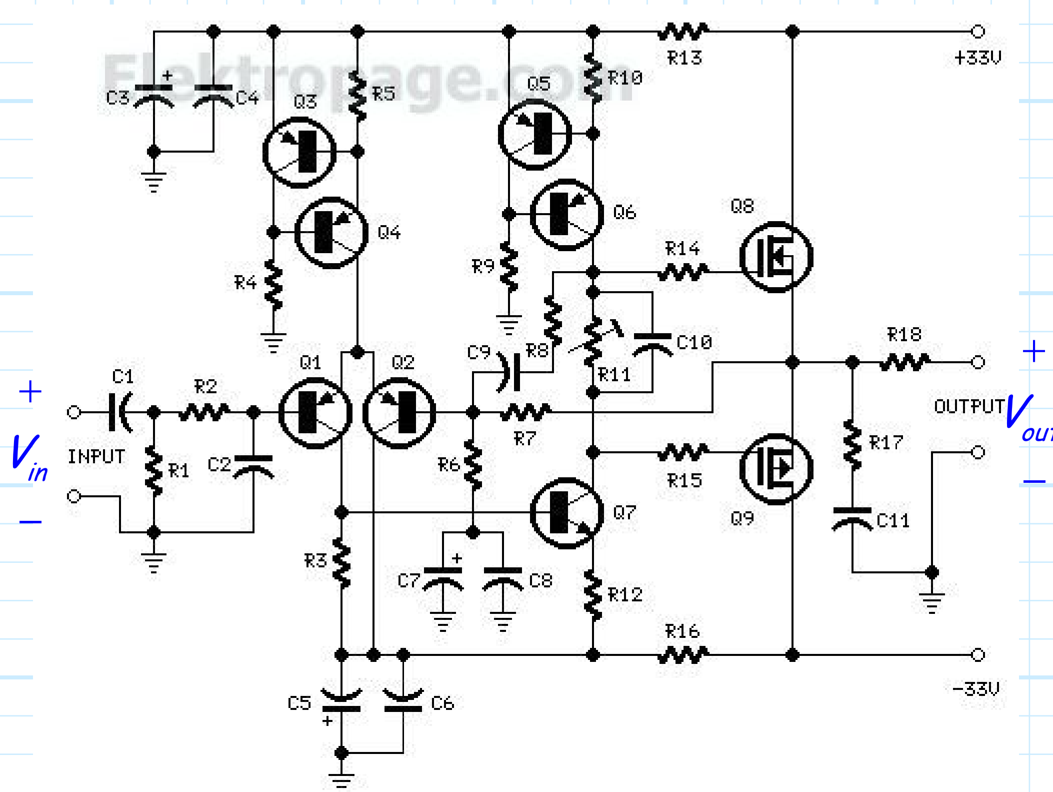
In addition, each equivalent circuit model uses the **same** two **impedance** values—the **input** impedance Z_{in} and **output** impedance Z_{out} .

Q: *So what are these models* ***good for?***

A: Say we wish to analyze a circuit in which an amplifier is but **one** component.

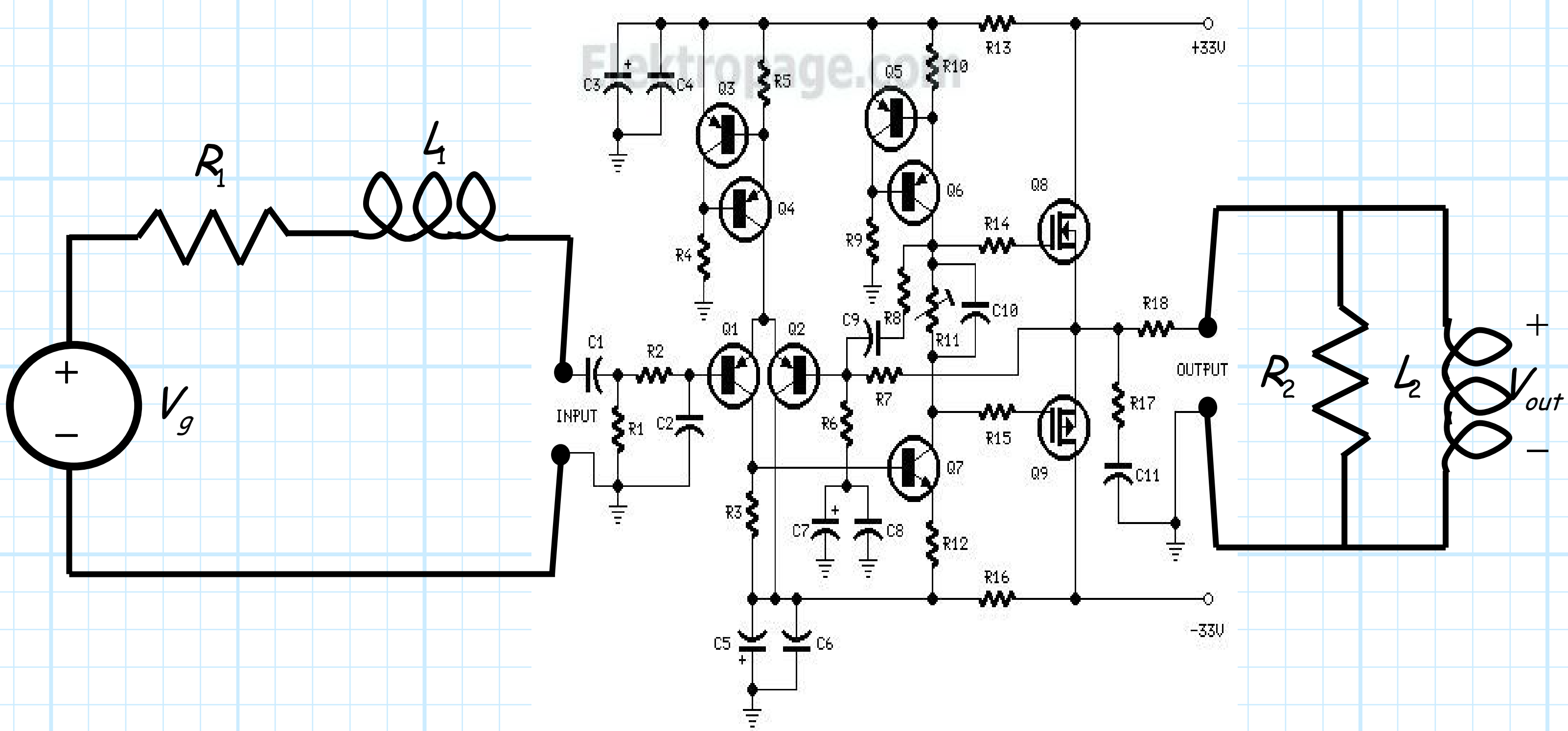
Instead of needing to analyze the **entire** amplifier circuit, we can analyze the circuit using the (far) **simpler** equivalent circuit model.

For **example**, consider **this** audio amplifier design:



This might be on the final

Say we wish to connect a **source** (e.g., microphone) to its **input**, and a **load** (e.g., speaker) to its **output**:



Let's say on the **EECS 412 final**, I ask **you** to determine V_{out} in the circuit above.

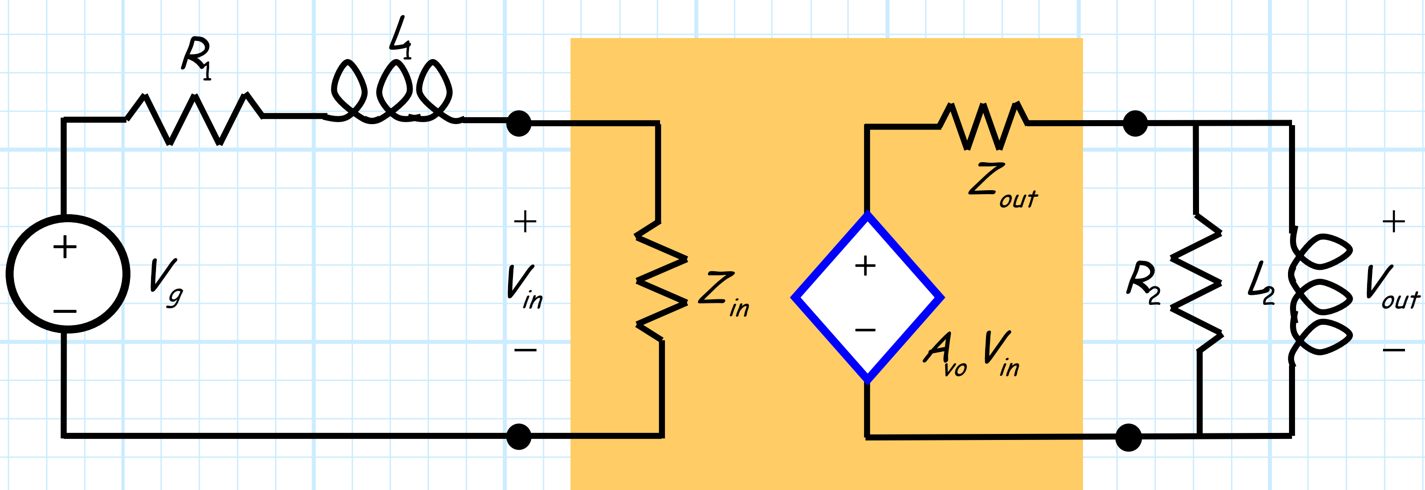
I'm not *quite* the jerk I appear to be!

Q: *Yikes! How could we **possibly** analyze this circuit on an exam—it would take way **too** much time (not to mention way **too** many pages of work)?*

A: Perhaps, but let's say that I also provide you with the amplifier **input impedance** Z_{in}, **output impedance** Z_{out}, and **open-circuit voltage gain** A_{vo}.

You thus know **everything** there is to know about the amplifier!

Just replace the amplifier with its **equivalent circuit**:



The relationship between input and output voltages

From **input** circuit, we can conclude (with a little help from voltage division):

$$V_{in} = V_g \left(\frac{Z_{in}}{R_1 + j\omega L_1 + Z_{in}} \right)$$

And the **output** circuit is likewise:

$$V_{out} = A_{vo} V_{in} \left(\frac{R_2 \| j\omega L_2}{Z_{out} + R_2 \| j\omega L_2} \right)$$

where:

$$R_2 \| j\omega L_2 = \frac{j\omega R_2 L_2}{R_2 + j\omega L_2}$$

The output is not open-circuited!

Q: *Wait! I **thought** we could determine the output voltage from the input voltage by simply multiplying by the voltage gain A_{vo}. I am **certain** that you told us:*

$$V_{out}^{oc} = A_{vo} V_{in}$$

A: I **did** tell you that! And this expression is **exactly correct**.

However, the voltage V_{out}^{oc} is the **open-circuit** output voltage of the amplifier—in **this** circuit (like most amplifier circuits!), the output is **not open**!

Hence $V_{out} \neq V_{out}^{oc}$, and so :

$$\begin{aligned} V_{out} &= A_{vo} V_{in} \left(\frac{R_2 \| j\omega L_2}{Z_{out} + R_2 \| j\omega L_2} \right) \\ &= V_{out}^{oc} \left(\frac{R_2 \| j\omega L_2}{Z_{out} + R_2 \| j\omega L_2} \right) \\ &\neq V_{out}^{oc} \end{aligned}$$

We can define a voltage gain

Now, combining the two expressions, we have our **answer**:

$$V_{out} = V_g A_{vo} \left(\frac{Z_{in}}{R_1 + j\omega L_1 + Z_{in}} \right) \left(\frac{R_2 \| j\omega L_2}{Z_{out} + R_2 \| j\omega L_2} \right)$$
$$= A_{vo} V_g \left(\frac{Z_{in}}{R_1 + j\omega L_1 + Z_{in}} \right) \left(\frac{j\omega R_2 L_2}{Z_{out} \left(R_2 + j\omega L_2 \right) + j\omega R_2 L_2} \right)$$

Now, be aware that we can (and often do!) **define** a voltage gain A_v, a value that is different from the **open-circuit** voltage gain of the **amplifier**.

For instance, in the above circuit example we could **define** a voltage gain as the ratio of the input voltage V_{in} and the output voltage V_{out}:

$$A_v \doteq \frac{V_{out}}{V_{in}} = A_{vo} \left(\frac{R_2 \| j\omega L_2}{Z_{out} + R_2 \| j\omega L_2} \right) = A_{vo} \left(\frac{j\omega R_2 L_2}{Z_{out} \left(R_2 + j\omega L_2 \right) + j\omega R_2 L_2} \right)$$

Or we can define a different gain

Or, we could alternatively **define** voltage gain as the ratio of the source voltage V_g and the output voltage V_{out}:

$$A_v \doteq \frac{V_{out}}{V_g} = A_{vo}\left(\frac{Z_{in}}{R_1 + j\omega L_1 + Z_{in}}\right)\left(\frac{j\omega R_2 L_2}{Z_{out}\left(R_2 + j\omega L_2\right) + j\omega R_2 L_2}\right)$$

Q: *Yikes! **Which** result is correct; which voltage gain is **"the"** voltage gain?*

A: Both are!

We can **define** a voltage gain A_v in **any** manner that is **useful** to us. However, we must make this definition explicit—**precisely** what two voltages are involved in the definition?

→ **No** voltage gain A_v is **"the"** voltage gain!

Note that the open-circuit voltage gain A_{vo} is a parameter of the **amplifier**—and of the amplifier **only**!

The open-circuit gain is the *amplifier* gain

Contrast A_{vo} to the two voltage gains defined above (i.e., V_{out}/V_{in} and V_{out}/V_g).

In each case, the result—of course—depends on **amplifier** parameters $(A_{vo}, Z_{in}, Z_{out})$.

However, the results **likewise** depend on the devices (source and load) **attached** to the amplifier (e.g., L_1, R_1, L_2, R_2).

→ The only **amplifier** voltage gain is its **open-circuit** voltage gain A_{vo}!

The low-frequency model

Now, let's switch gears and consider **low-frequency** (e.g., audio and video) applications.

At these frequencies, parasitic elements are typically **too small** to have any practical significance.

Additionally, low-frequency circuits **frequently** employ **no** reactive circuit elements (no capacitor or inductors).

As a result, we find that the input and output **impedances** exhibit almost **no imaginary** (i.e., reactive) components:

$$Z_{in}(\omega) \cong R_{in} + j\,0$$

$$Z_{out}(\omega) \cong R_{out} + j\,0$$

We can express this in the time domain

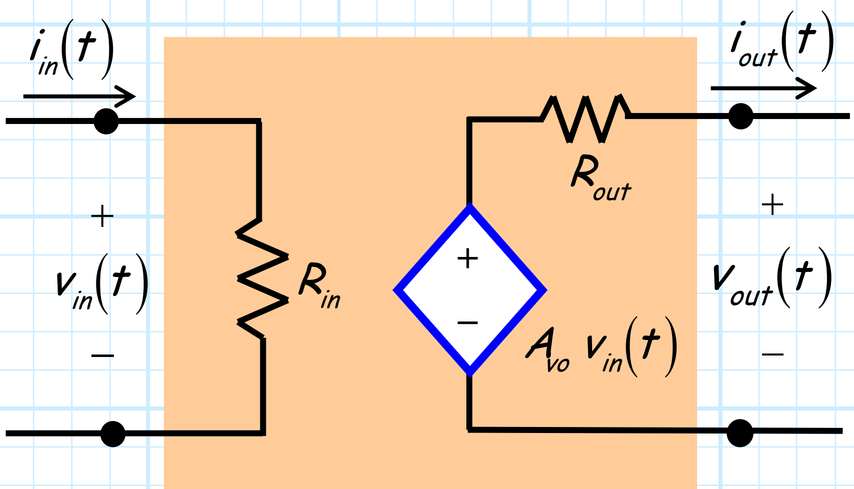
Likewise, the voltage and current **gains** of the amplifier are (almost) purely **real**:

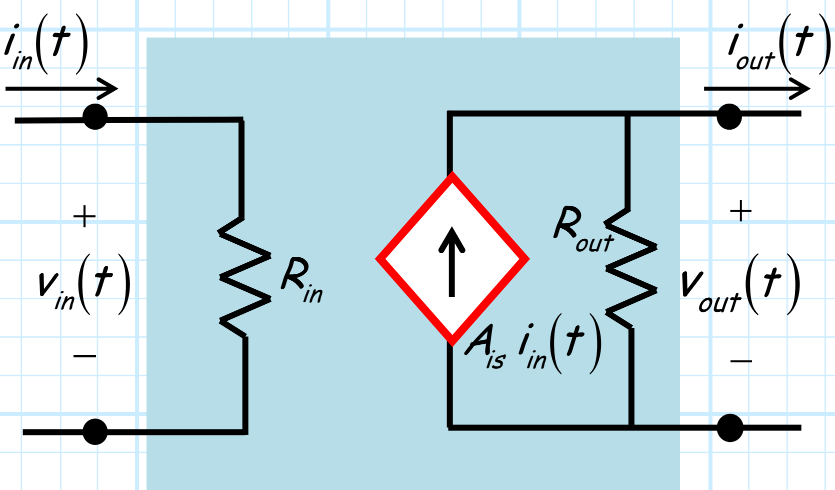
$$A_{vo}(\omega) \cong A_{vo} + j0$$

$$A_{is}(\omega) \cong A_{is} + j0$$

Note that these real values can be **positive** or **negative**.

The amplifier **circuit models** can thus be **simplified**—to the point that we can easily consider arbitrary **time-domain** signals (e.g., $v_{in}(t)$ or $i_{out}(t)$):





All real-valued

For this case, we find that the (approximate) relationships between the input and output are that of an **ideal** amplifier:

$$v_{out}^{oc}(t) = \int_{-\infty}^{t} A_{vo}\delta(t-t')\, v_{in}(t') = A_{vo}\, v_{in}(t)$$

$$i_{out}^{sc}(t) = \int_{-\infty}^{t} A_{is}\delta(t-t')\, i_{in}(t') = A_{is}\, i_{in}(t)$$

Specifically, we find that for these low-frequency **models**:

$$R_{in} = \frac{v_{in}(t)}{i_{in}(t)} \qquad R_{out} = \frac{v_{out}^{oc}(t)}{i_{out}^{sc}(t)}$$

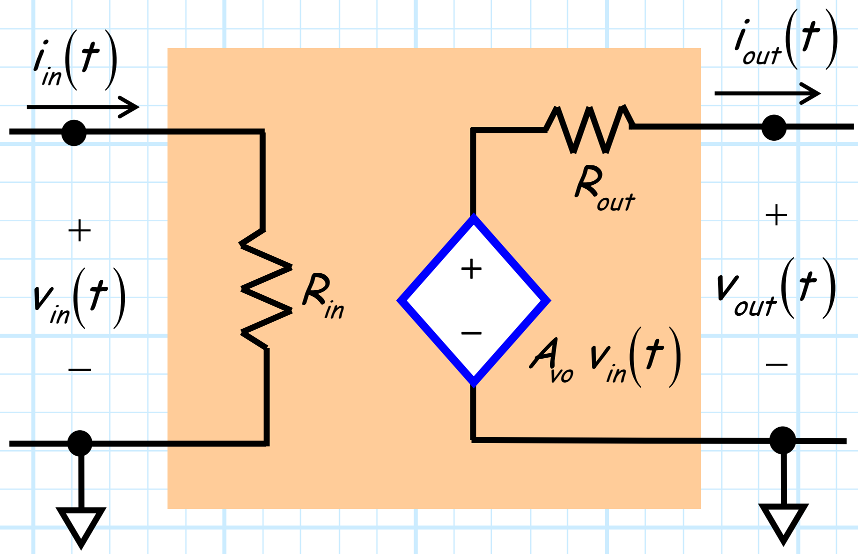
$$A_{vo} = \frac{v_{out}^{oc}(t)}{v_{in}(t)} \qquad A_{is} = \frac{i_{out}^{sc}(t)}{i_{in}(t)}$$

One important **caveat** here; this “low-frequency” model is applicable only for **input signals** that are **likewise** low-frequency—the input signal spectrum must **not** extend beyond the amplifier **bandwidth**.

Voltage is referenced to ground potential

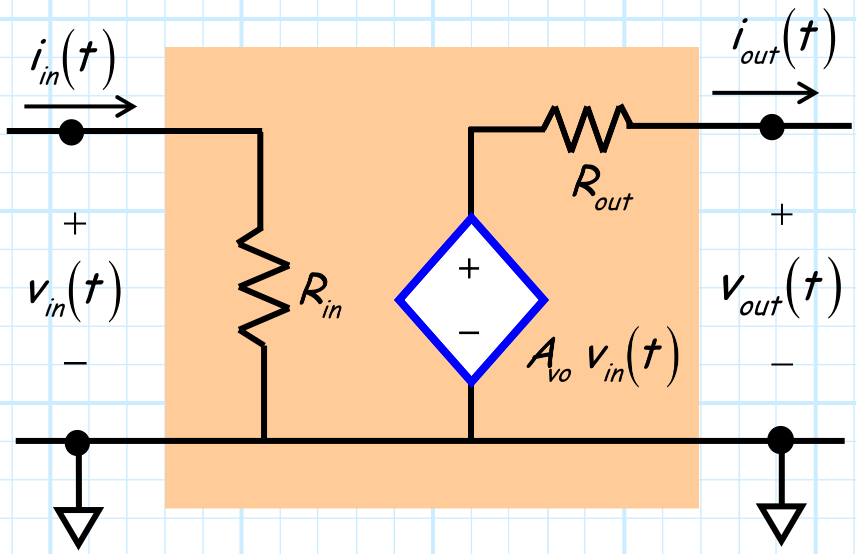
Now one **last topic**.

Frequently, both the input and output **voltages** are expressed with respect to **ground potential**, a situation expressed in the circuit **model** as:

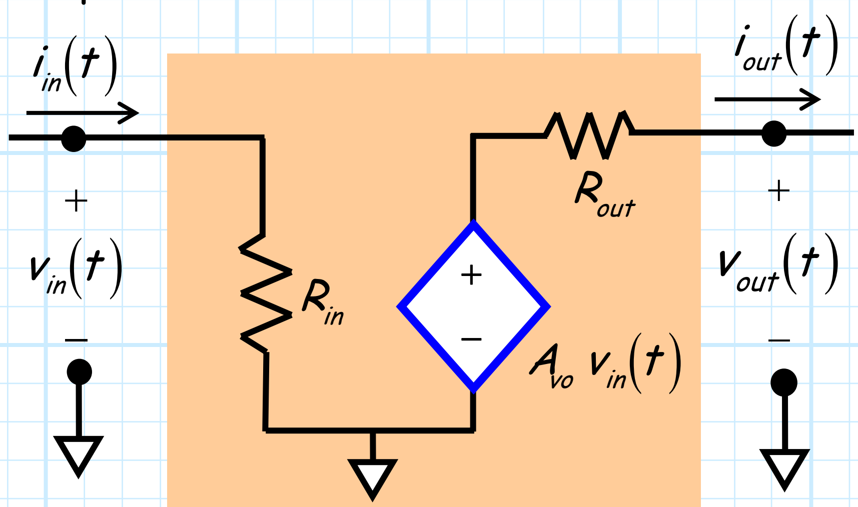


You'll often see this notation

Now, two nodes at ground potential are two nodes that are **connected** together! Thus, an **equivalent** model to the one above is:



Which is **generally** simplified to this model:



Current and Voltage Amplifiers

Q: *I'll admit to being **dog-gone** confused.*

*You say that **every** amplifier can be described **equally** well in terms of **either** its open-circuit voltage gain A_{vo}, **or** its short-circuit current gain A_{is}.*

*Yet, amps I have seen are denoted **specifically** as either a dad-gum **current** amplifier or a gul-darn **voltage** amplifier.*

***Are** voltage and current amplifiers **separate** devices, and if so, **what** are the differences between them?*

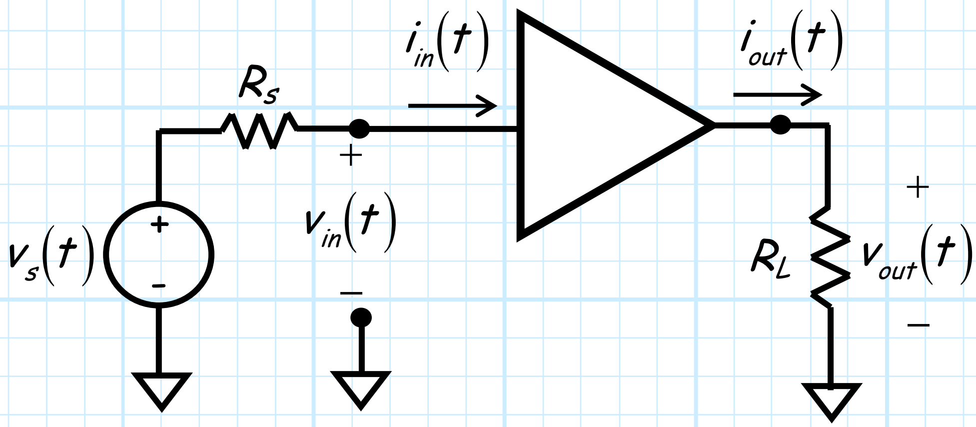
A: **Any** amplifier can be used as **either** a current amp **or** as a voltage amp.

However, we will find that an amp that works well as **one** does not generally work well as the **other**! Hence, we can in general **classify** amps as either voltage amps or current amps.

Define a gain

To see the difference we first need to provide some **definitions**.

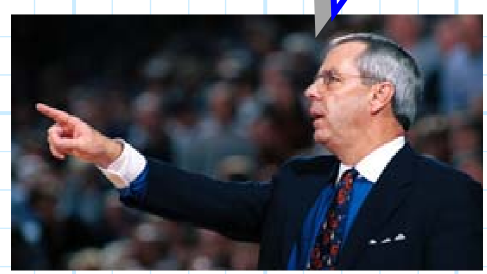
First, consider the following circuit:



We **define** a voltage gain A_v as:

$$A_v \doteq \frac{v_{out}(t)}{v_s(t)}$$

Q: *Isn't that just A_{vo} ??*



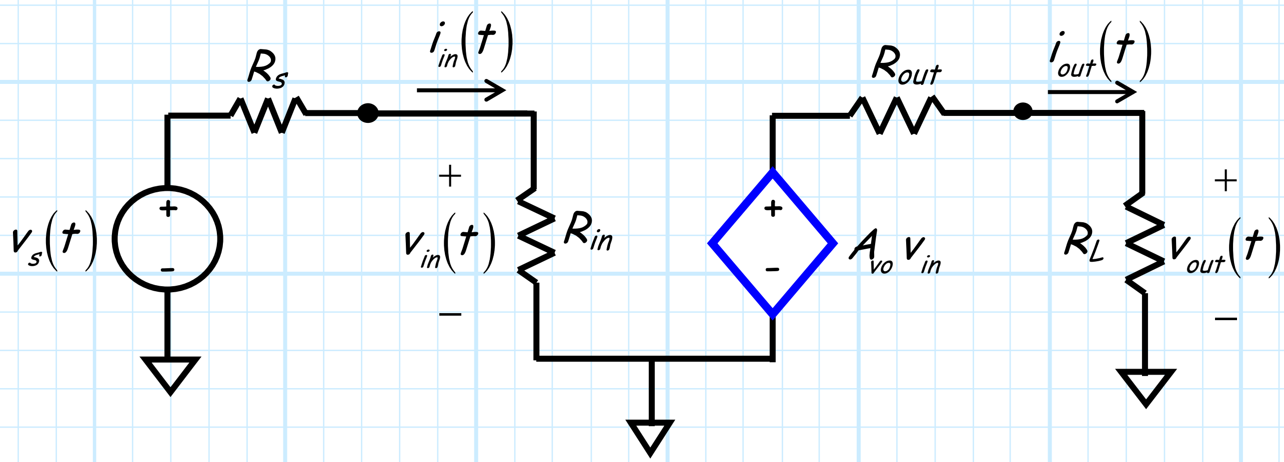
A: NO! Notice that the output of the amplifier is **not open circuited**.

This is what the model is for

Likewise, the **source** voltage v_s is **not** generally equal to the **input** voltage v_{in}.

We must use a **circuit model** to determine voltage gain A_v .

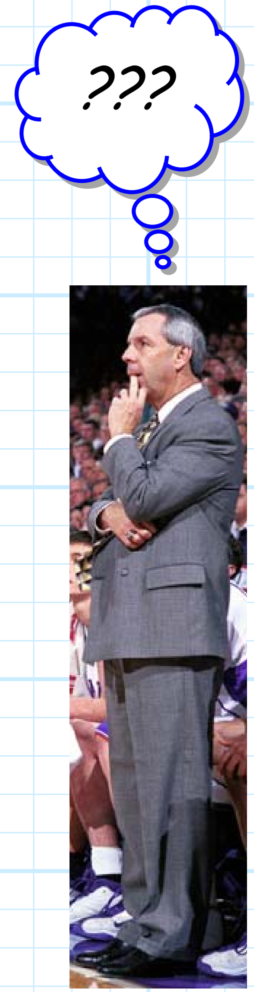
Although we can use **either** model, we will find it easier to analyze the **voltage** gain if we use the model with the dependent **voltage** source:



The result

Analyzing the **input** section of this circuit, we find:

$$v_{in} = \left(\frac{R_{in}}{R_s + R_{in}} \right) v_s$$



and analyzing the **output**:

$$v_{out} = \left(\frac{R_L}{R_{out} + R_L} \right) A_{vo} \, v_{in}$$

combining the two expressions we get:

$$v_{out} = \left(\frac{R_L}{R_{out} + R_L} \right) A_{vo} \left(\frac{R_{in}}{R_s + R_{in}} \right) v_s$$

and therefore the **voltage gain** A_v is:

$$A_v \doteq \frac{v_{out}(t)}{v_s(t)} = \left(\frac{R_L}{R_{out} + R_L} \right) A_{vo} \left(\frac{R_{in}}{R_s + R_{in}} \right)$$

How to maximize voltage gain

Note in the above expression that the first and third product terms are **limited**:

$$0 \leq \left(\frac{R_L}{R_{out}+R_L}\right) \leq 1 \qquad \text{and} \qquad 0 \leq \left(\frac{R_{in}}{R_s+R_{in}}\right) \leq 1$$

We find that each of these terms will approach their **maximum** value (i.e., one) when:

$$R_{out} \ll R_L \qquad \text{and} \qquad R_{in} \gg R_s$$

Thus, if the **input** resistance is very **large** (>>R_s) and the **output** resistance is very **small** (<<R_L), the voltage gain for this circuit will be **maximized** and have a value approximately **equal** to the **open-circuit voltage gain**!

$$v_o \approx A_{vo}\, v_s \qquad \text{iff} \quad R_{out} \ll R_L \;\; \text{and} \;\; R_{in} \gg R_s$$

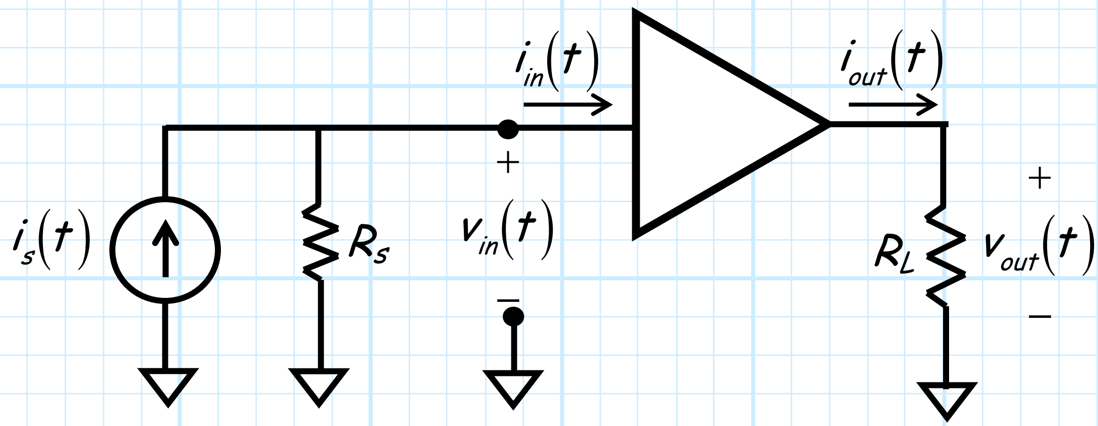
A good voltage amplifier

Thus, we can infer **three** characteristics of a good **voltage amplifier**:

1. Very **large input** resistance ($R_{in} \gg R_s$).
2. Very **small output** resistance ($R_{out} \ll R_L$).
3. Large open-circuit **voltage gain** ($A_{vo} \gg 1$).

Now for current gain

Now let's consider a **second** circuit:



We define **current gain** A_i as:

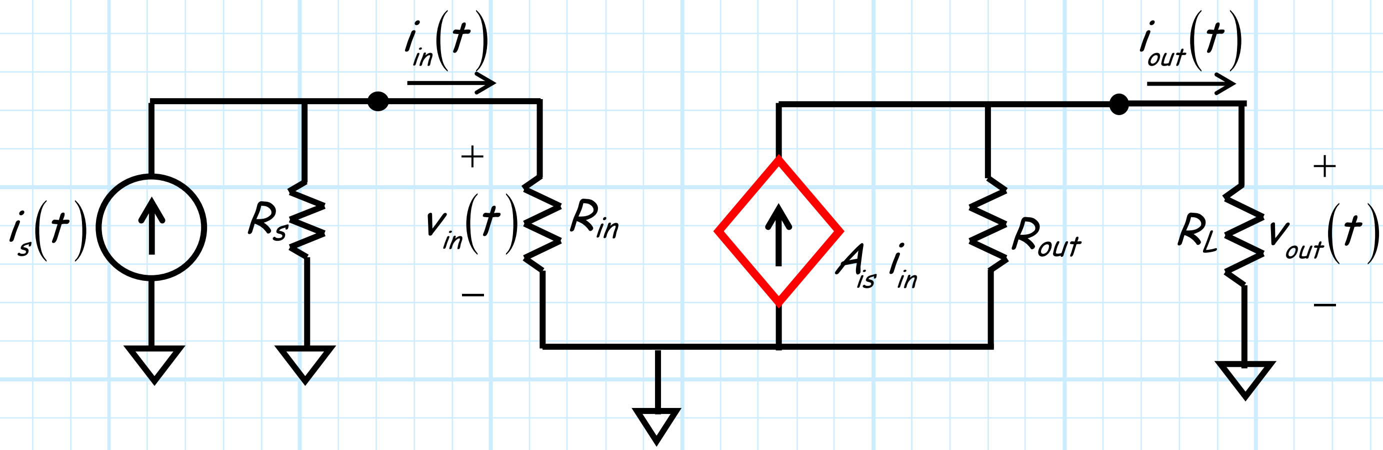
$$A_i \doteq \frac{i_{out}(t)}{i_s(t)}$$

Note that this gain is **not** equal to the **short-circuit** current gain A_{is}. This current gain A_i depends on the **source** and **load** resistances, as well as the amplifier parameters.

Therefore, we must use a **circuit model** to determine current gain A_i .

Use the other model

Although we can use **either** model, we will find it easier to analyze the **current** gain if we use the model with the dependent **current** source:



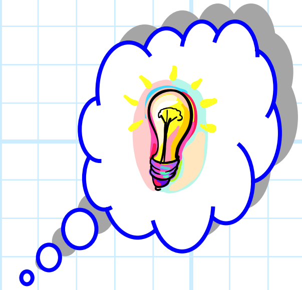
Analyzing the **input** section, we can use **current division** to determine:

$$i_{in} = \left(\frac{R_s}{R_s + R_{in}}\right) i_s$$

We likewise can use current division to analyze the **output** section:

$$i_{out} = \left(\frac{R_{out}}{R_{out} + R_L}\right) A_{is}\, i_{in}$$

How to maximize current gain



Combining these results, we find that:

$$i_{out} = \left(\frac{R_{out}}{R_{out}+R_L}\right) A_{is} \left(\frac{R_s}{R_s+R_{in}}\right) i_s$$

and therefore the **current gain** A_i is:

$$A_i \doteq \frac{i_o(t)}{i_s(t)} = \left(\frac{R_{out}}{R_{out}+R_L}\right) A_{is} \left(\frac{R_s}{R_s+R_{in}}\right)$$

Note in the above expression that the first and third product terms are **limited**:

$$0 \le \left(\frac{R_{out}}{R_{out}+R_L}\right) \le 1 \qquad \text{and} \qquad 0 \le \left(\frac{R_s}{R_s+R_{in}}\right) \le 1$$

We find that each of these terms will approach their **maximum** value (i.e., one) when:

$$R_{out} \gg R_L \qquad \text{and} \qquad R_{in} \ll R_s$$

The ideal current amp

Thus, if the **input** resistance is very **small** (<<R_s) and the **output** resistance is very **large** (>>R_L), the voltage gain for this circuit will be maximized and have a value approximately equal to the short-circuit current gain!

$$i_{out} \approx A_{is}\, i_s \quad \text{iff} \quad R_{out} \gg R_L \;\text{ and }\; R_{in} \ll R_s$$

Thus, we can infer **three** characteristics of a good **current amplifier**:

1. Very **small input** resistance ($R_i \ll R_s$).
2. Very **large output** resistance ($R_o \gg R_L$).
3. Large short-circuit **current gain** ($A_{is} \gg 1$).

Note the ideal resistances are **opposite** to those of the ideal **voltage** amplifier!

You can trust ol' Roy!

It's actually quite ***simple****.*

An amplifier with ***low*** *input resistance and* ***high*** *output resistance will typically provide great* ***current*** *gain but lousy* ***voltage*** *gain.*

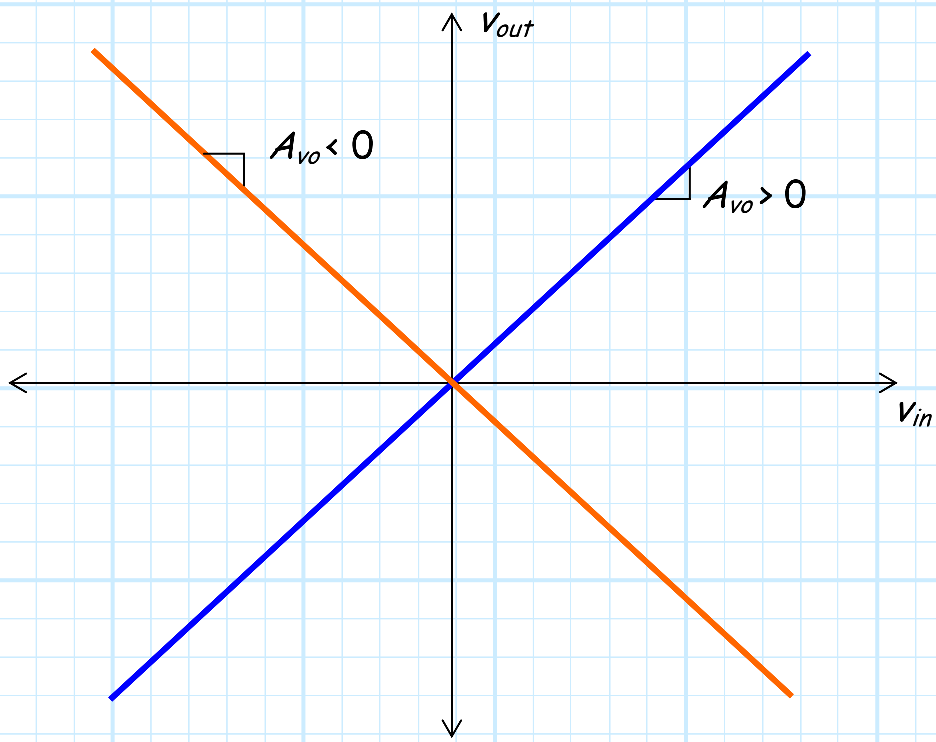
Conversely, an amplifier with ***high*** *input resistance and* ***low*** *output resistance will typically make a great* ***voltage*** *amplifier but a dog-gone poor* ***current*** *amp.*

Non-Linear Behavior of Amplifiers

Note that the **ideal** amplifier transfer function:

$$v_{out}^{oc}(t) = A_{vo}\, v_i(t)$$

is an equation of a **line** (with slope = A_{vo} and y-intercept = 0).



The output voltage is limited

This **ideal** transfer function implies that the **output voltage** can be **very** large, provided that the gain A_{vo} and the input voltage v_{in} are large.

However, we find in a "real" amplifier that there are **limits** on how large the output voltage can become.

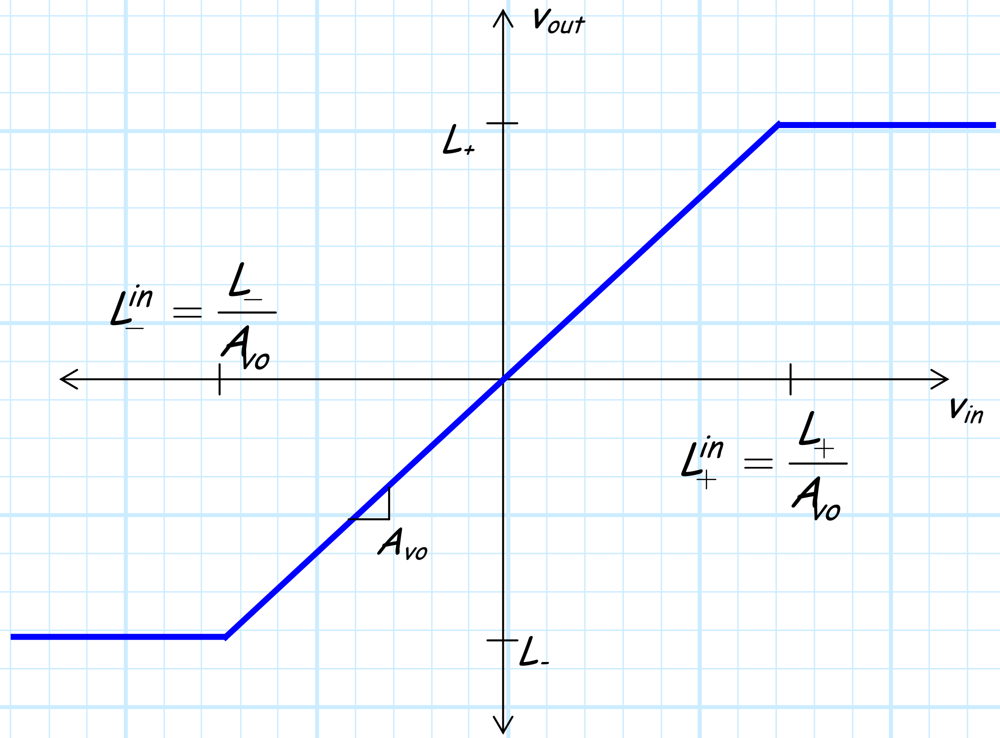
The transfer function of an amplifier is more **accurately** expressed as:

$$v_{out}(t) = \begin{cases} L_+ & v_{in}(t) > L_+^{in} \\ A_{vo}\, v_{in}(t) & L_-^{in} < v_{in}(t) < L_+^{in} \\ L_- & v_{in}(t) < L_-^{in} \end{cases}$$

Amplifier saturation

This expression is shown **graphically** as:

This expression (and graph) shows that electronic amplifiers have a **maximum** and **minimum** output voltage (L_+ and L_-).



If the **input** voltage is either too large or too small (too negative), then the amplifier **output** voltage will be equal to either L_+ or L_- .

If $v_{out} = L_+$ or $v_{out} = L_-$, we say the amplifier is in **saturation** (or compression).

Make sure the input isn't too large!

Amplifier saturation occurs when the **input** voltage is **greater** than:

$$v_{in} > \frac{L_+}{A_{vo}} \doteq L_+^{in}$$

or when the **input** voltage is **less** than:

$$v_{in} < \frac{L_-}{A_{vo}} \doteq L_-^{in}$$

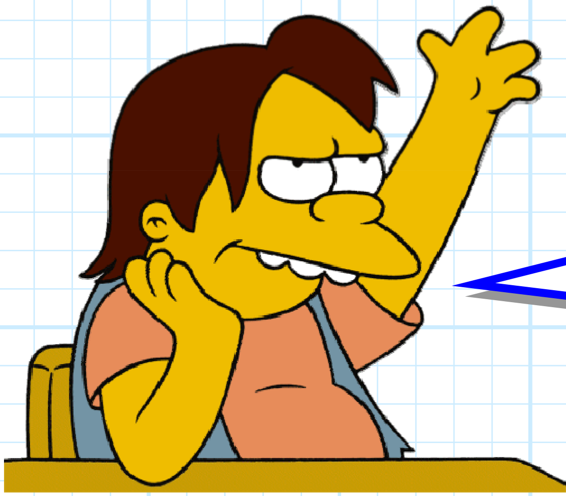
Often, we find that these voltage limits are **symmetric**, i.e.:

$$L_- = -L_+ \quad \text{and} \quad L_-^{in} = -L_+^{in}$$

For example, the output limits of an amplifier might be L_+ = 15 V and L_- = -15 V.

However, we find that these limits are also often **asymmetric** (e.g., L_+ = +15 V and L_- = +5 V).

Saturation: Who really cares?

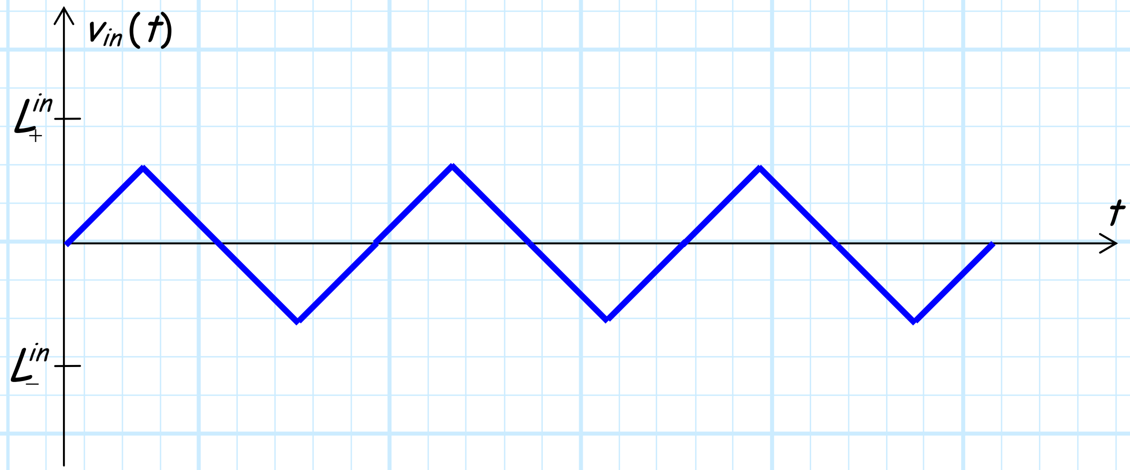


Q: *Why do we **care** if an amplifier saturates? Does it cause any **problems**, or otherwise result in performance **degradation**??*

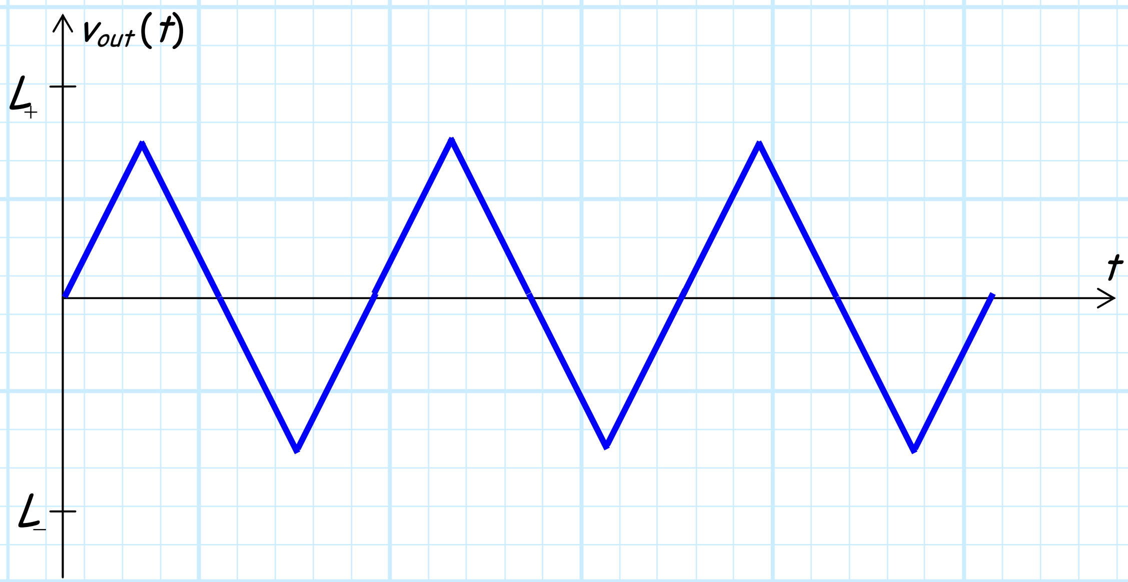
A: **Absolutely**! If an amplifier saturates—even momentarily—the unavoidable result will be a **distorted** output signal.

A distortion free example

For example, consider a case where the input to an amplifier is a **triangle** wave:

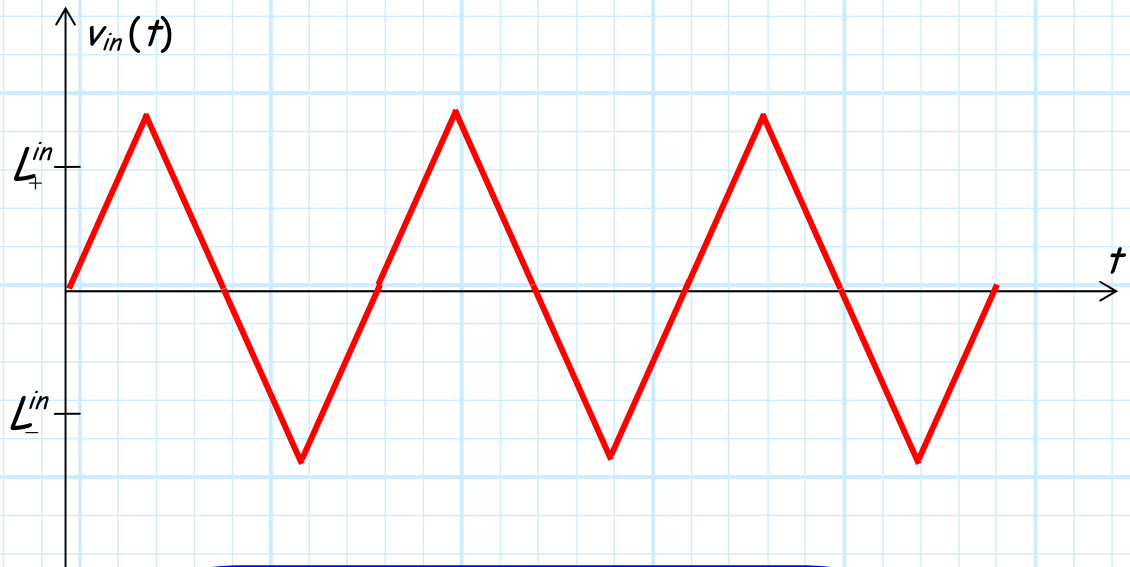


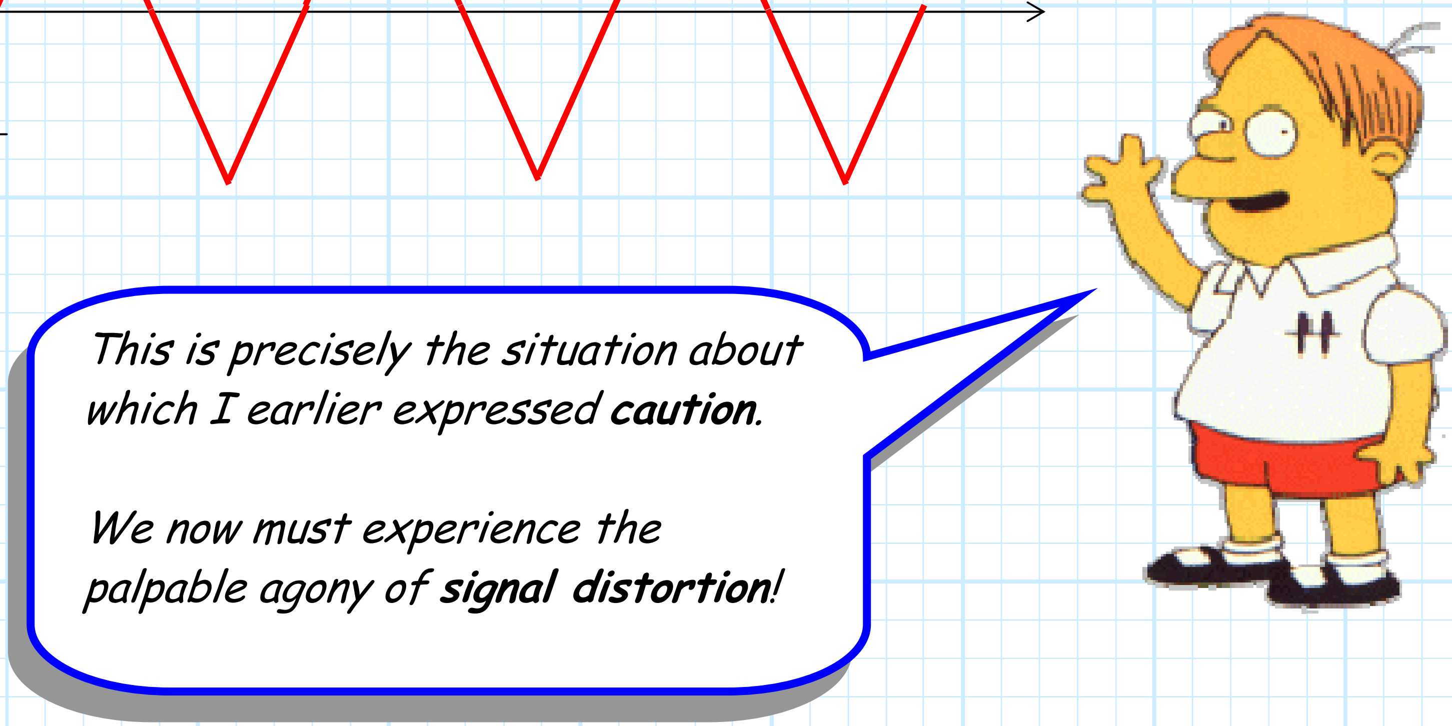
Since $L_-^{in} < v_{in}(t) < L_+^{in}$ for all time t, the **output** signal will be within the limits L_+ and L_- for all time t, and thus the amplifier output will be $v_{out}(t) = A_{vo}\, v_{in}(t)$:



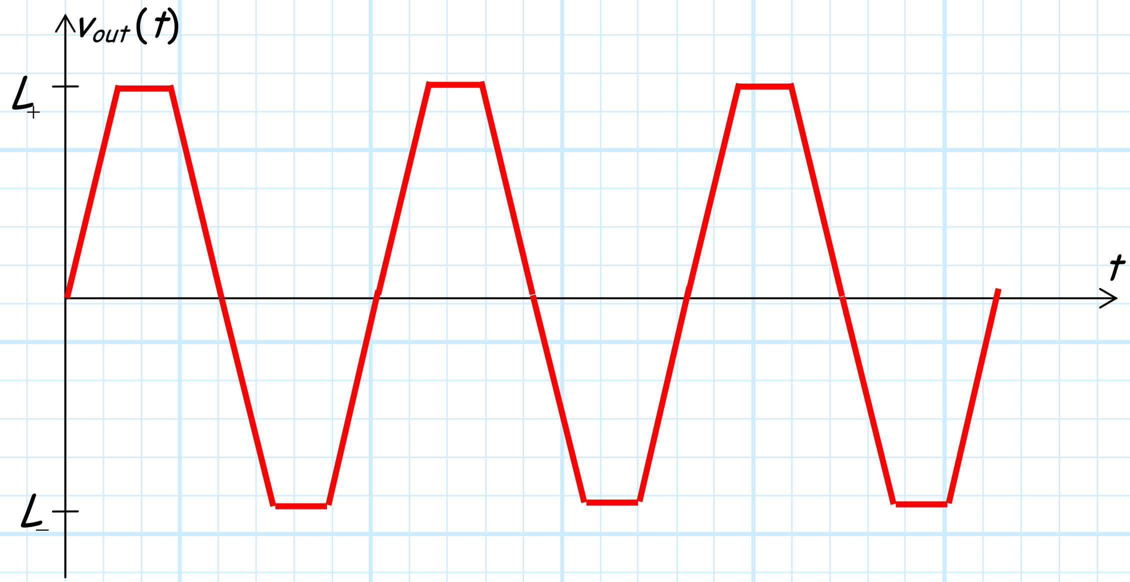
The input is too darn big!

Consider now the case where the input signal is much **larger**, such that $v_{in}(t) > L_+^{in}$ and $v_{in}(t) < L_-^{in}$ for some time t (e.g., the input triangle wave **exceeds** the voltage limits L_+^{in} and L_-^{in} some of the time):





Palpable agony



Note that this output signal is **not** a triangle wave!

For time t where $v_{in}(t) > L_+^{in}$ and $v_{in}(t) < L_-^{in}$, the value $A_{vo}\, v_{in}(t)$ is greater than L_+ and less than L_-, respectively.

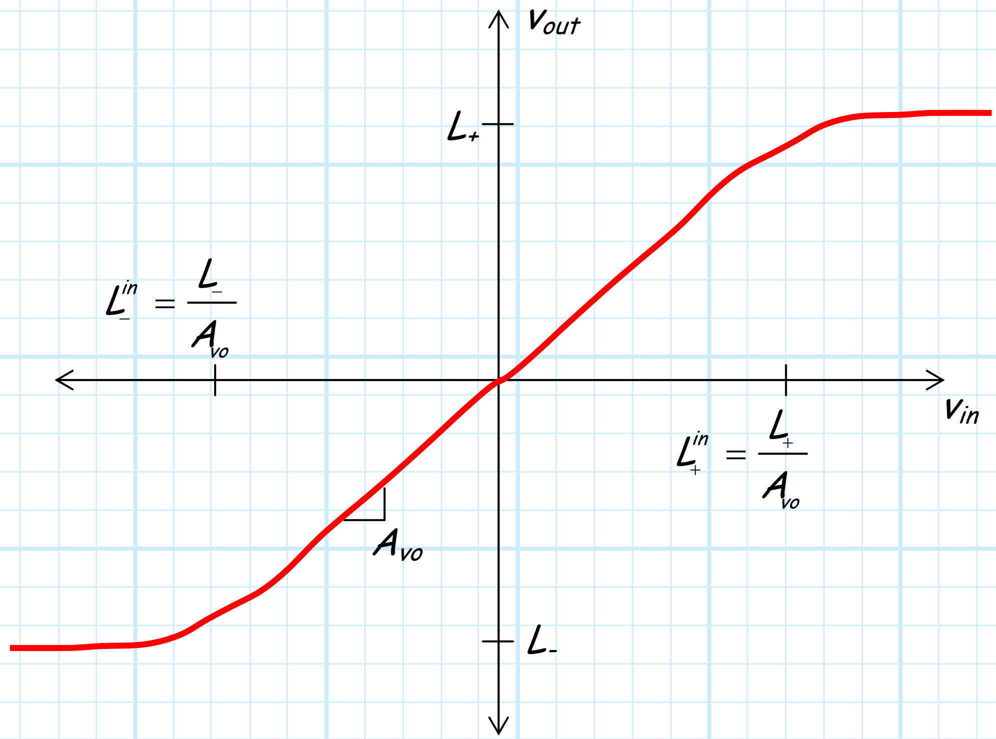
Thus, the output voltage is limited to $v_{out}(t) = L_+$ and $v_{out}(t) = L_-$ for these times.

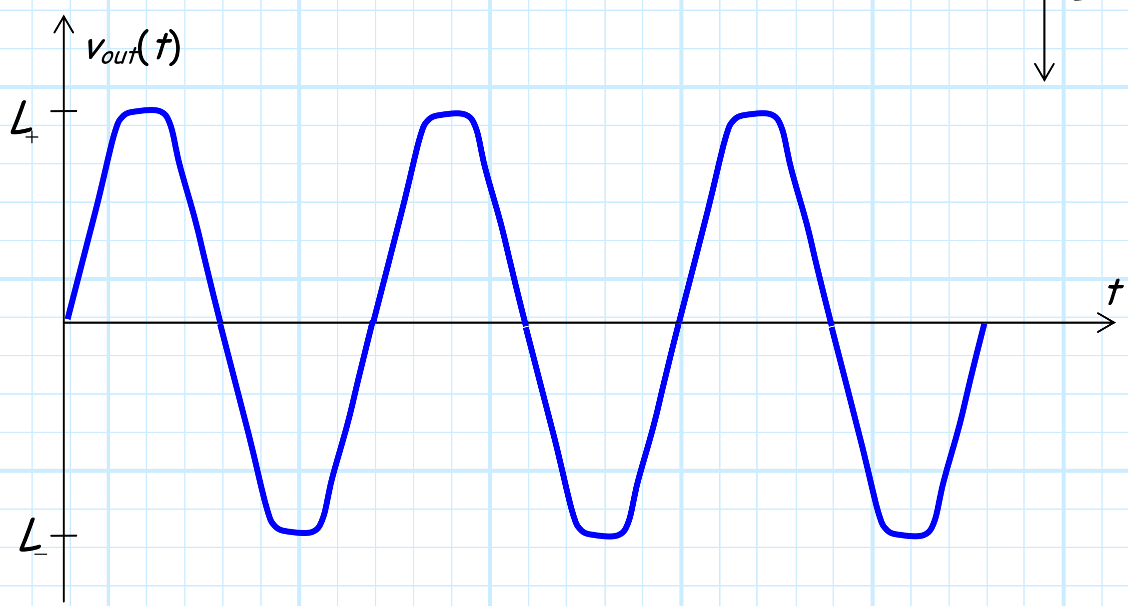
As a result, we find that output $v_{out}(t)$ does **not** equal $A_{vo}\, v_{in}(t)$ —the output signal is **distorted**!

"Soft" Saturation

In reality, the **saturation** voltages L_+, L_-, L_+^{in}, and L_-^{in} are not so **precisely** defined.

The transition from the linear amplifier region to the saturation region is **gradual**, and cannot be unambiguously defined at a precise point.

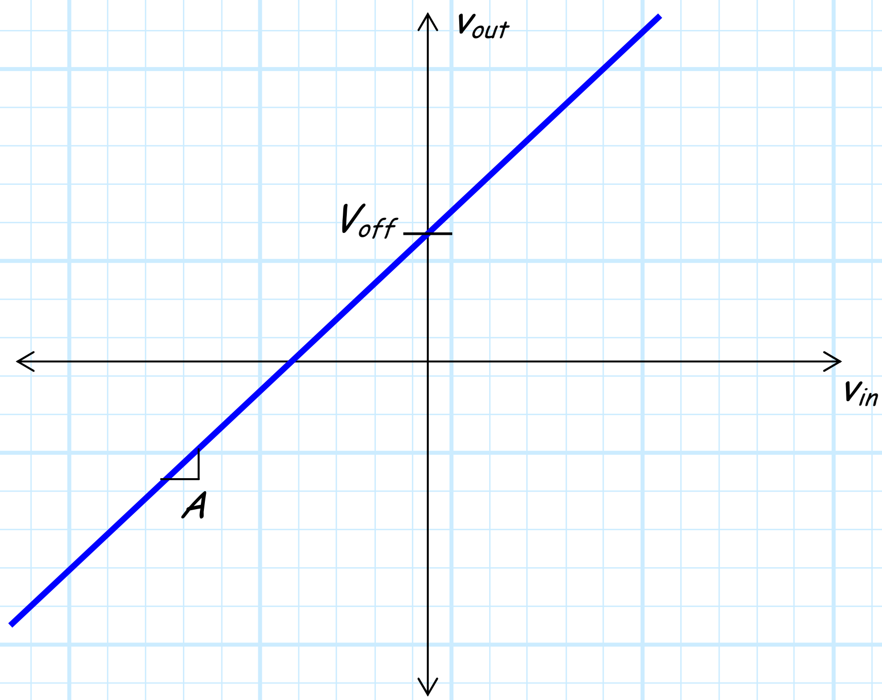




Yet another problem: DC offset

Now for another **non-linear** problem!

We will find that many amplifiers exhibit a **DC offset** (i.e., a DC **bias**) at their output.



How do we define gain?

The output of these amplifiers can be expressed as:

$$v_{out}(t) = A\, v_{in}(t) + V_{off}$$

where A and V_{off} are constants.

It is evident that if the input is **zero**, the output voltage will **not** be (zero, that is)!

$$\text{i.e.,} \quad v_{out} = V_{off} \quad \text{if} \quad v_{in} = 0$$

Q: ***Yikes!*** *How do we determine the* ***gain*** *of such an amplifier?*

If: $v_{out}(t) = A\, v_{in}(t) + V_{off}$

then what is:

$$\frac{v_{out}(t)}{v_{in}(t)} = ?????$$

The ***ratio*** *of the output voltage to input voltage is* ***not a constant****!*

Calculus: is there anything it can't do?

A: The gain of **any** amplifier can be defined more precisely using the **derivative** operator:

$$A_{vo} \doteq \frac{d\,v_{out}}{d\,v_{in}}$$

Thus, for an amplifier with an output DC offset, we find the voltage gain to be:

$$A_{vo} = \frac{d\,v_{out}}{d\,v_{in}} = \frac{d\left(A\,v_{in} + V_{off}\right)}{d\,v_{in}} = A$$

In other words, the gain of an amplifier is determined by the **slope** of the transfer function!

This sort of makes sense!

For an amplifier with **no** DC offset (i.e., $v_o = A_{vo} v_i$), it is easy to see that the gain is **likewise** determined from this definition:

$$A_{vo} = \frac{d\, v_{out}}{d\, v_{in}} = \frac{d\, A_{vo}\, v_{in}}{d\, v_{in}} = A_{vo}$$

Hey, hey! This definition makes sense if you think about it—gain is the ***change*** *of the output voltage with respect to a* ***change*** *at the input.*

For example, of small change Δv_{in} at the ***input*** *will result in a change of $A_{vo}\, \Delta v_{in}$ at the* ***output****.*

If A_{vo} is ***large****, this change at the output will be* ***large****!*

Both problems collide

OK, here's **another** problem.

The derivative of the transfer curve for **real** amplifiers will **not be a constant**.

We find that the gain of a amplifier will often be **dependent** on the input voltage!

The main reason for this is amplifier **saturation**.

Consider again the transfer function of an amplifier that **saturates**:

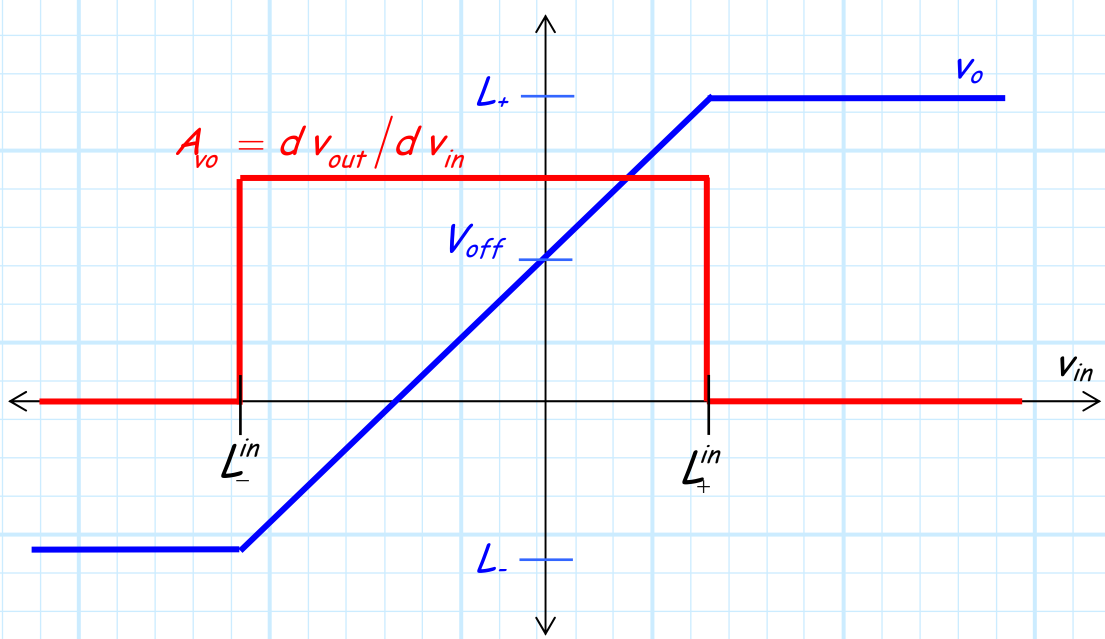
$$v_{out} = \begin{cases} L_+ & v_{in} > L_+^{in} \\ A v_{in} + V_{off} & L_-^{in} < v_i < L_+^{in} \\ L_- & v_{in} < L_-^{in} \end{cases}$$

Gain is a function of v_{in}

We find the **gain** of this amplifier by taking the **derivative** with respect to v_{in}:

$$A_{vo} = \frac{d\,v_{out}}{d\,v_{in}} = \begin{cases} 0 & v_{in} > L_+^{in} \\ A & L_-^{in} < v_{in} < L_+^{in} \\ 0 & v_{in} < L_-^{in} \end{cases}$$

Graphically, this result is:

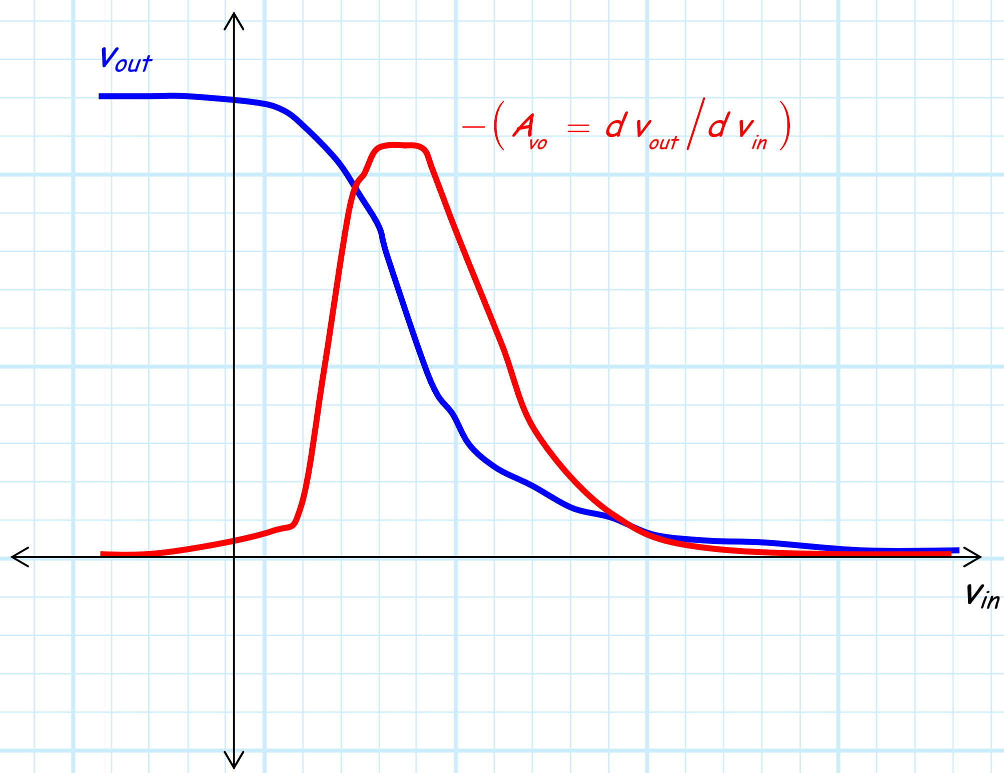


You'll see this transfer function again!

Thus, the gain of this amplifier when in saturation is **zero**. A change in the input voltage will result in **no change** on the output—the output voltage will simply be $v_o = L_{\pm}$.

Again, the transition into saturation is **gradual** for real amplifiers.

In fact, we will find that many of the amplifiers studied in this class have a **transfer function** that looks something like this→



We will find that the voltage gain of many amplifiers is **dependent** on the input voltage.

Thus, a **DC bias** at the input of the amplifier is often required to **maximize** the amplifier gain.