Filters

A RF/microwave filter is (typically) a passive, reciprocal, 2-port linear device.

If port 2 of this device is terminated in a matched load, then we can relate the incident and output power as:

$$P_{\text{out}} = |S_{21}|^2 P_{\text{inc}}$$

We define this power transmission through a filter in terms of the power transmission coefficient $T$:

$$T = \frac{P_{\text{out}}}{P_{\text{inc}}} = |S_{21}|^2$$

Since microwave filters are typically passive, we find that:

$$0 \leq T \leq 1$$

in other words, $P_{\text{out}} \leq P_{\text{inc}}$. 
Q: What happens to the “missing” power $P_{inc} - P_{out}$?

A: Two possibilities: the power is either absorbed ($P_{abs}$) by the filter (converted to heat), or is reflected ($P_r$) at the input port.

I.E.:

Thus, by conservation of energy:

$$P_{inc} = P_r + P_{abs} + P_{out}$$

Now ideally, a microwave filter is lossless, therefore $P_{abs} = 0$ and:

$$P_{inc} = P_r + P_{out}$$

which alternatively can be written as:

$$\frac{P_{inc}}{P_{inc}} = \frac{P_r + P_{out}}{P_{inc}}$$

$$1 = \frac{P_r}{P_{inc}} + \frac{P_{out}}{P_{inc}}$$
Recall that $\frac{P_{\text{out}}}{P_{\text{inc}}} = T$, and we can likewise define $\frac{P_r}{P_{\text{inc}}}$ as the **power reflection coefficient**:

$$\Gamma \doteq \frac{P_r}{P_{\text{inc}}} = |S_{11}|^2$$

We again emphasize that the filter output port is terminated in a **matched** load.

Thus, we can conclude that for a **lossless** filter:

$$1 = \Gamma + T$$

Which is simply **another** way of saying for a lossless device that $1 = |S_{11}|^2 + |S_{21}|^2$.

Now, **here’s** the important part!

For a microwave **filter**, the coefficients $\Gamma$ and $T$ are **functions of frequency**! I.E.,:

$$\Gamma(\omega) \quad \text{and} \quad T(\omega)$$

The **behavior** of a microwave filter is described by these functions!
We find that for most signal frequencies $\omega_s$, these functions will have a value equal to one of two different approximate values.

Either:

$$\Gamma(\omega = \omega_s) \approx 0 \quad \text{and} \quad T(\omega = \omega_s) \approx 1$$

or

$$\Gamma(\omega = \omega_s) \approx 1 \quad \text{and} \quad T(\omega = \omega_s) \approx 0$$

In the first case, the signal frequency $\omega_s$ is said to lie in the pass-band of the filter. Almost all of the incident signal power will pass through the filter.

In the second case, the signal frequency $\omega_s$ is said to lie in the stop-band of the filter. Almost all of the incident signal power will be reflected at the input—almost no power will appear at the filter output.
Consider then these four types of functions of $\Gamma(\omega)$ and $T(\omega)$:

1. Low-Pass Filter

![Diagram of Low-Pass Filter]

Note for this filter:

$$T(\omega) = \begin{cases} 
1 & \omega < \omega_c \\
0 & \omega > \omega_c 
\end{cases}$$

$$\Gamma(\omega) = \begin{cases} 
0 & \omega < \omega_c \\
1 & \omega > \omega_c 
\end{cases}$$

This filter is a low-pass type, as it "passes" signals with frequencies less than $\omega_c$, while "rejecting" signals at frequencies greater than $\omega_c$.

Q: This frequency $\omega_c$ seems to be very important! What is it?
A: Frequency $\omega_c$ is a filter parameter known as the cutoff frequency; a value that approximately defines the frequency region where the filter pass-band transitions into the filter stop band.

According, this frequency is defined as the frequency where the power transmission coefficient is equal to $\frac{1}{2}$:

$$T(\omega = \omega_c) = 0.5$$

Note for a lossless filter, the cutoff frequency is likewise the value where the power reflection coefficient is $\frac{1}{2}$:

$$\Gamma(\omega = \omega_c) = 0.5$$

2. High-Pass Filter

![Diagram of high-pass filter with transmission and reflection coefficients](image)
Note for this filter:

\[
T(\omega) = \begin{cases} 
\approx 0 & \omega < \omega_c \\
\approx 1 & \omega > \omega_c 
\end{cases}
\]

\[
\Gamma(\omega) = \begin{cases} 
\approx 1 & \omega < \omega_c \\
\approx 0 & \omega > \omega_c 
\end{cases}
\]

This filter is a high-pass type, as it “passes” signals with frequencies greater than \(\omega_c\), while “rejecting” signals at frequencies less than \(\omega_c\).

3. Band-Pass Filter

Note for this filter:

\[
T(\omega) = \begin{cases} 
\approx 1 & |\omega - \omega_0| < \Delta\omega/2 \\
\approx 0 & |\omega - \omega_0| > \Delta\omega/2 
\end{cases}
\]

\[
\Gamma(\omega) = \begin{cases} 
\approx 0 & |\omega - \omega_0| < \Delta\omega/2 \\
\approx 1 & |\omega - \omega_0| > \Delta\omega/2 
\end{cases}
\]
This filter is a band-pass type, as it “passes” signals within a frequency bandwidth $\Delta \omega$, while “rejecting” signals at all frequencies outside this bandwidth.

In addition to filter bandwidth $\Delta \omega$, a fundamental parameter of bandpass filters is $\omega_0$, which defines the center frequency of the filter bandwidth.

3. Band-Stop Filter

Note for this filter:

$$T(\omega) = \begin{cases} 
\approx 0 & |\omega - \omega_0| < \Delta \omega/2 \\
\approx 1 & |\omega - \omega_0| > \Delta \omega/2 
\end{cases}$$

$$\Gamma(\omega) = \begin{cases} 
\approx 1 & |\omega - \omega_0| < \Delta \omega/2 \\
\approx 0 & |\omega - \omega_0| > \Delta \omega/2 
\end{cases}$$

This filter is a band-stop type, as it “rejects” signals within a frequency bandwidth $\Delta \omega$, while “passing” signals at all frequencies outside this bandwidth.