Matched, Lossless, Reciprocal Devices

Often, we describe a device or network as matched, lossless, or reciprocal.

**Q:** What do these three terms mean??

**A:** Let’s explain each of them one at a time!

**Matched**

A matched device is another way of saying that the input impedance at each port is equal to $Z_0$ when all other ports are terminated in matched loads. As a result, the reflection coefficient of each port is zero—no signal will be come out of a port if a signal is incident on that port (and only that port).

In other words, we want:

$$V_m^- = S_{mm}V_m^+ = 0 \quad \text{for all } m$$

a result that occurs when:

$$S_{mm} = 0 \quad \text{for all } m$$

We find therefore that a matched device will exhibit a scattering matrix where all diagonal elements are zero.
Therefore:

\[
\bar{S} = \begin{bmatrix}
0 & 0.1 & j0.2 \\
0.1 & 0 & 0.3 \\
-j0.2 & 0.3 & 0 \\
\end{bmatrix}
\]

is an example of a scattering matrix for a **matched**, three port device.

**Lossless**

For a lossless device, all of the power that delivered to each device port must eventually finds its way out!

In other words, power is not absorbed by the network—no power to be converted to heat!

Consider, for example, a **four-port** device. Say a signal is incident on port 1, and that all other ports are terminated. The power incident on port 1 is therefore:

\[
P_1^+ = \frac{|V_1^+|^2}{2Z_0}
\]

while the power leaving the device at each port is:

\[
P_m^- = \frac{|V_m^-|^2}{2Z_0} = \frac{|S_{m1}V_1^-|^2}{2Z_0} = |S_{m1}|^2 P_1^+
\]
The total power leaving the device is therefore:

\[ P_{\text{out}} = P_1^- + P_2^- + P_3^- + P_4^- \]

\[ = |S_{11}|^2 P_1^+ + |S_{21}|^2 P_1^+ + |S_{31}|^2 P_1^+ + |S_{41}|^2 P_1^+ \]

\[ = \left( |S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 + |S_{41}|^2 \right) P_1^+ \]

Note therefore that if the device is **lossless**, the output power will be **equal** to the input power, i.e., \( P_{\text{out}} = P_1^+ \). This is true only if:

\[ |S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 + |S_{41}|^2 = 1 \]

If the device is lossless, this will likewise be true for each of the other ports:

\[ |S_{12}|^2 + |S_{22}|^2 + |S_{32}|^2 + |S_{42}|^2 = 1 \]

\[ |S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 + |S_{43}|^2 = 1 \]

\[ |S_{14}|^2 + |S_{24}|^2 + |S_{34}|^2 + |S_{44}|^2 = 1 \]

We can state in general then:

\[ \sum_{m=1}^{N} |S_{mn}|^2 = 1 \quad \text{for all } n \]

In fact, it can be shown that a lossless device will have a **unitary** scattering matrix, i.e.:

\[ \bar{\bar{S}} H \bar{\bar{S}} = \bar{\bar{I}} \]
where $H$ indicates \textbf{conjugate transpose} and $\bar{I}$ is the identity matrix.

The columns of a unitary matrix form an \textbf{orthonormal set}—that is, the \textbf{magnitude} of each column is 1 (as shown above) and dissimilar column vector are mutually \textbf{orthogonal}. In other words, the inner product (i.e., dot product) of dissimilar vectors is zero:

$$\sum_{n=1}^{N} s_{i,j}^* s_{i,j} = s_{i,j}^* s_{i,j} + s_{2,j}^* + \cdots + s_{N,j}^* s_{N,j} = 0 \quad \text{for all} \; i \neq j$$

Reciprocal

Reciprocity results when we build a \textbf{passive} (i.e., unpowered) device with \textbf{simple} materials.

For a reciprocal network, we find that the elements of the scattering matrix are \textbf{related} as:

$$s_{mn} = s_{nm}$$

For example, a reciprocal device will have $s_{21} = s_{12}$ or $s_{32} = s_{23}$. We can write reciprocity in matrix form as:

$$\bar{S}^T = \bar{S}$$

where $T$ indicates (non-conjugate) transpose.