The Complex Propagation Constant $\gamma$

Recall that the current and voltage along a transmission line have the form:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{+\gamma z}$$

where $Z_0$ and $\gamma$ are complex constants that describe the properties of a transmission line. Since $\gamma$ is complex, we can consider both its real and imaginary components.

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta$$

where $\alpha = \text{Re}\{\gamma\}$ and $\beta = \text{Im}\{\gamma\}$. Therefore, we can write:

$$e^{-\gamma z} = e^{-(\alpha + j\beta)z} = e^{-\alpha z} e^{-j\beta z}$$

Since $|e^{j\beta z}| = 1$, then $e^{-\alpha z}$ alone determines the magnitude of $e^{-\gamma z}$. 
I.E., $|e^{-\gamma z}| = e^{-\alpha z}$.

Therefore, $\alpha$ expresses the **attenuation** of the signal due to the loss in the transmission line.

Since $e^{-\alpha z}$ is a real function, it expresses the **magnitude** of $e^{-\gamma z}$ only. The **relative phase** $\phi(z)$ of $e^{-\gamma z}$ is therefore determined by $e^{-j\beta z} = e^{-j\phi(z)}$ only (recall $|e^{-j\beta z}| = 1$).

From Euler's equation:

$$e^{j\phi(z)} = e^{j\beta z} = \cos(\beta z) + j \sin(\beta z)$$

Therefore, $\beta z$ represents the **relative phase** $\phi(z)$ of the oscillating signal, as a function of transmission line position $z$. Since phase $\phi(z)$ is expressed in radians, and $z$ is distance (in meters), the value $\beta$ must have units of:

$$\beta = \frac{\phi}{z} \quad \text{radians/meter}$$
The **wavelength** \( \lambda \) of the signal is the distance \( \Delta z \) over which the relative phase changes by \( 2\pi \) radians. So:

\[
2\pi = \phi(z + \Delta z) - \phi(z) = \beta \Delta z = \beta \lambda
\]

or, rearranging:

\[
\beta = \frac{2\pi}{\lambda}
\]

Since the signal is oscillating in time at rate \( \omega \) rad/sec, the **propagation velocity** of the wave is:

\[

\nu_p = \frac{\omega}{\beta} = \frac{\omega \lambda}{2\pi} = f \lambda
\]

where \( f \) is **frequency** in cycles/sec.

Recall we originally considered the transmission line current and voltage as a function of time and position (i.e., \( v(z,t) \) and \( i(z,t) \)). We assumed the time function was sinusoidal, oscillating with frequency \( \omega \):

\[
v(z,t) = Re\{V(z)e^{i\omega t}\}
\]

\[
i(z,t) = Re\{I(z)e^{i\omega t}\}
\]
Now that we know $V(z)$ and $I(z)$, we can write the original functions as:

$$v(z, t) = \text{Re} \left\{ V_0^+ e^{-az} e^{-j(\beta z + \omega t)} + V_0^- e^{az} e^{j(\beta z + \omega t)} \right\}$$

$$i(z, t) = \text{Re} \left\{ \frac{V_0^+}{Z_0} e^{-az} e^{-j(\beta z + \omega t)} - \frac{V_0^-}{Z_0} e^{az} e^{j(\beta z + \omega t)} \right\}$$

The first term in each equation describes a wave propagating in the $+z$ direction, while the second describes a wave propagating in the opposite $(-z)$ direction.

Each wave has wavelength:

$$\lambda = \frac{2\pi}{\beta}$$

And velocity:

$$v_p = \frac{\omega}{\beta}$$