The Poynting Vector

Recall that plane waves and spherical waves are electromagnetic waves.

In other words, they consist of both electric and magnetic fields!

Q: Just what is the magnetic field $\mathbf{H}(\vec{r})$?

A: Use Faraday's Law!

i.e. $\nabla \times \mathbf{E}(\vec{r},t) = -\frac{\partial \mathbf{H}(\vec{r},t)}{\partial t}$

If $\mathbf{E}(\vec{r},t) = \vec{e} \exp[\frac{1}{j}(kx-\omega t)]$

then we find $\mathbf{H}(\vec{r},t) = \frac{\vec{e}}{\hbar} \frac{1}{\hbar} \exp[\frac{1}{j}(kx-\omega t)]$
where \( |\vec{E}| = |\vec{H}| \), \( \vec{H} \cdot \vec{E} = 0 \), \( \vec{H} \cdot \vec{H} = 0 \)

and

\[ N = \sqrt{\frac{M_0}{\varepsilon_0}} = 377 \, \Omega \]

This value, \( N = 377 \, \Omega \), is known as the characteristic impedance of free-space.

**Q:** Why \( 377 \, \Omega \) (i.e., impedance)?

**A:** Note \( N = \frac{|\vec{E}|}{|\vec{H}|} \).

Recall the unit of an electric field is \( \text{V/m} \) and for magnetic field \( \text{A/m} \).

\[ \frac{|\vec{E}|}{|\vec{H}|} \text{ has units of } \frac{\text{V/m}}{\text{A/m}} = \frac{V}{A} = \text{ohms} \]

\[ \Rightarrow \frac{|\vec{E}|}{|\vec{H}|} = \text{impedance}! \]
Now, let's examine the Poynting Vector:

$$ \overrightarrow{W}(\vec{r}) = \frac{1}{2} \text{Re} \left\{ \overrightarrow{E}(\vec{r},t) \times \overrightarrow{H}^*(\vec{r},t) \right\} $$

where \( \ast \) denotes the complex conjugate and \( \times \) denotes the vector cross product.

For our plane-wave example, we find:

$$ \overrightarrow{W}(\vec{r}) = \frac{1}{2} \left| \overrightarrow{E}(\vec{r},t) \right| \left| \overrightarrow{H}(\vec{r},t) \right| \times $$

$$ = \frac{1}{2} \left| \overrightarrow{E}(\vec{r},t) \right|^2 \times $$

$$ = \frac{2 \pi \hbar}{\nu} \times $$

$$ = \frac{2 \pi e^2}{\nu} \times $$

Q: What the heck does this mean?

A: Check out the units of \( \overrightarrow{W}(\vec{r}) \)!

$$ \left| \overrightarrow{E}(\vec{r},t) \right| \Rightarrow \text{Volts/m} \quad \left| \overrightarrow{H}(\vec{r},t) \right| \Rightarrow \text{Amps/m} $$
$|\overline{W}|$ has units of $\frac{V}{m} \cdot A = \frac{V\cdot A}{m^2}$

But, $(\text{volts})(\text{amps}) = \text{Watts} \implies \text{power}$!

$|\overline{W}|$ has units of $\frac{\text{watts}}{m^2}$

$\implies \text{Power/unit area}$

$\therefore$

1) The magnitude of $\overline{W}(\vec{r})$ is the power density of the e.m. wave!

2) The direction of $\overline{W}(\vec{r})$ (i.e., $\hat{x}$ for our example) describes the direction of power flow.

* Note for a plane-wave, the power density is constant, i.e., $|\overline{W}(\vec{r})|$ is independent of $x$. $|\overline{W}(\vec{r})| = \frac{|\vec{E}|^2}{2\mu}$
* But, this is not necessarily true for all propagating e.m. waves!

* For example, the Poynting vector for a spherical wave is

\[ \overrightarrow{W}(\vec{r}) = \frac{|\vec{E}|^2}{\varepsilon_0} \frac{\hat{r}}{r^2} \]

So the power density decreases as \( r^{-2} \)

i.e., \( |\overrightarrow{W}(\vec{r})| \propto \frac{1}{r^2} \) for spherical wave.