The Scattering Matrix

At “low” frequencies, we can completely characterize a linear device or network using an **impedance** matrix, which relates the currents and voltages at each device terminal to the currents and voltages at all other terminals.

But, at microwave frequencies, it is **difficult** to measure total currents and voltages!

* Instead, we can measure the **magnitude** and **phase** of each of the two transmission line waves $V^+(z)$ and $V^-(z)$.

* In other words, we can determine the relationship between the incident and reflected wave at each device terminal to the incident and reflected waves at all other terminals.

These relationships are completely represented by the **scattering matrix**. It **completely** describes the behavior of a linear, multi-port device at a **given frequency** $\omega$. 
Consider the 4-port microwave device shown below:

Note in this example, there are four identical transmission lines connected to the same “box”. Inside this box there may be a very simple linear device/circuit, or it might contain a very large and complex linear microwave system.

- Either way, the “box” can be fully characterized by its scattering parameters!
First, note that each transmission line has a specific location that effectively defines the input to the device (i.e., \( z_{1p}, z_{2p}, z_{3p}, z_{4p} \)). These often arbitrary positions are known as the port locations, or port planes of the device.

Say there exists an incident wave on port 1 (i.e., \( V_1^+(z_1) \neq 0 \)), while the incident waves on all other ports are known to be zero (i.e., \( V_2^+(z_2) = V_3^+(z_3) = V_4^+(z_4) = 0 \)).

Say we measure/determine the voltage of the wave flowing into port 1, at the port 1 plane (i.e., determine \( V_1^+(z_1 = z_{1p}) \)).

Say we then measure/determine the voltage of the wave flowing out of port 2, at the port 2 plane (i.e., determine \( V_2^-(z_2 = z_{2p}) \)).

The complex ratio between \( V_1^+(z_1 = z_{1p}) \) and \( V_2^-(z_2 = z_{2p}) \) is known as the scattering parameter \( S_{21} \):

\[
S_{21} = \frac{V_2^-(z = z_2)}{V_1^+(z = z_1)} = \frac{V_{02} e^{j \beta z_{2p}}}{V_{01} e^{-j \beta z_{1p}}} = \frac{V_{02}^* e^{j \beta (z_{2p} + z_{1p})}}{V_{01}^* e^{j \beta z_{1p}}} = \frac{V_{02} e^{j \beta (z_{2p} - z_{1p})}}{V_{01} e^{j \beta z_{1p}}}
\]

Likewise, the scattering parameters \( S_{31} \) and \( S_{41} \) are:

\[
S_{31} = \frac{V_3^-(z_3 = z_{3p})}{V_1^+(z_1 = z_{1p})} \quad \text{and} \quad S_{41} = \frac{V_4^-(z_4 = z_{4p})}{V_1^+(z_1 = z_{1p})}
\]
We of course could also define, say, scattering parameter $S_{34}$ as the ratio between the complex values $V_4^+(z_4 = z_{4p})$ (the wave into port 4) and $V_3^-(z_3 = z_{3p})$ (the wave out of port 3), given that the input to all other ports (1,2, and 3) are zero. Thus, more generally, the ratio of the wave incident on port $n$ to the wave emerging from port $m$ is:

$$S_{mn} = \frac{V_m^-(z_m = z_{mp})}{V_n^+(z_n = z_{np})} \quad \text{(given that } V_k^+(z_k) = 0 \text{ for all } k \neq n)$$

Note that frequently the port positions are assigned a zero value (e.g., $z_{1p} = 0$, $z_{2p} = 0$). This of course simplifies the scattering parameter calculation:

$$S_{mn} = \frac{V_m^-(z_m = 0)}{V_n^+(z_n = 0)} = \frac{V_{0m}^- e^{j\beta_0}}{V_{0n}^+ e^{-j\beta_0}} = \frac{V_{0m}^-}{V_{0n}^+}$$

We will generally assume that the port locations are defined as $z_{np} = 0$, and thus use the above notation. But remember where this expression came from!
Q: But how do we ensure that only one incident wave is non-zero?

A: Terminate all other ports with a matched load!
Note that if the ports are terminated in a matched load (i.e., \( Z_L = Z_0 \)), then \( \Gamma_{nL} = 0 \) and therefore:

\[
V^{+}_n(z_n) = 0
\]

In other words, terminating a port ensures that there will be no signal incident on that port!

**Q:** Just between you and me, I think you've messed this up! In all previous handouts you said that if \( \Gamma_L = 0 \), the wave in the minus direction would be zero:

\[
V^{-}(z) = 0 \quad \text{if} \quad \Gamma_L = 0
\]

but just now you said that the wave in the positive direction would be zero:

\[
V^{+}(z) = 0 \quad \text{if} \quad \Gamma_L = 0
\]

Of course, there is no way that both statements can be correct!

**A:** Actually, both statements are correct! You must be careful to understand the physical definitions of the plus and minus directions—in other words, the propagation directions of waves \( V^{+}_n(z_n) \) and \( V^{-}_n(z_n) \)!
For example, we originally analyzed this case:

\[ V^+(z) \rightarrow \Gamma_L \rightarrow \left\{ \begin{array}{l} V^-(z) = 0 \quad \text{if} \quad \Gamma_L = 0 \\ V^-(z) \leftarrow \end{array} \right. \]

In this original case, the wave incident on the load is \( V^+(z) \) (plus direction), while the reflected wave is \( V^-(z) \) (minus direction).

**Contrast** this with the case we are now considering:

For this current case, the situation is reversed. The wave incident on the load is now denoted as \( V^-_n(z_n) \) (coming out of port \( n \)), while the wave reflected off the load is now denoted as \( V^+_n(z_n) \) (going into port \( n \)).

As a result, \( V^+_n(z_n) = 0 \) when \( \Gamma_{nL} = 0 \)!
Perhaps we could more \textbf{generally} state that:

\[
V_{\text{reflected}} (z = z_L) = \Gamma_L V_{\text{incident}} (z = z_L)
\]

For each case, \textbf{you must be able to correctly identify the mathematical statement describing the wave incident on, and reflected from, some passive load.}

Like most equations in engineering, the \textbf{variable names can change}, but the \textbf{physics described by the mathematics will not}!

Now, \textbf{back} to our discussion of \textbf{S-parameters}. We found that if \(z_{np} = 0\) for all ports \(n\), the scattering parameters could be directly written in terms of wave \textbf{amplitudes} \(V_{0n}^+\) and \(V_{0m}^-\).

\[
S_{mn} = \frac{V_{0m}^-}{V_{0n}^+} \quad \text{(given that } V_k^+(z_k) = 0 \text{ for all } k \neq n)\]

Which we can now \textbf{equivalently} state as:

\[
S_{mn} = \frac{V_{0m}^-}{V_{0n}^+} \quad \text{(given that all ports, except port } n, \text{ are matched)}
\]
One more **important** note—notice that for the **matched** ports (i.e., those ports with **no** incident wave), the voltage of the exiting **wave** is also the **total** voltage!

\[
V_m(z_m) = V_{0m}^* e^{-j\beta z_m} + V_{0m}^- e^{+j\beta z_m} \\
= 0 + V_{0m}^- e^{+j\beta z_m} \\
= V_{0m}^- e^{+j\beta z_m} \quad \text{(for all terminated ports)}
\]

Thus, the value of the exiting wave at each terminated port is likewise the value of the total voltage at those ports:

\[
V_m(0) = V_{0m}^- \quad \text{(for all terminated ports)}
\]

And so, we can express some of the scattering parameters equivalently as:

\[
S_{mn} = \frac{V_m(0)}{V_{0n}^+} \quad \text{(for matched port } m, \text{ i.e., for } m \neq n)\]

You might find this result helpful if attempting to determine scattering parameters where \( m \neq n \) (e.g., \( S_{21}, S_{43}, S_{13} \)), as we can often use traditional **circuit theory** to easily determine the **total** port voltage \( V_m(0) \).

However, we **cannot** use the expression above to determine the scattering parameters when \( m = n \) (e.g., \( S_{11}, S_{22}, S_{33} \)).

**Think** about this! The scattering parameters for these cases are:
Therefore, port \( n \) is a port where there actually is some incident wave \( V_{0n}^+ \) (port \( n \) is not terminated in a matched load!). Thus, the total voltage is not simply the value of the exiting wave, as both an incident wave and exiting wave exists at port \( n \).

Typically, it is much more difficult to determine/measure the scattering parameters of the form \( S_{nn} \), as opposed to scattering parameters of the form \( S_{mn} \) (where \( m \neq n \)) where there is only an exiting wave from port \( m \).
A: OK, say that our ports are not matched, such that we have waves simultaneously incident on each of the four ports of our device.

Since the device is linear, the output at any port due to all the incident waves is simply the coherent sum of the output at that port due to each wave!

For example, the output wave at port 3 can be determined by (assuming $z_{np} = 0$):

$$V_{03}^- = S_{34} V_{04}^+ + S_{33} V_{03}^+ + S_{32} V_{02}^+ + S_{31} V_{01}^+$$

More generally, the output at port $m$ of an $N$-port device is:

$$V_{0m}^- = \sum_{n=1}^{N} S_{mn} V_{0n}^+ \quad (z_{np} = 0)$$
This expression can be written in matrix form as:

\[ \bar{V}^- = \bar{S} \bar{V}^+ \]

Where \( \bar{V}^- \) is the vector:

\[ \bar{V}^- = [V_{01}^-, V_{02}^-, V_{03}^-, \ldots, V_{0N}^-]^T \]

and \( \bar{V}^+ \) is the vector:

\[ \bar{V}^+ = [V_{01}^+, V_{02}^+, V_{03}^+, \ldots, V_{0N}^+]^T \]

Therefore \( \bar{S} \) is the scattering matrix:

\[
\bar{S} = \begin{bmatrix}
S_{11} & \cdots & S_{1n} \\
\vdots & \ddots & \vdots \\
S_{m1} & \cdots & S_{mn}
\end{bmatrix}
\]

The scattering matrix is a \( N \) by \( N \) matrix that completely characterizes a linear, \( N \)-port device. Effectively, the scattering matrix describes a multi-port device the way that \( \Gamma_L \) describes a single-port device (e.g., a load)!

But beware! The values of the scattering matrix for a particular device or network, just like \( \Gamma_L \), are frequency dependent! Thus, it may be more instructive to explicitly write:

\[
\bar{S}(\omega) = \begin{bmatrix}
S_{11}(\omega) & \cdots & S_{1n}(\omega) \\
\vdots & \ddots & \vdots \\
S_{m1}(\omega) & \cdots & S_{mn}(\omega)
\end{bmatrix}
\]