The Terminated, Lossless Transmission Line

Now let’s attach something to our transmission line. Consider a lossless line, length $\ell$, terminated with a load $Z_L$.

Q: What is the current and voltage at each and every point on the transmission line (i.e., what is $I(z)$ and $V(z)$ for all points $z$ where $z_L - \ell \leq z \leq z_L$)?

A: To find out, we must apply boundary conditions!

In other words, at the end of the transmission line ($z = z_L$) where the load is attached—we have many requirements that all must be satisfied!
1. To begin with, the voltage and current \( I(z = z_L) \) and \( V(z = z_L) \) must be consistent with a valid transmission line solution:

\[
V(z = z_L) = V^+ (z = z_L) + V^- (z = z_L) = V_0^+ e^{-j\beta z_L} + V_0^- e^{+j\beta z_L}
\]

\[
I(z = z_L) = \frac{V_0^+ (z = z_L)}{Z_0} - \frac{V_0^- (z = z_L)}{Z_0} = \frac{V_0^+}{Z_0} e^{-j\beta z_L} - \frac{V_0^-}{Z_0} e^{+j\beta z_L}
\]

2. Likewise, the load voltage and current must be related by Ohm's law:

\[
V_L = Z_L I_L
\]

3. Most importantly, we recognize that the values \( I(z = z_L) \), \( V(z = z_L) \) and \( I_L, V_L \) are not independent, but in fact are strictly related by Kirchoff’s Laws!
From KVL and KCL we find these requirements:

\[ V(z = z_L) = V_L \]

\[ I(z = z_L) = I_L \]

These are the boundary conditions for this particular problem.

→ **Careful**! Different transmission line problems lead to different boundary conditions—you must access each problem individually and independently!

Combining these equations and boundary conditions, we find that:

\[ V_L = Z_L I_L \]

\[ V(z = z_L) = Z_L I(z = z_L) \]

\[ V^+(z = z_L) + V^-(z = z_L) = \frac{Z_L}{Z_0} \left( V^+(z = z_L) - V^-(z = z_L) \right) \]

Rearranging, we can conclude:

\[ \frac{V^-(z = z_L)}{V^+(z = z_L)} = \frac{Z_L - Z_0}{Z_L + Z_0} \]
Q: Hey wait as second! We earlier defined $V^-(z)/V^+(z)$ as reflection coefficient $\Gamma(z)$. How does this relate to the expression above?

A: Recall that $\Gamma(z)$ is a function of transmission line position $z$. The value $V^-(z = z_L)/V^+(z = z_L)$ is simply the value of function $\Gamma(z)$ evaluated at $z = z_L$ (i.e., evaluated at the end of the line):

$$\frac{V^-(z = z_L)}{V^+(z = z_L)} = \Gamma(z = z_L) = \frac{Z_L - Z_0}{Z_L + Z_0}$$

This value is of fundamental importance for the terminated transmission line problem, so we provide it with its own special symbol ($\Gamma_L$):

$$\Gamma_L = \Gamma(z = z_L) = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Q: Wait! We earlier determined that:

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0}$$

so it would seem that:

$$\Gamma_L = \Gamma(z = z_L) = \frac{Z(z = z_L) - Z_0}{Z(z = z_L) + Z_0}$$

Which expression is correct??
They both are! It is evident that the two expressions:

\[ \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \quad \text{and} \quad \Gamma_L = \frac{Z(z = z_L) - Z_0}{Z(z = z_L) + Z_0} \]

are equal if:

\[ Z(z = z_L) = Z_L \]

And since we know that from Ohm's Law:

\[ Z_L = \frac{V_L}{I_L} \]

and from Kirchoff's Laws:

\[ \frac{V_L}{I_L} = \frac{V(z = z_L)}{I(z = z_L)} \]

and that line impedance is:

\[ \frac{V(z = z_L)}{I(z = z_L)} = Z(z = z_L) \]

we find it apparent that the line impedance at the end of the transmission line is equal to the load impedance:

\[ Z(z = z_L) = Z_L \]

The above expression is essentially another expression of the boundary condition applied at the end of the transmission line.
A: We are trying to find $V(z)$ and $I(z)$ when a lossless transmission line is terminated by a load $Z_L$!

We can now determine the value of $V_0^-$ in terms of $V_0^+$. Since:

$$\Gamma_L = \frac{V_0^-(z = Z_L)}{V_0^+(z = Z_L)} = \frac{V_0^- e^{j \beta z_L}}{V_0^+ e^{-j \beta z_L}}$$

We find:

$$V_0^- = e^{-2j \beta z_L} \Gamma_L V_0^+$$

And therefore we find:

$$V^-(z) = (e^{-2j \beta z_L} \Gamma_L V_0^+) e^{+j \beta z}$$

$$V(z) = V_0^+ \left[ e^{-j \beta z} + (e^{-2j \beta z_L} \Gamma_L) e^{+j \beta z} \right]$$

$$I(z) = \frac{V_0^+}{Z_0} \left[ e^{-j \beta z} - (e^{-2j \beta z_L} \Gamma_L) e^{+j \beta z} \right]$$

where:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$
Now, we can further *simplify* our analysis by arbitrarily assigning the end point $z_L$ a *zero* value (i.e., $z_L = 0$):

If the load is located at $z=0$ (i.e., if $z_L = 0$), we find that:

$$V(z = 0) = V^+(z = 0) + V^-(z = 0)$$

$$= V_0^+ e^{-j\beta(0)} + V_0^- e^{+j\beta(0)}$$

$$= V_0^+ + V_0^-$$

$$I(z = 0) = \frac{V_0^+(z = 0)}{Z_0} - \frac{V_0^-(z = 0)}{Z_0}$$

$$= \frac{V_0^+}{Z_0} e^{-j\beta(0)} - \frac{V_0^-}{Z_0} e^{+j\beta(0)}$$

$$= \frac{V_0^+ - V_0^-}{Z_0}$$

$$Z(z = 0) = Z_0 \left( \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} \right)$$
Likewise, it is apparent that if $z_L = 0$, $\Gamma_L$ and $\Gamma_0$ are the same:

$$\Gamma_L = \Gamma(z = z_L) = \frac{V^-(z = 0)}{V^+(z = 0)} = \frac{V_0^-}{V_0^+} = \Gamma_0$$

Therefore:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_0$$

Thus, we can write the line current and voltage simply as:

$$V(z) = V_0^+ \left[ e^{-j\beta z} + \Gamma_L e^{+j\beta z} \right]$$

[for $z_L = 0$]

$$I(z) = \frac{V_0^+}{Z_0} \left[ e^{-j\beta z} - \Gamma_L e^{+j\beta z} \right]$$

Q: But, how do we determine $V_0^+$?  

A: We require a second boundary condition to determine $V_0^+$.  
The only boundary left is at the other end of the transmission line.  Typically, a source of some sort is located there.  This makes physical sense, as something must generate the incident wave!