Consider this circuit:

\[ I_1(z) \quad Z_1, \beta_1 \quad Z_2, \beta_2 \quad I_2(z) \]

I.E., a transmission line with characteristic impedance \( Z_1 \) transitions to a different transmission line at location \( z = 0 \). This second transmission line has different characteristic impedance \( Z_2 \) (\( Z_1 \neq Z_2 \)). This second line is terminated with a load \( Z_L = Z_2 \) (i.e., the second line is matched).

**Q:** What is the voltage and current along each of these two transmission lines? More specifically, what are \( V_{01}^+, V_{01}^-, V_{02}^+, V_{02}^- \) ?

**A:** Since a source has not been specified, we can only determine \( V_{01}^+, V_{02}^+ \) and \( V_{02}^- \) in terms of complex constant \( V_{01}^+ \). To accomplish this, we must apply a boundary condition at \( z = 0 \).
We know that the voltage along the first transmission line is:

$$V_1(z) = V_{01}^+ e^{-j \beta_1 z} + V_{01}^- e^{+j \beta_1 z} \quad [\text{for } z < 0]$$

while the current along that same line is described as:

$$I_1(z) = \frac{V_{01}^+}{Z_1} e^{-j \beta_1 z} - \frac{V_{01}^-}{Z_1} e^{+j \beta_1 z} \quad [\text{for } z < 0]$$

We likewise know that the voltage along the second transmission line is:

$$V_2(z) = V_{02}^+ e^{-j \beta_2 z} + V_{02}^- e^{+j \beta_2 z} \quad [\text{for } z > 0]$$

while the current along that same line is described as:

$$I_2(z) = \frac{V_{02}^+}{Z_2} e^{-j \beta_2 z} - \frac{V_{02}^-}{Z_2} e^{+j \beta_2 z} \quad [\text{for } z > 0]$$

Moreover, since the second line is terminated in a matched load, we know that the reflected wave from this load must be zero:

$$V_2^-(z) = V_{02}^- e^{-j \beta_2 z} = 0$$
The voltage and current along the second transmission line is thus simply:

\[ V_2(z) = V_2^+(z) = V_{02}^+ e^{-j\beta_2 z} \quad [\text{for } z > 0] \]

\[ I_2(z) = I_2^+(z) = \frac{V_{02}^+}{Z_2} e^{-j\beta_2 z} \quad [\text{for } z > 0] \]

At the location where these two transmission lines meet, the current and voltage expressions each must satisfy some specific boundary conditions:

The first boundary condition comes from KVL, and states that:

\[ V_1(z = 0) = V_2(z = 0) \]

\[ V_{01}^+ e^{-j\beta_1(0)} + V_{01}^- e^{+j\beta_1(0)} = V_{02}^+ e^{-j\beta_2(0)} \]

\[ V_{01}^+ + V_{01}^- = V_{02}^+ \]
while the second boundary condition comes from KCL, and states that:

\[ I_1(z = 0) = I_2(z = 0) \]
\[
\frac{V_{01}^+}{Z_1} e^{-j\beta_1(0)} - \frac{V_{01}^-}{Z_1} e^{+j\beta_1(0)} = \frac{V_{02}^+}{Z_2} e^{-j\beta_2(0)} 
\]
\[
\frac{V_{01}^+}{Z_1} - \frac{V_{01}^-}{Z_1} = \frac{V_{02}^+}{Z_2}
\]

We now have two equations and two unknowns \((V_{01}^-\) and \(V_{02}^+)\) ! We can solve for each in terms of \(V_{01}^+\) (i.e., the incident wave).

From the first boundary condition we can state:

\[ V_{01}^- = V_{02}^+ - V_{01}^+ \]

Inserting this into the second boundary condition, we find an expression involving only \(V_{02}^+\) and \(V_{01}^+\):

\[
\frac{V_{01}^+}{Z_1} - \frac{V_{01}^-}{Z_1} = \frac{V_{02}^+}{Z_2}
\]
\[
\frac{V_{01}^+}{Z_1} - \frac{V_{02}^+ - V_{01}^+}{Z_1} = \frac{V_{02}^+}{Z_2}
\]
\[
\frac{2V_{01}^+}{Z_1} = \frac{V_{02}^+}{Z_2} + \frac{V_{02}^+}{Z_1}
\]

Solving this expression, we find:

\[ V_{02}^+ = \left( \frac{2Z_2}{Z_1 + Z_2} \right) V_{01}^+ \]
We can therefore define a transmission coefficient, which relates $V_{02}^+$ to $V_{01}^+$:

$$T_0 = \frac{V_{02}^+}{V_{01}^+} = \frac{2Z_2}{Z_1 + Z_2}$$

Meaning that $V_{02}^+ = T V_{01}^+$, and thus:

$$V_2(z) = V_2^+(z) = T V_{01}^+ e^{-j\beta z} \quad \text{[for } z > 0 \text{]}$$

We can likewise determine the constant $V_{01}^-$ in terms of $V_{01}^+$. We again start with the first boundary condition, from which we concluded:

$$V_{02}^+ = V_{01}^+ + V_{01}^-$$

We can insert this into the second boundary condition, and determine an expression involving $V_{01}^-$ and $V_{01}^+$ only:

$$\frac{V_{01}^+}{Z_1} - \frac{V_{01}^-}{Z_1} = \frac{V_{02}^+}{Z_2}$$

$$\frac{V_{01}^+}{Z_1} - \frac{V_{01}^-}{Z_1} = \frac{V_{01}^+ + V_{01}^-}{Z_2}$$

$$\left(\frac{1}{Z_1} - \frac{1}{Z_2}\right) V_{01}^+ = \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right) V_{01}^-$$
Solving this expression, we find:

\[ V_{01}^- = \left( \frac{Z_2 - Z_1}{Z_2 + Z_1} \right) V_{01}^+ \]

We can therefore define a reflection coefficient, which relates \( V_{01}^- \) to \( V_{01}^+ \):

\[ \Gamma_0 = \frac{V_{01}^-}{V_{01}^+} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \]

This result should not surprise us!

Note that because the second transmission line is matched, its input impedance is equal to \( Z_1 \):

\[ Z_{in} = Z_2 \]
and thus we can *equivalently* write the entire circuit as:

![Circuit Diagram]

We have already analyzed *this* circuit! We know that:

\[
V_{01}^- = \Gamma_{01} V_{01}^+
\]

\[
= \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right) V_{01}^+
\]

Which is *exactly* the same result as we determined earlier!

The values of the reflection coefficient \( \Gamma_0 \) and the transmission coefficient \( T_0 \) are *not* independent, but in fact are directly *related*. Recall the *first* boundary expressed was:

\[
V_{01}^+ + V_{01}^- = V_{02}^+
\]

Dividing this by \( V_{01}^+ \):

\[
1 + \frac{V_{01}^-}{V_{01}^+} = \frac{V_{02}^+}{V_{01}^+}
\]
Since $\Gamma_0 = \mathcal{V}_{01}^- / \mathcal{V}_{01}^+$ and $\mathcal{T}_0 = \mathcal{V}_{02}^+ / \mathcal{V}_{01}^+$:

$$1 + \Gamma_0 = \mathcal{T}_0$$

Note the result $\mathcal{T}_0 = 1 + \Gamma_0$ is true for this particular circuit, and therefore is not a universally valid expression for two-port networks!