II Transmitter and Receiver Design

We design radio systems using **RF/microwave** components.

Q: Why don't we use the "usual" circuit components (e.g., resistors, capacitors, op-amps, transistors) ??

A: We **do** use these! But we require new devices because:

1.

2.

A. Microwave Components

Let's carefully examine each of the microwave devices that are useful for radio design:

1)
2)
3)
4)
5)
6)
7)
8)

1. Transmission Lines

Q: *So just what is a **transmission line**?*

A:

→

Q: *Oh, so it's simply a conducting **wire**, right?*

A:

HO: The Telegraphers Equations

HO: Time-Harmonic Solutions for Linear Circuits

a) *Basic Transmission Line Theory*

Q: *So, what complex functions I(z) and V(z) **do** satisfy both telegrapher equations?*

A:

HO: The Transmission Line Wave Equations

Q: *Are the solutions for I(z) and V(z) completely **independent**, or are they **related** in any way ?*

A:

HO: The Transmission Line Characteristic Impedance

Q: *So what is the significance of the constant β? What does it tell us?*

A:

HO: The Propagation Constant

Q: *Is **characteristic** impedance Z_0 the same as the concept of impedance I learned about in **circuits** class?*

A:

HO: Line Impedance

Q: *These **wave** functions $V^{+}(z)$ and $V^{-}(z)$ seem to be important. How are **they** related?*

A:

HO: The Reflection Coefficient

HO: V, I, Z or V^+, V^-, Γ??

b) ***The Terminated, Lossless Transmission Line***

We now know that a **lossless** transmission line is **completely** characterized by **real** constants Z_0 and β.

Likewise, the **2 waves** propagating on a transmission line are **completely** characterized by **complex** constants V_0^+ and V_0^-.

Q: *Z_0 and β are determined from L, C, and ω. How do we find V_0^+ and V_0^- ?*

A:

Every transmission line has **2** "boundaries"

1)
2)

Typically, there is a **source** at one end of the line, and a **load** at the other.

→

Let's apply the **load** boundary condition!

HO: The Terminated, Lossless Transmission Line

HO: Special Values of Load Impedance

Q: *So what is the significance of the constant β? What does it tell us?*

A:

HO: The Propagation Constant

Q: *So the line impedance at the **end** of a line must be load impedance Z_L (i.e., $Z(z=z_L)=Z_L$); what is the line impedance at the **beginning** of the line (i.e., $Z(z=z_L-\ell)=?$)?*

A:

HO: Transmission Line Input Impedance

Q: *You said the purpose of the transmission line is to transfer **E.M. energy** from the source to the load. Exactly how much **power** is flowing in the transmission line, and how much is **delivered** to the load?*

A: ***HO: Power Flow and Return Loss***

Note that we can **specify** a load with:

1)
2)
3)

A fourth alternative is VSWR.

HO: VSWR

c) *A second boundary condition: Applying a generator to the transmission line*

Q: *A passive load Z_L specifies $Z(z)$ and $\Gamma(z)$, but we still don't explicitly know $V(z)$, $I(z)$, $V^+(z)$, or $V^-(z)$. How are these functions determined?*

A:

HO: A Transmission Line Connecting Source and Load

Q: *OK, we can **finally** ask the question that we have been concerned with since the very beginning: How much **power** is delivered **to** the load **by** the source?*

A: HO: Delivered Power

Q: *So the power transferred depends on the **source**, the **transmission line**, and the **load**. What combination of these devices will result in **maximum** power transfer?*

A: ***HO: Special Cases of Source and Input Impedances***

Q: *Yikes! The signal source is generally a Thevenin's equivalent of the **output** of some **useful device**, while the load impedance is generally the **input** impedance of some other **useful device**. I do **not** want to—nor typically can I—**change** these devices or **alter** their characteristics.*

*Must I then just **accept** the fact that I will achieve **suboptimum** power transfer?*

A:

HO: Matching Networks

Q: But in microwave circuits, a source and load are connected by a transmission line. Can we implement matching networks in transmission line circuits?

A: ***HO: Matching Networks and Transmission Lines***

Q: *Matching networks seem almost too good to be true; can we really design and construct them to provide a perfect match?*

A: It is relatively easy to provide a near perfect match at precisely **one frequency**!

But, since lossless matching networks are made entirely of **reactive** elements (not to mention the reactive components of the source and load impedance), we find that changing the signal frequency will typically "**mismatch**" our circuit!

Thus a difficult challenge for any microwave component designer is to provide a **wideband** match to a transmission line with characteristic impedance Z_0.

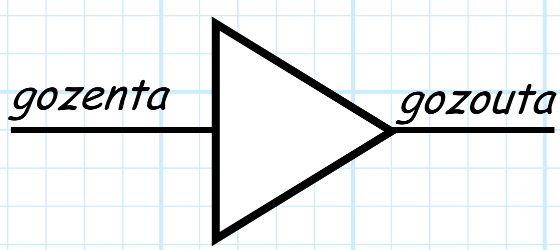
→

d) *Scattering Parameters*

Note that a passive load is a one-port device—a device that can be characterized (at one frequency) by impedance Z_L or load reflection coefficient Γ_L .

However, many microwave devices have **multiple** ports!

Most common are **two-port devices** (e.g., amplifiers and filters), devices with both a gozenta and a gozouta.



Note that a transmission line is also two-port device!

Q: *Are there any known ways to characterize a **multi-port** device?*

A: Yes! **Two** methods are:

1.

2.

HO: The Impedance Matrix

Q: *You say that the impedance matrix characterizes a multi-port device. But is this characterization helpful? Can we actually use it to solve real problems?*

A: *Example: Using the Impedance Matrix*

Q: The impedance matrix relates the quantities $V(z)$ and $I(z)$, is there an equivalent matrix that relates $V^+(z)$ and $V^-(z)$?

A:

HO: The Scattering Matrix

Q: Can the scattering matrix likewise be used to solve real problems?

A: Of course!

Example: The Scattering Matrix

Example: Scattering Parameters

Q: *But, can the scattering matrix by itself tell us anything about the device it characterizes?*

A: Yes! It can tell us if the device is matched, or lossless, or reciprocal.

HO: Matched, Lossless, Reciprocal

e) *Types of Transmission Lines*

Perhaps the most common transmission line structure is coaxial transmission line.

HO:Coaxial Transmission Lines

Coaxial transmission lines are used with connectorized devices.

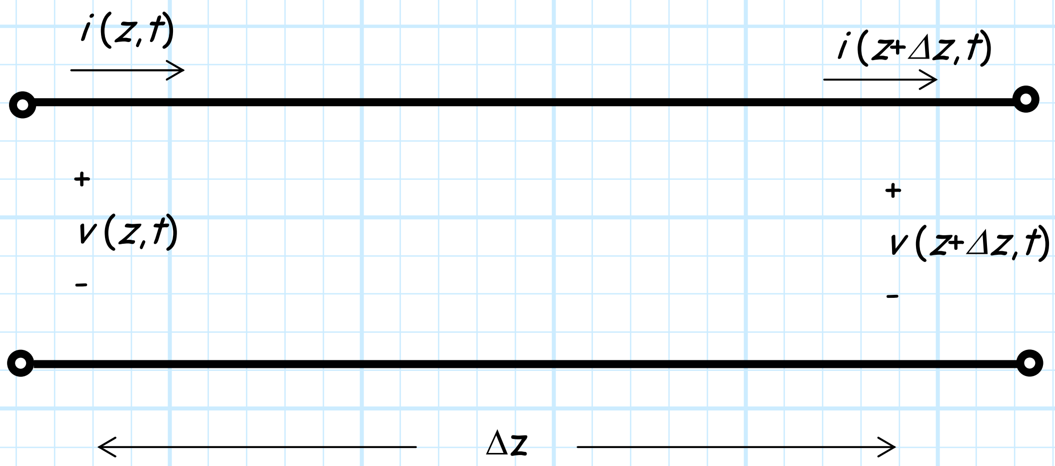
HO: Coax Connectors

We can also construct transmission lines on printed circuit boards.

HO: Printed Circuit Board Transmission Lines

The Telegrapher Equations

Consider a section of "wire":



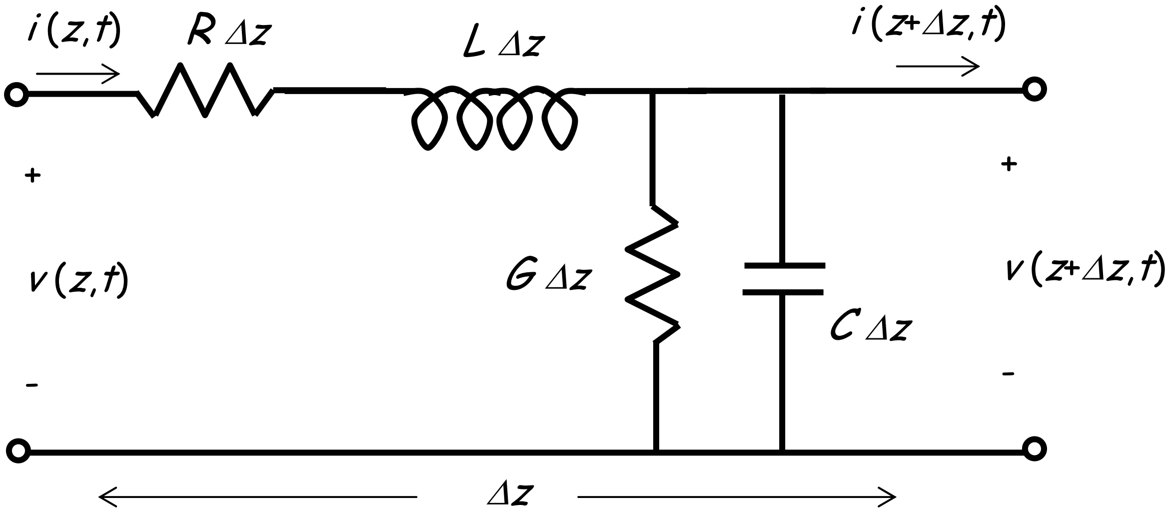
Q: *Huh ?! Current i and voltage v are a function of* ***position z*** *??*
Shouldn't $i(z,t) = i(z+\Delta z,t)$ *and* $v(z,t) = v(z+\Delta z,t)$ *?*

A: NO ! Because a wire is never a **perfect** conductor.

A "wire" will have:

1) Inductance
2) Resistance
3) Capacitance
4) Conductance

i.e.,



Where:

R = resistance/unit length
L = inductance/unit length
C = capacitance/unit length
G = conductance/unit length

$\therefore$ resistance of wire length Δz is $R\Delta z$.

Using KVL, we find:

$$v(z+\Delta z,t)-v(z,t)=-R\Delta z\, i(z,t)-L\Delta z\frac{\partial i(z,t)}{\partial t}$$

and from KCL:

$$i(z+\Delta z,t)-i(z,t)=-G\Delta z\, v(z,t)-C\Delta z\frac{\partial v(z,t)}{\partial t}$$

Dividing the first equation by Δz, and then taking the limit as $\Delta z \rightarrow 0$:

$$\lim_{\Delta z \rightarrow 0} \frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = -R\, i(z,t) - L\frac{\partial i(z,t)}{\partial t}$$

which, by definition of the derivative, becomes:

$$\frac{\partial v(z,t)}{\partial z} = -R\, i(z,t) - L\frac{\partial i(z,t)}{\partial t}$$

Similarly, the KCL equation becomes:

$$\frac{\partial i(z,t)}{\partial z} = -G\, v(z,t) - C\frac{\partial v(z,t)}{\partial t}$$

These equations are known as the **telegrapher's equations** !

$$\boxed{\begin{aligned} \frac{\partial v(z,t)}{\partial z} &= -R\, i(z,t) - L\frac{\partial i(z,t)}{\partial t} \\ \frac{\partial i(z,t)}{\partial z} &= -G\, v(z,t) - C\frac{\partial v(z,t)}{\partial t} \end{aligned}}$$

Time-Harmonic Solutions for Linear Circuits

There are an unaccountably **infinite** number of solutions $v(z,t)$ and $i(z,t)$ for the telegrapher's equations! However, we can simplify the problem by assuming that the function of time is **time harmonic** (i.e., sinusoidal), oscillating at some radial **frequency** ω (e.g., $\cos \omega t$).

Q: *Why on earth would we assume a **sinusoidal** function of time? Why not a **square wave**, or **triangle wave**, or a "sawtooth" function?*

A: We assume **sinusoids** because they have a **very** special property!

Sinusoidal time functions—and **only** a sinusoidal time functions—are the **eigen functions** of **linear, time-invariant** systems.

Q: *???*

A: If a sinusoidal voltage source with frequency ω is used to excite a linear, time-invariant circuit (and a transmission line is **both** linear **and** time invariant!), then the voltage at each and **every** point with the circuit will likewise vary sinusoidally—at the same frequency ω!

Q: *So what? Isn't that **obvious**?*

A: Not at all! If you were to excite a linear circuit with a **square wave**, or **triangle wave**, or **sawtooth**, you would find that—generally speaking—**nowhere** else in the circuit is the voltage a perfect square wave, triangle wave, or sawtooth. The linear circuit will effectively **distort** the input signal into something **else**!

Q: *Into what function will the input signal be distorted?*

A: It depends—both on the original form of the **input signal,** and the parameters of the **linear circuit**. At **different** points within the circuit we will discover **different** functions of time—**unless**, of course, we use a **sinusoidal** input. Again, for a sinusoidal excitation, we find at **every** point within circuit an **undistorted** sinusoidal function!

Q: *So, the sinusoidal function at every point in the circuit is **exactly** the same as the input sinusoid?*

A: Not quite **exactly** the same. Although at every point within the circuit the voltage will be precisely sinusoidal (with frequency ω), the **magnitude** and **relative phase** of the sinusoid will generally be different at each and every point within the circuit.

Thus, the voltage along a transmission line—when excited by a sinusoidal source—**must** have the form:

$$v(z,t) = v(z)\cos\left(\omega t + \varphi(z)\right)$$

Thus, at some arbitrary location z along the transmission line, we **must** find a time-harmonic oscillation of **magnitude** $v(z)$ and **relative phase** $\varphi(z)$.

Now, consider Euler's equation, which states:

$$e^{j\psi} = \cos\psi + j\sin\psi$$

Thus, it is apparent that:

$$Re\left\{e^{j\psi}\right\} = \cos\psi$$

and so we conclude that the voltage on a transmission line can be expressed as:

$$\begin{aligned} v(z,t) &= v(z)\cos\left(\omega t + \varphi(z)\right) \\ &= Re\left\{v(z)\, e^{j(\omega t + \varphi(z))}\right\} \\ &= Re\left\{v(z)\, e^{+j\varphi(z)} e^{j\omega t}\right\} \end{aligned}$$

Thus, we can specify the time-harmonic voltage at each an every location z along a transmission line with the **complex** function $V(z)$:

$$V(z) = v(z)\, e^{-j\varphi(z)}$$

where the **magnitude** of the complex function is the **magnitude** of the sinusoid:

$$v(z) = |V(z)|$$

and the phase of the complex function is the relative phase of the sinusoid :

$$\varphi(z) = arg\{V(z)\}$$

Q: *Hey wait a minute! What happened to the time-harmonic function $e^{j\omega t}$??*

A: There really is no reason to **explicitly** write the complex function $e^{j\omega t}$, since we know in fact (being the eigen function of linear systems and all) that if this is the time function at any **one** location (such as qt the excitation source) then this must be time function at **all** transmission line locations z!

The only **unknown** is the **complex** function $V(z)$. Once we determine $V(z)$, we can always (if we so desire) "recover" the **real** function $v(z,t)$ as:

$$v(z,t) = Re\{V(z)e^{j\omega t}\}$$

Thus, if we assume a **time-harmonic source**, finding the transmission line solution $v(z,t)$ reduces to solving for the **complex function** $V(z)$.

The Transmission Line Wave Equation

Let's assume that $v(z,t)$ and $i(z,t)$ each have the time-harmonic form:

$$v(z,t) = \mathrm{Re}\left\{V(z)e^{j\omega t}\right\} \quad \text{and} \quad i(z,t) = \mathrm{Re}\left\{I(z)e^{j\omega t}\right\}$$

The time-derivative of these functions are:

$$\frac{\partial v(z,t)}{\partial t} = \mathrm{Re}\left\{V(z)\frac{\partial e^{j\omega t}}{\partial t}\right\} = \mathrm{Re}\left\{j\omega V(z)e^{j\omega t}\right\}$$

$$\frac{\partial i(z,t)}{\partial t} = \mathrm{Re}\left\{I(z)\frac{\partial e^{j\omega t}}{\partial t}\right\} = \mathrm{Re}\left\{j\omega I(z)e^{j\omega t}\right\}$$

The telegrapher's equations thus become:

$$Re\left\{\frac{\partial V(z)}{\partial z}e^{j\omega t}\right\} = Re\left\{-(R+j\omega L)I(z)\,e^{j\omega t}\right\}$$

$$Re\left\{\frac{\partial I(z)}{\partial z}e^{j\omega t}\right\} = Re\left\{-(G+j\omega C)V(z)\,e^{j\omega t}\right\}$$

And then simplifying, we have the **complex** form of **telegrapher's equations**:

$$\frac{\partial V(z)}{\partial z} = -(R + j\omega L)\, I(z)$$

$$\frac{\partial I(z)}{\partial z} = -(G + j\omega C)\, V(z)$$

Note that these complex differential equations are **not** a function of time t !

* The functions *I(z)* and *V(z)* are **complex,** where the **magnitude** and **phase** of the complex functions describe the **magnitude** and **phase** of the sinusoidal time function $e^{j\omega t}$.

* Thus, *I(z)* and *V(z)* describe the current and voltage along the transmission line, as a function as position *z*.

* **Remember**, not just **any** function *I(z)* and *V(z)* can exist on a transmission line, but rather **only** those functions that satisfy the **telegraphers equations**.

Our task, therefore, is to **solve** the telegrapher equations and find **all** solutions $I(z)$ and $V(z)$!

Q: *So, what functions $I(z)$ and $V(z)$* ***do*** *satisfy both telegrapher's equations??*

A: To make this easier, we will combine the telegrapher equations to form **one** differential equation for $V(z)$ and **another** for $I(z)$.

First, take the **derivative** with respect to z of the **first** telegrapher equation:

$$\frac{\partial}{\partial z}\left\{\frac{\partial V(z)}{\partial z} = -(R + j\omega L)\, I(z)\right\}$$

$$= \frac{\partial^2 V(z)}{\partial z^2} = -(R + j\omega L)\frac{\partial I(z)}{\partial z}$$

Note that the **second** telegrapher equation expresses the derivative of $I(z)$ in terms of $V(z)$:

$$\frac{\partial I(z)}{\partial z} = -(G + j\omega C)\, V(z)$$

Combining these two equations, we get an equation involving $V(z)$ **only**:

$$\frac{\partial^2 V(z)}{\partial z^2} = (R + j\omega L)(G + j\omega C)\, V(z)$$

Now, we find at high frequencies that:

$$R \ll j\omega L \quad \text{and} \quad G \ll j\omega C$$

and so we can **approximate** the differential equation as:

$$\frac{\partial^2 V(z)}{\partial z^2} = (j\omega L)(j\omega C)V(z) = \omega^2 LC\, V(z) = \beta^2\, V(z)$$

where it is apparent that:

$$\boxed{\beta^2 \doteq \omega^2 LC}$$

In a **similar** manner (i.e., begin by taking the derivative of the **second** telegrapher equation), we can derive the differential equation:

$$\frac{\partial^2 I(z)}{\partial z} = \beta^2\, I(z)$$

We have **decoupled** the telegrapher's equations, such that we now have **two** equations involving **one** function only:

$$\boxed{\begin{gathered}\frac{\partial^2 V(z)}{\partial z} = \beta^2\, V(z)\\[2ex] \frac{\partial^2 I(z)}{\partial z} = \beta^2\, I(z)\end{gathered}}$$

These are known as the **transmission line wave equations**.

Note only **special** functions satisfy these equations: if we take the double derivative of the function, the result is the **original function** (to within a constant)!

Q: *Yeah right! Every function that I know is **changed** after a double differentiation. What kind of "magical" function could possibly satisfy this differential equation?*

A: Such functions **do** exist !

For example, the functions $V(z) = e^{-j\beta z}$ and $V(z) = e^{+j\beta z}$ each satisfy this transmission line wave equation (**insert** these into the differential equation and see for **yourself**!).

Likewise, since the transmission line wave equation is a **linear** differential equation, a weighted **superposition** of the two solutions is **also a solution** (again, **insert** this solution to and see for **yourself**!):

$$V(z) = V_0^+ \, e^{-j\beta z} + V_0^- \, e^{+j\beta z}$$

In fact, it turns out that **any** and **all** possible solutions to the differential equations can be expressed in **this** simple form!

Therefore, the **general** solution to these wave equations (and thus the telegrapher equations) are:

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

$$I(z) = I_0^+ e^{-j\beta z} + I_0^- e^{+j\beta z}$$

where V_0^+, V_0^-, I_0^+, and I_0^- are **complex constants**.

→ It is **unfathomably** important that **you** understand what this result means!

It means that the functions $V(z)$ and $I(z)$, describing the current and voltage at **all** points z along a transmission line, can **always** be **completely** specified with just **four complex constants** ($V_0^+, V_0^-, I_0^+, I_0^-$)!!

We can **alternatively** write these solutions as:

$$V(z) = V^+(z) + V^-(z)$$

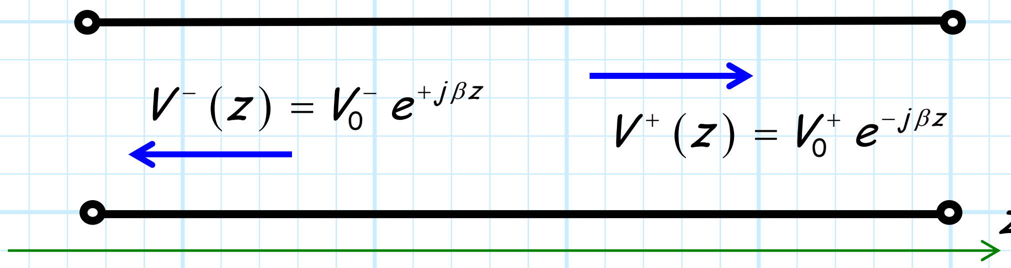
$$I(z) = I^+(z) + I^-(z)$$

where:

$$V^+(z) \doteq V_0^+ e^{-j\beta z} \qquad V^-(z) \doteq V_0^- e^{+j\beta z}$$

$$I^+(z) \doteq I_0^+ e^{-j\beta z} \qquad I^-(z) \doteq I_0^- e^{+j\beta z}$$

The two terms in each solution describe **two waves** propagating in the transmission line, **one** wave ($V^+(z)$ or $I^+(z)$) propagating in one direction (**+z**) and the **other** wave ($V^-(z)$ or $I^-(z)$) propagating in the **opposite** direction (**-z**).



Q: *So just what **are** the complex values V_0^+, V_0^-, I_0^+, I_0^- ?*

A: Consider the wave solutions at **one** specific point on the transmission line—the point $z = 0$. For example, we find that:

$$\begin{aligned} V^+(z=0) &= V_0^+ e^{-j\beta(z=0)} \\ &= V_0^+ e^{-(0)} \\ &= V_0^+ (1) \\ &= V_0^+ \end{aligned}$$

In other words, V_0^+ is simply the **complex** value of the wave function $V^+(z)$ at the point z =0 on the transmission line!

Likewise, we find:

$$V_0^- = V^-(z=0)$$

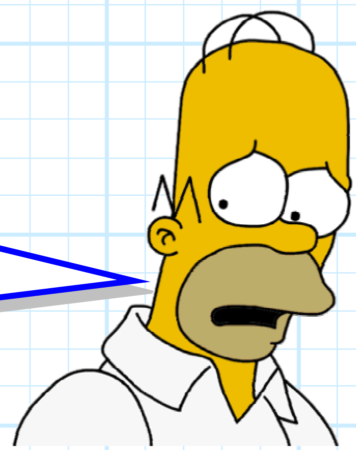
$$I_0^+ = I^+(z=0)$$

$$I_0^- = I^-(z=0)$$

Again, the four complex values V_0^+, I_0^+, V_0^-, I_0^- are **all** that is needed to determine the voltage and current at any and all points on the transmission line.

More specifically, **each** of these four complex constants completely specifies **one** of the four transmission line wave functions $V^+(z)$, $I^+(z)$, $V^-(z)$, $I^-(z)$.

Q: *But what **determines** these wave functions? How do we **find** the values of constants V_0^+, I_0^+, V_0^-, I_0^-?*



A: As you might expect, the voltage and current on a transmission line is determined by the devices **attached** to it on either end (e.g., active sources and/or passive loads)!

The precise values of V_0^+, I_0^+, V_0^-, I_0^- are therefore determined by satisfying the **boundary conditions** applied at **each end** of the transmission line—much more on this **later**!

The Characteristic Impedance of a Transmission Line

So, from the telegrapher's differential equations, we know that the complex current $I(z)$ and voltage $V(z)$ **must** have the form:

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

$$I(z) = I_0^+ e^{-j\beta z} + I_0^- e^{+j\beta z}$$

Let's insert the expression for $V(z)$ into the first telegrapher's equation, and **see what happens** !

$$\frac{dV(z)}{dz} = -j\beta V_0^+ e^{-j\beta z} + j\beta V_0^- e^{+j\beta z} = -j\omega L\, I(z)$$

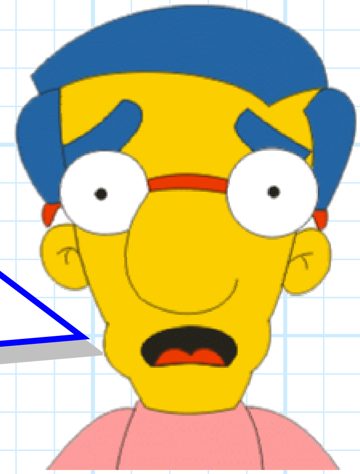
Therefore, rearranging, $I(z)$ must be:

$$I(z) = \frac{\beta}{\omega L}\left(V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}\right)$$

Q: ***But wait*** *! I thought we already knew current I(z). Isn't it:*

$$I(z) = I_0^+ e^{-j\beta z} + I_0^- e^{+j\beta z} \quad ??$$

How can ***both*** *expressions for I(z) be true??*



A: Easy ! Both expressions for current are **equal** to each other.

$$I(z) = I_0^+ e^{-j\beta z} + I_0^- e^{+j\beta z} = \frac{\beta}{\omega L}\left(V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}\right)$$

For the above equation to be true for **all** z, I_0 and V_0 must be related as:

$$I_0^+ e^{-\gamma z} = \left(\frac{\beta}{\omega L}\right) V_0^+ e^{-\gamma z} \qquad \text{and} \qquad I_0^- e^{+\gamma z} = \left(\frac{-\beta}{\omega L}\right) V_0^- e^{+\gamma z}$$

Or—recalling that $V_0^+ e^{-j\beta z} = V^+(z)$ (etc.)—we can express this in terms of the **two propagating waves**:

$$I^+(z) = \left(\frac{\beta}{\omega L}\right) V^+(z) \qquad \text{and} \qquad I^-(z) = \left(\frac{-\beta}{\omega L}\right) V^-(z)$$

Now, we note that since:

$$\beta = \omega\sqrt{LC}$$

We find that:

$$\frac{\beta}{\omega L} = \frac{\omega\sqrt{LC}}{\omega L} = \sqrt{\frac{C}{L}}$$

Thus, we come to the **startling** conclusion that:

$$\boxed{\frac{V^+(z)}{I^+(z)} = \sqrt{\frac{L}{C}} \quad \text{and} \quad \frac{-V^-(z)}{I^-(z)} = \sqrt{\frac{L}{C}}}$$

Q: *What's so startling about **this** conclusion?*

A: Note that although the magnitude and phase of each propagating wave is a **function** of transmission line **position** z (e.g., $V^+(z)$ and $I^+(z)$), the **ratio** of the voltage and current of **each wave** is independent of position—a **constant** with respect to position z!

Although $V_0^{\pm}$ and $I_0^{\pm}$ are determined by **boundary conditions** (i.e., what's connected to either end of the transmission line), the **ratio** $V_0^{\pm}/I_0^{\pm}$ is determined by the parameters of the transmission line **only** (R, L, G, C).

→ This ratio is an important **characteristic of a transmission line**, called its **Characteristic Impedance Z_0**.

$$Z_0 \doteq \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-} = \sqrt{\frac{L}{C}}$$

We can therefore describe the current and voltage along a transmission line as:

$$V(z) = V_0^+ \, e^{-j\beta z} + V_0^- \, e^{+j\beta z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z}$$

or equivalently:

$$V(z) = Z_0 \, I_0^+ \, e^{-j\beta z} - Z_0 \, I_0^- \, e^{+j\beta z}$$

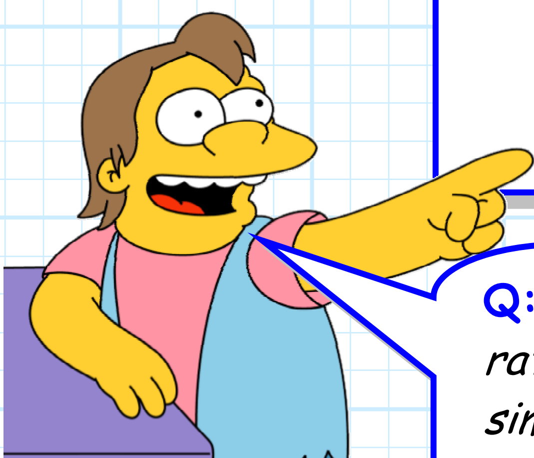
$$I(z) = I_0^+ \, e^{-j\beta z} + I_0^- \, e^{+j\beta z}$$

Note that instead of characterizing a transmission line with **real** parameters L and C, we can (and typically do!) describe a lossless transmission line using real parameters Z_0 and β.

Line Impedance

Now let's define **line impedance** $Z(z)$, a **complex** function which is simply the ratio of the complex line **voltage** and complex line **current**:

$$\boxed{Z(z) = \frac{V(z)}{I(z)}}$$



Q: *Hey! I know what this is! The ratio of the voltage to current is simply the **characteristic impedance** Z_0, right ???*

A: NO! The line impedance $Z(z)$ is (generally speaking) **NOT** the transmission line **characteristic** impedance Z_0 !!!

→ It is **unfathomably important** that you understand this!!!!

To see why, recall that:

$$V(z) = V^+(z) + V^-(z)$$

And that:

$$I(z) = \frac{V^+(z) - V^-(z)}{Z_0}$$

Therefore:

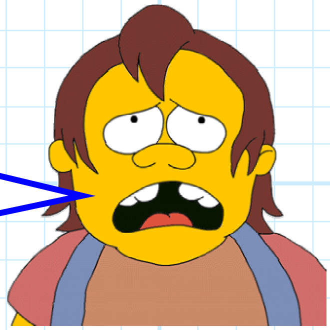
$$\boxed{Z(z) = \frac{V(z)}{I(z)} = Z_0 \left(\frac{V^+(z) + V^-(z)}{V^+(z) - V^-(z)} \right) \neq Z_0}$$

Or, more specifically, we can write:

$$Z(z) = Z_0 \left(\frac{V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}}{V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}} \right)$$

Q: *I'm confused! Isn't:*

$$V^+(z)/I^+(z) = Z_0 \;???$$



A: Yes! That **is** true! The ratio of the voltage to current for **each** of the two propagating waves is $\pm Z_0$. However, the ratio of the **sum** of the two voltages to the **sum** of the two currents is **not** equal to Z_0 (generally speaking)!

This is actually confirmed by the equation above. Say that $V^-(z) = 0$, so that only **one** wave ($V^+(z)$) is propagating on the line.

In this case, the ratio of the **total** voltage to the total current is simply the ratio of the voltage and current of the **one** remaining wave—the **characteristic impedance** Z_0!

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \left(\frac{V^+(z)}{V^+(z)} \right) = \frac{V^+(z)}{I^+(z)} = Z_0 \quad \text{(when } V^-(z) = 0\text{)}$$

Q: *So, it appears to me that characteristic impedance Z_0 is a **transmission line parameter**, depending **only** on the transmission line values L and C.*

*Whereas **line impedance** is $Z(z)$ depends the magnitude and phase of the two propagating waves $V^+(z)$ and $V^-(z)$--values that depend **not only** on the transmission line, but also on the two things **attached** to either **end** of the transmission line!*

Right !?

A: Exactly! Moreover, note that characteristic impedance Z_0 is simply a **number**, whereas line impedance $Z(z)$ is a **function** of position (z) on the transmission line.

The Reflection Coefficient

So, we know that the transmission line **voltage** $V(z)$ and the transmission line **current** $I(z)$ can be related by the **line impedance** $Z(z)$:

$$V(z) = Z(z)\, I(z)$$

or equivalently:

$$I(z) = \frac{V(z)}{Z(z)}$$

Q: *Piece of cake! I fully understand the concepts of **voltage**, **current** and **impedance** from my **circuits** classes. Let's move on to something more important (or, at the very least, more **interesting**).*

Expressing the "activity" on a transmission line in terms of **voltage**, **current** and **impedance** is of course **perfectly** valid.

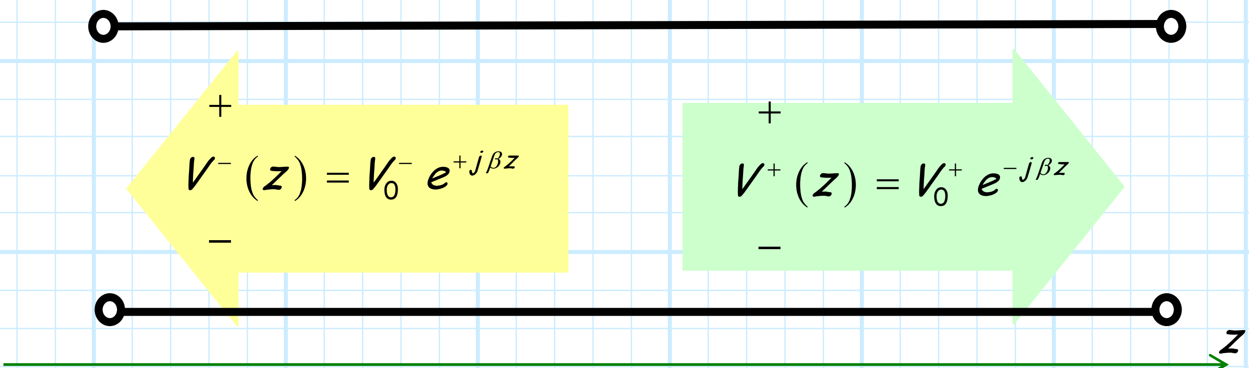
However, let us look **closer** at the expression for each of these quantities:

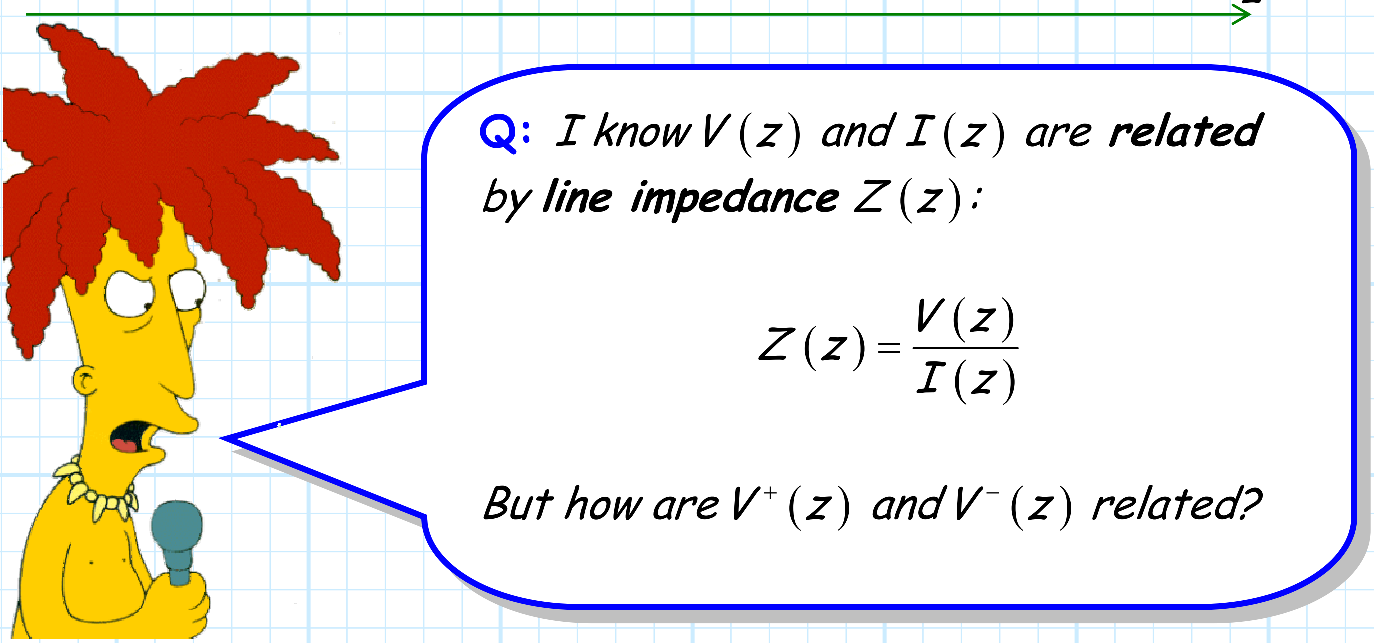
$$V(z) = V^+(z) + V^-(z)$$

$$I(z) = \frac{V^+(z) - V^-(z)}{Z_0}$$

$$Z(z) = Z_0 \left(\frac{V^+(z) + V^-(z)}{V^+(z) - V^-(z)} \right)$$

It is evident that we can **alternatively** express all "activity" on the transmission line in terms of the two transmission line **waves** $V^+(z)$ and $V^-(z)$.





A: Similar to line impedance, we can define a new parameter—the **reflection coefficient** $\Gamma(z)$—as the **ratio** of the two quantities:

$$\boxed{\Gamma(z) \doteq \frac{V^-(z)}{V^+(z)} \quad \Rightarrow \quad V^-(z) = \Gamma(z) V^+(z)}$$

More specifically, we can express $\Gamma(z)$ as:

$$\Gamma(z) = \frac{V_0^- \, e^{+j\beta z}}{V_0^+ \, e^{-j\beta z}} = \frac{V_0^-}{V_0^+} \; e^{+j2\beta z}$$

Note then, the value of the reflection coefficient at z=0 is:

$$\Gamma(z=0) = \frac{V^-(z=0)}{V_0^+(z=0)} \; e^{+j2\beta(0)} = \frac{V_0^-}{V_0^+}$$

We define this value as Γ_0, where:

$$\boxed{\Gamma_0 \doteq \Gamma(z=0) = \frac{V_0^-}{V_0^+}}$$

Note then that we can alternatively write $\Gamma(z)$ as:

$$\boxed{\Gamma(z) = \Gamma_0 \; e^{+j2\beta z}}$$

So now we have **two different** but equivalent ways to describe transmission line activity!

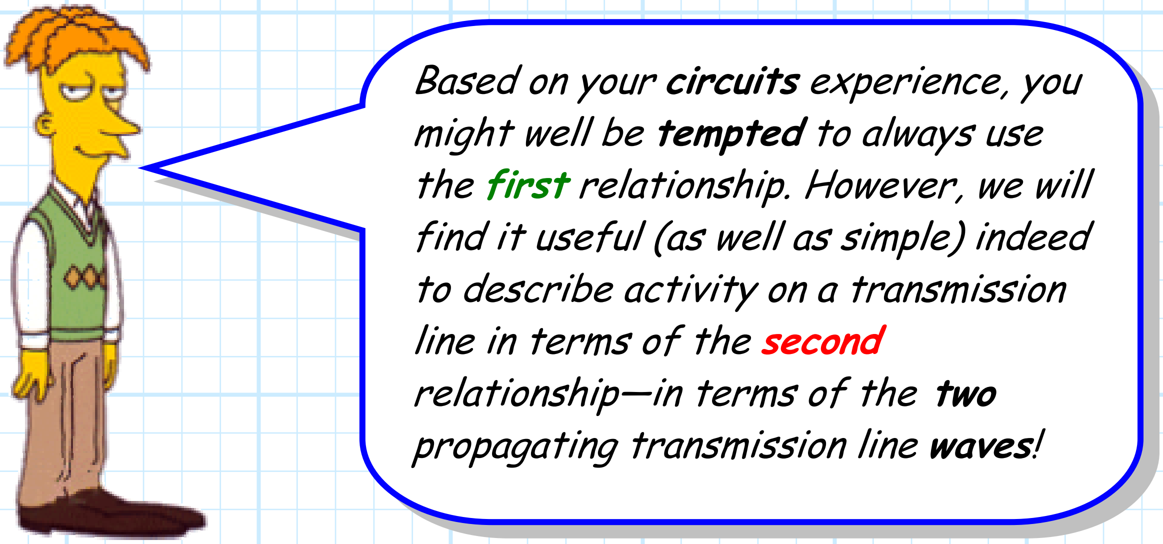
We can use (total) **voltage** and **current**, related by **line impedance**:

$$Z(z) = \frac{V(z)}{I(z)} \quad \therefore \quad V(z) = Z(z)\, I(z)$$

Or, we can use the two propagating **voltage waves**, related by the **reflection coefficient**:

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} \quad \therefore \quad V^-(z) = \Gamma(z)\, V^+(z)$$

These are **equivalent** relationships—we can use **either** when describing a transmission line.



V, I, Z or V^+, V^-, Γ ?

Q: *How do I choose **which** relationship to use when describing/analyzing transmission line activity? What if I make the **wrong** choice? How will I know **if** my analysis is correct?*

A: Remember, the two relationships are **equivalent**. There is **no** explicitly wrong or right choice—**both** will provide you with precisely the **same** correct answer!

For example, we know that the total voltage and current can be determined from knowledge wave representation:

$$
\begin{aligned}
V(z) &= V^+(z) + V^+(z) \\
&= V^+(z)\big(1+\Gamma(z)\big)
\end{aligned}
$$

$$
\begin{aligned}
I(z) &= \frac{V^+(z) - V^+(z)}{Z_0} \\
&= \frac{V^+(z)\big(1-\Gamma(z)\big)}{Z_0}
\end{aligned}
$$

Or explicitly using the wave solutions $V^+(z) = V_0^+ e^{-j\beta z}$ and $V^-(z) = V_0^- e^{+j\beta z}$:

$$\begin{aligned} V(z) &= V_0^+ \, e^{-j\beta z} + V_0^- \, e^{+j\beta z} \\ &= V_0^+ \left(e^{-j\beta z} + \Gamma_0 \, e^{+j\beta z} \right) \end{aligned}$$

$$\begin{aligned} I(z) &= \frac{V_0^+ \, e^{-j\beta z} - V_0^- \, e^{+j\beta z}}{Z_0} \\ &= \frac{V_0^+ \left(e^{-j\beta z} - \Gamma_0 \, e^{+j\beta z} \right)}{Z_0} \end{aligned}$$

More importantly, we find that **line impedance** $Z(z) = V(z)/I(z)$ can be expressed as:

$$\boxed{\begin{aligned} Z(z) &= Z_0 \frac{V^+(z) + V^+(z)}{V^+(z) - V^+(z)} \\ &= Z_0 \left(\frac{1 + \Gamma(z)}{1 - \Gamma(z)} \right) \end{aligned}}$$

Look what happened—the line impedance can be **completely** and unambiguously expressed in terms of **reflection coefficient** $\Gamma(z)$!

More explicitly:

$$Z(z) = Z_0 \frac{V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}}{V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}}$$

$$= Z_0 \frac{1 + \Gamma_0 e^{+j2\beta z}}{1 - \Gamma_0 e^{+j2\beta z}}$$

With a little algebra, we find likewise that the wave functions can be determined from $V(z), I(z)$ and $Z(z)$:

$$\boxed{\begin{aligned} V^+(z) &= \frac{V(z) + I(z)Z_0}{2} \\ &= \frac{V(z)}{Z(z)}\left(\frac{Z(z) + Z_0}{2}\right) \\ V^-(z) &= \frac{V(z) - I(z)Z_0}{2} \\ &= \frac{V(z)}{Z(z)}\left(\frac{Z(z) - Z_0}{2}\right) \end{aligned}}$$

From this result we easily find that the reflection coefficient $\Gamma(z)$ can **likewise** be written directly in terms of **line impedance**:

$$\boxed{\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0}}$$

Thus, the values $\Gamma(z)$ and $Z(z)$ are **equivalent** parameters—if we know **one**, then we can directly determine the **other**!

Q: *So, if they are equivalent, why wouldn't I **always** use the current, voltage, line impedance representation? After all, I am more **familiar** and more confident those quantities. The **wave** representation sort of **scares** me!*

A: Perhaps I can **convince** you of the value of the **wave** representation.

Remember, the time-harmonic solution to the telegraphers equation simply boils down to **two complex constants**—V_0^+ and V_0^-. Once these complex values have been determined, we can describe **completely** the activity **all** points along our transmission line.

For the **wave representation** we find:

$$V^+(z) = V_0^+ \, e^{-j\beta z}$$

$$V^-(z) = V_0^+ \, e^{+j\beta z}$$

$$\Gamma(z) = \frac{V_0^-}{V_0^+} e^{+j2\beta z}$$

Note that the **magnitudes** of the complex functions are in fact **constants** (with respect to position *z*):

$$\left|V^{+}(z)\right| = \left|V_0^{+}\right|$$

$$\left|V^{-}(z)\right| = \left|V_0^{+}\right|$$

$$\left|\Gamma(z)\right| = \left|\frac{V_0^{-}}{V_0^{+}}\right|$$

While the **relative phase** of these complex functions are expressed as a **simple** linear relationship with respect to z:

$$\boldsymbol{arg}\left\{V^{+}(z)\right\} = -\beta z$$

$$\boldsymbol{arg}\left\{V^{-}(z)\right\} = +\beta z$$

$$\boldsymbol{arg}\left\{\Gamma(z)\right\} = +2\beta z$$

Now, **contrast** this with the complex current, voltage, impedance functions:

$$V(z) = V_0^+ \, e^{-j\beta z} + V_0^- \, e^{+j\beta z}$$

$$I(z) = \frac{V_0^+ \, e^{-j\beta z} - V_0^- \, e^{+j\beta z}}{Z_0}$$

$$Z(z) = Z_0 \, \frac{V_0^+ \, e^{-j\beta z} + V_0^- \, e^{+j\beta z}}{V_0^+ \, e^{-j\beta z} - V_0^- \, e^{+j\beta z}}$$

With magnitude:

$$|V(z)| = \left| V_0^+ \, e^{-j\beta z} + V_0^- \, e^{+j\beta z} \right| = ??$$

$$|I(z)| = \frac{\left| V_0^+ \, e^{-j\beta z} - V_0^- \, e^{+j\beta z} \right|}{Z_0} = ??$$

$$|Z(z)| = Z_0 \, \frac{\left| V_0^+ \, e^{-j\beta z} + V_0^- \, e^{+j\beta z} \right|}{\left| V_0^+ \, e^{-j\beta z} - V_0^- \, e^{+j\beta z} \right|} = ??$$

and phase:

$$\textbf{arg}\{V(z)\} = \textbf{arg}\left\{V_0^+ \, e^{-j\beta z} + V_0^- \, e^{+j\beta z}\right\} = ??$$

$$\textbf{arg}\{I(z)\} = \textbf{arg}\left\{V_0^+ \, e^{-j\beta z} - V_0^- \, e^{+j\beta z}\right\} = ??$$

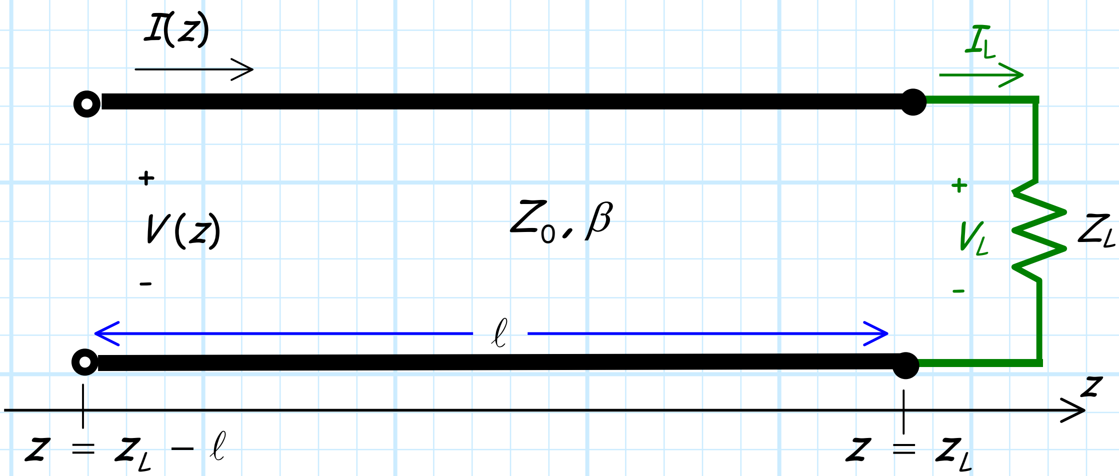
$$\begin{aligned} \textbf{arg}\{Z(z)\} &= \textbf{arg}\left\{V_0^+ \, e^{-j\beta z} + V_0^- \, e^{+j\beta z}\right\} \\ &\quad - \textbf{arg}\left\{V_0^+ \, e^{-j\beta z} - V_0^- \, e^{+j\beta z}\right\} \\ &= ?? \end{aligned}$$

Q: *It appears to me that when attempting to describe the activity along a transmission line—as a function of **position** z—it is much **easier** and more **straightforward** to use the **wave** representation. Is my insightful conclusion **correct** (nyuck, nyuck, nyuck)?*

A: Yes it is! However, this does **not** mean that we **never** determine $V(z)$, $I(z)$, or $Z(z)$; these quantities are still **fundamental** and very important—particularly at each **end** of the transmission line!

The Terminated, Lossless Transmission Line

Now let's **attach** something to our transmission line. Consider a **lossless** line, length ℓ, terminated with a **load** Z_L.



Q: *What is the **current** and **voltage** at each and **every** point on the transmission line (i.e., what is $I(z)$ and $V(z)$ for **all** points z where $z_L - \ell \leq z \leq z_L$?)?*

A: To find out, we must apply **boundary conditions**!

In other words, at the **end** of the transmission line ($z = z_L$)—where the load is **attached**—we have **many** requirements that **all** must be satisfied!

1. To begin with, the voltage and current ($I(z=z_L)$ and $V(z=z_L)$) must be consistent with a valid **transmission line solution**:

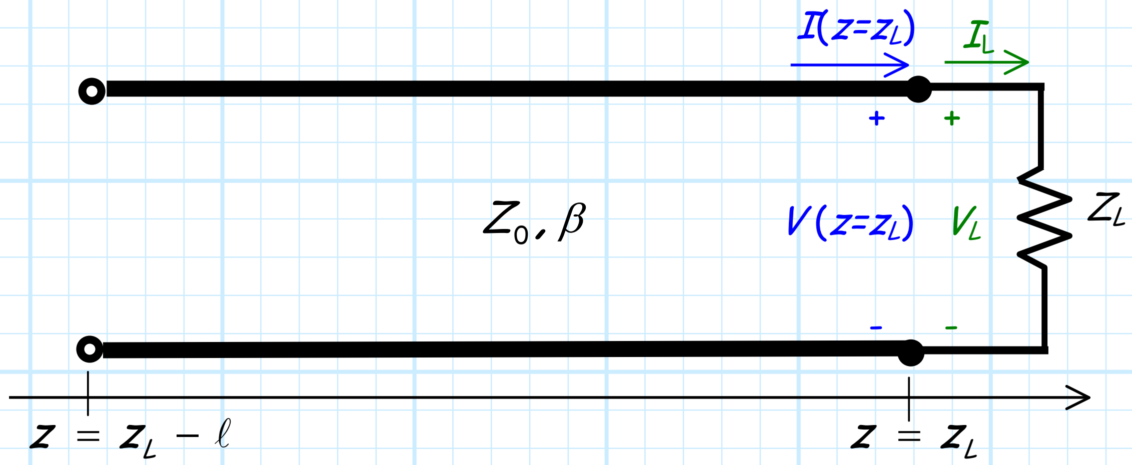
$$\begin{aligned} V(z=z_L) &= V^+(z=z_L) + V^-(z=z_L) \\ &= V_0^+ e^{-j\beta z_L} + V_0^- e^{+j\beta z_L} \end{aligned}$$

$$\begin{aligned} I(z=z_L) &= \frac{V_0^+(z=z_L)}{Z_0} - \frac{V_0^-(z=z_L)}{Z_0} \\ &= \frac{V_0^+}{Z_0} e^{-j\beta z_L} - \frac{V_0^-}{Z_0} e^{+j\beta z_L} \end{aligned}$$

2. Likewise, the load voltage and current must be related by **Ohm's law**:

$$V_L = Z_L I_L$$

3. Most importantly, we recognize that the values $I(z=z_L)$, $V(z=z_L)$ and I_L, V_L are **not** independent, but in fact are strictly related by **Kirchoff's Laws**!

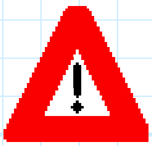


From KVL and KCL we find these requirements:

$$V(z = z_L) = V_L$$

$$I(z = z_L) = I_L$$

These are the **boundary conditions** for **this** particular problem.



→ **Careful**! Different transmission line problems lead to **different** boundary conditions—**you** must access each problem **individually** and **independently**!

Combining these equations and boundary conditions, we find that:

$$V_L = Z_L \, I_L$$

$$V(z = z_L) = Z_L \, I(z = z_L)$$

$$V^+(z = z_L) + V^-(z = z_L) = \frac{Z_L}{Z_0} \left(V^+(z = z_L) - V^-(z = z_L) \right)$$

Rearranging, we can conclude:

$$\frac{V^-(z = z_L)}{V^+(z = z_L)} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Q: *Hey wait as second! We earlier defined $V^-(z)/V^+(z)$ as **reflection coefficient** $\Gamma(z)$. How does this relate to the expression **above**?*

A: Recall that $\Gamma(z)$ is a **function** of transmission line position z. The value $V^-(z=z_L)/V^+(z=z_L)$ is simply the value of function $\Gamma(z)$ **evaluated** at $z=z_L$ (i.e., evaluated at the **end** of the line):

$$\frac{V^-(z=z_L)}{V^+(z=z_L)} = \Gamma(z=z_L) = \frac{Z_L - Z_0}{Z_L + Z_0}$$

This value is of **fundamental** importance for the terminated transmission line problem, so we provide it with its **own** special symbol (Γ_L)!

$$\boxed{\Gamma_L \doteq \Gamma(z=z_L) = \frac{Z_L - Z_0}{Z_L + Z_0}}$$

Q: *Wait! We **earlier** determined that:*

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0}$$

so it would seem that:

$$\Gamma_L = \Gamma(z=z_L) = \frac{Z(z=z_L) - Z_0}{Z(z=z_L) + Z_0}$$

***Which** expression is correct??*

A: They **both** are! It is evident that the two expressions:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \quad \text{and} \quad \Gamma_L = \frac{Z(z = z_L) - Z_0}{Z(z = z_L) + Z_0}$$

are **equal** if:

$$Z(z = z_L) = Z_L$$

And since we know that from **Ohm's Law**:

$$Z_L = \frac{V_L}{I_L}$$

and from **Kirchoff's Laws**:

$$\frac{V_L}{I_L} = \frac{V(z = z_L)}{I(z = z_L)}$$

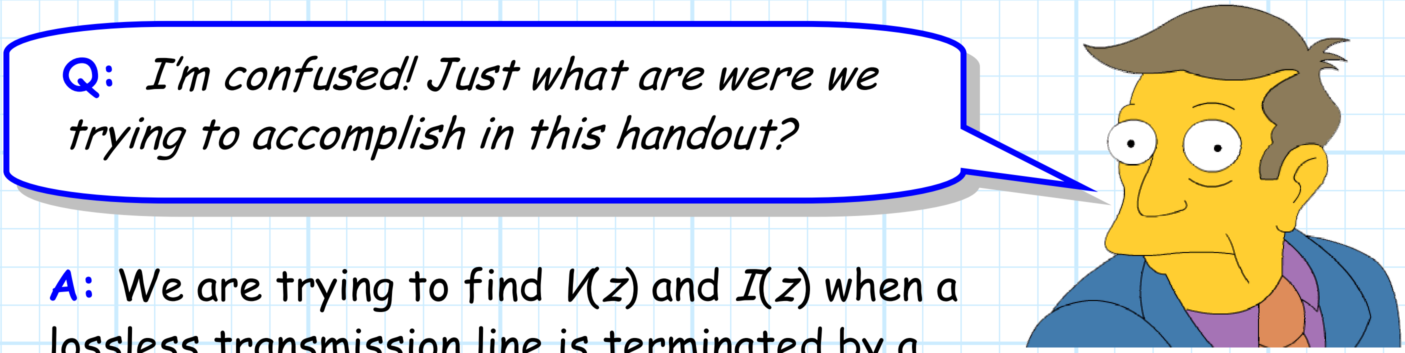
and that **line impedance** is:

$$\frac{V(z = z_L)}{I(z = z_L)} = Z(z = z_L)$$

we find it apparent that the **line impedance** at the **end** of the transmission line is **equal** to the **load impedance**:

$$Z(z = z_L) = Z_L$$

The above expression is essentially **another** expression of the **boundary condition** applied at the **end** of the transmission line.



A: We are trying to find $V(z)$ and $I(z)$ when a lossless transmission line is terminated by a load Z_L!

We can now determine the value of V_0^- in terms of V_0^+. Since:

$$\Gamma_L = \frac{V^-(z=z_L)}{V^+(z=z_L)} = \frac{V_0^- e^{+j\beta z_L}}{V_0^+ e^{-j\beta z_L}}$$

We find:

$$V_0^- = e^{-2j\beta z_L}\,\Gamma_L\,V_0^+$$

And therefore we find:

$$V^-(z) = \left(e^{-2j\beta z_L}\,\Gamma_L\,V_0^+\right)e^{+j\beta z}$$

$$V(z) = V_0^+\left[e^{-j\beta z} + \left(e^{-2j\beta z_L}\,\Gamma_L\right)e^{+j\beta z}\right]$$

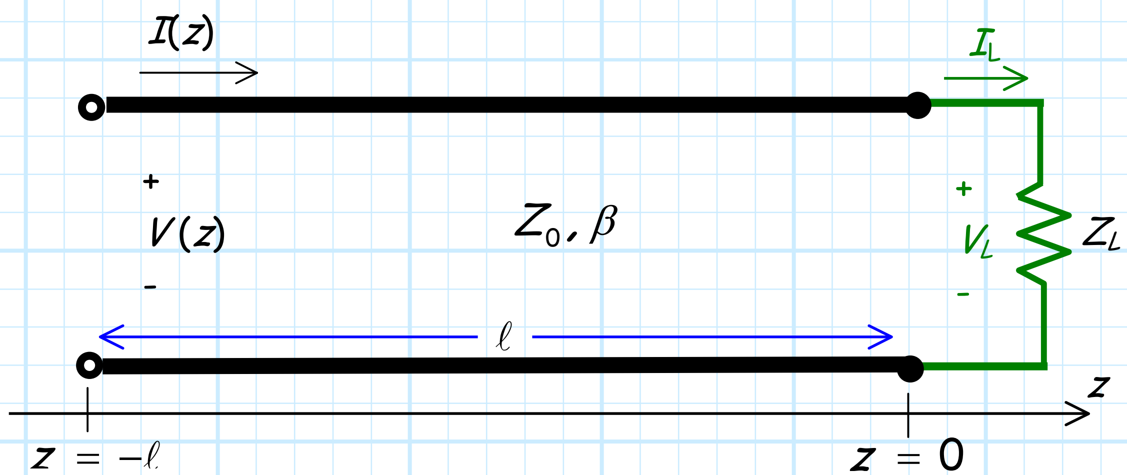
$$I(z) = \frac{V_0^+}{Z_0}\left[e^{-j\beta z} - \left(e^{-2j\beta z_L}\,\Gamma_L\right)e^{+j\beta z}\right]$$

where:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$z_L = 0$

Now, we can further **simplify** our analysis by **arbitrarily** assigning the end point z_L a **zero** value (i.e., $z_L = 0$):



If the load is located at z=0 (i.e., **if** $z_L = 0$), we find that:

$$\begin{aligned} V(z=0) &= V^+(z=0) + V^-(z=0) \\ &= V_0^+ e^{-j\beta(0)} + V_0^- e^{+j\beta(0)} \\ &= V_0^+ + V_0^- \end{aligned}$$

$$\begin{aligned} I(z=0) &= \frac{V_0^+(z=0)}{Z_0} - \frac{V_0^-(z=0)}{Z_0} \\ &= \frac{V_0^+}{Z_0} e^{-j\beta(0)} - \frac{V_0^-}{Z_0} e^{+j\beta(0)} \\ &= \frac{V_0^+ - V_0^-}{Z_0} \end{aligned}$$

$$Z(z=0) = Z_0 \left(\frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} \right)$$

Likewise, it is apparent that **if** $z_L = 0$, Γ_L and Γ_0 are the same:

$$\Gamma_L = \Gamma(z = z_L) = \frac{V^-(z=0)}{V^+(z=0)} = \frac{V_0^-}{V_0^+} = \Gamma_0$$

Therefore:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_0$$

Thus, we can write the line current and voltage simply as:

$$V(z) = V_0^+ \left[e^{-j\beta z} + \Gamma_L \, e^{+j\beta z} \right]$$

$$\left[\text{for } z_L = 0\right]$$

$$I(z) = \frac{V_0^+}{Z_0} \left[e^{-j\beta z} - \Gamma_L \, e^{+j\beta z} \right]$$

Q: *But, how do we determine V_0^+ ??*

A: We require a **second** boundary condition to determine V_0^+. The only boundary left is at the **other end** of the transmission line. Typically, a **source** of some sort is located there. This makes physical sense, as something must generate the **incident** wave !

Special Values of Load Impedance

It's interesting to note that the load Z_L enforces a boundary condition that explicitly determines neither $V(z)$ nor $I(z)$—but **completely** specifies **line impedance** $Z(z)$!

$$Z(z) = Z_0 \frac{e^{-j\beta z} + \Gamma_L e^{+j\beta z}}{e^{-j\beta z} - \Gamma_L e^{+j\beta z}} = Z_0 \frac{Z_L \cos\beta z - jZ_0 \sin\beta z}{Z_0 \cos\beta z - jZ_L \sin\beta z}$$

$$\Gamma(z) = \Gamma_L e^{+j2\beta z} = \frac{Z_L - Z_0}{Z_L + Z_0} e^{+j2\beta z}$$

Likewise, the load boundary condition leaves $V^+(z)$ and $V^-(z)$ undetermined, but **completely** determines **reflection coefficient function** $\Gamma(z)$!

Let's look at some **specific** values of load impedance $Z_L = R_L + jX_L$ and see what functions $Z(z)$ and $\Gamma(z)$ result!

1. $Z_L = Z_0$

In this case, the **load** impedance is **numerically** equal to the **characteristic** impedance of the transmission line. Assuming the line is **lossless**, then Z_0 is **real**, and thus:

$$R_L = Z_0 \qquad \text{and} \qquad X_L = 0$$

It is evident that the resulting load reflection coefficient is **zero**:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_0 - Z_0}{Z_0 + Z_0} = 0$$

This result is very interesting, as it means that there is **no reflected wave** $V^-(z)$!

Thus, the **total** voltage and current along the transmission line is simply voltage and current of the **incident** wave:

$$V(z) = V^+(z) = V_0^+ e^{-j\beta z}$$

$$I(z) = I^+(z) = \frac{V_0^+}{Z_0} e^{-j\beta z}$$

Meaning that the **line** impedance is likewise **numerically** equal to the **characteristic** impedance of the transmission line for **all** line position z:

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{V_0^+ e^{-j\beta z}}{V_0^+ e^{-j\beta z}} = Z_0$$

And likewise, the reflection coefficient is **zero** at **all** points along the line:

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{0}{V^+(z)} = 0$$

We call this condition (when $Z_L = Z_0$) the **matched** condition, and the load $Z_L = Z_0$ a **matched load**.

2. $Z_L = jX_L$

For this case, the load impedance is **purely reactive** (e.g. a capacitor of inductor), the real (resistive) portion of the load is zero:

$$R_L = 0$$

The resulting load reflection coefficient is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{jX_L - Z_0}{jX_L + Z_0}$$

Given that Z_0 is real (i.e., the line is **lossless**), we find that this load reflection coefficient is generally some **complex** number.

We **can** rewrite this value explicitly in terms of its **real** and **imaginary** part as:

$$\Gamma_L = \frac{jX_L - Z_0}{jX_L + Z_0} = \left(\frac{X_L^2 - Z_0^2}{X_L^2 + Z_0^2}\right) + j\left(\frac{2\,Z_0\,X_L}{X_L^2 + Z_0^2}\right)$$

*Yuck! This **isn't** much help!*

Let's **instead** write this complex value Γ_L in terms of its **magnitude** and **phase**. For **magnitude** we find a much more straightforward result!

$$|\Gamma_L|^2 = \frac{|jX_L - Z_0|^2}{|jX_L + Z_0|^2} = \frac{X_L^2 + Z_0^2}{X_L^2 + Z_0^2} = 1$$

Its magnitude is **one**! Thus, we find that for reactive loads, the reflection coefficient can be simply expressed as:

$$\Gamma_L = e^{j\theta_\Gamma}$$

where

$$\theta_\Gamma = tan^{-1}\left[\frac{2\,Z_0\,X_L}{X_L^2 - Z_0^2}\right]$$

We can therefore conclude that for a **reactive load**:

$$V_0^- = e^{j\theta_\Gamma}\,V_0^+$$

As a result, the **total** voltage and current along the transmission line is simply (assuming $z_L = 0$):

$$\begin{aligned} V(z) &= V_0^+\left(e^{-j\beta z} + e^{+j\theta_L} e^{+j\beta z}\right) \\ &= V_0^+\; e^{+j\theta_\Gamma/2}\left(e^{-j(\beta z + \theta_\Gamma/2)} + e^{+j(\beta z + \theta_\Gamma/2)}\right) \\ &= 2V_0^+\; e^{+j\theta_\Gamma/2}\,\cos\left(\beta z + \theta_\Gamma/2\right) \end{aligned}$$

$$\begin{aligned} I(z) &= \frac{V_0^+}{Z_0}\left(e^{-j\beta z} - e^{+j\beta z}\right) \\ &= \frac{V_0^+}{Z_0} e^{+j\theta_L/2}\left(e^{-j(\beta z+\theta_L/2)} - e^{+j(\beta z+\theta_L/2)}\right) \\ &= -j\,\frac{2V_0^+}{Z_0} e^{+j\theta_L/2}\sin\left(\beta z + \theta_L/2\right) \end{aligned}$$

Meaning that the **line impedance** can be written in terms of a trigonometric function:

$$Z(z) = \frac{V(z)}{I(z)} = j\,Z_0 \cot\left(\beta z + \theta_\Gamma/2\right)$$

Note that this impedance is **purely reactive**— *V(z)* and *I(z)* are 90° out of phase!

We also note that the line impedance at the **end** of the transmission line is:

$$Z(z=0) = jZ_0\ cot\left(\theta_\Gamma/2\right)$$

With a little trigonometry, we can show (trust me!) that:

$$cot\left(\theta_\Gamma/2\right) = \frac{X_L}{Z_0}$$

and therefore:

$$Z(z=0) = jZ_0\ cot\left(\theta_\Gamma/2\right) = j\ X_L = Z_L$$

Just as we **expected** (and our boundary condition **demanded**)!

Finally, the reflection coefficient **function** is:

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{V_0^+ e^{+j\theta_\Gamma} e^{+j\beta z}}{V_0^+ e^{-j\beta z}} = e^{+j2(\beta z + \theta_\Gamma/2)}$$

Meaning that for purely reactive loads:

$$|\Gamma(z)| = \left| e^{+j2(\beta z + \theta_\Gamma/2)} \right| = 1$$

In other words, the **magnitude** reflection coefficient function is equal to one—at each and **every** point on the transmission line.

3. $Z_L = R_L$

For this case, the load impedance is **purely real** (e.g. a **resistor**), and thus there is no reactive component:

$$X_L = 0$$

The resulting **load reflection coefficient** is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{R - Z_0}{R + Z_0}$$

Given that Z_0 is real (i.e., the line is **lossless**), we find that this load reflection coefficient must be a purely **real** value! In other words:

$$Re\{\Gamma_L\} = \frac{R - Z_0}{R + Z_0} \qquad\qquad Im\{\Gamma_L\} = 0$$

So a real-valued load Z_L results in a real valued load reflection coefficient G_L .

Now let's consider the line impedance $Z(z)$ and reflection coefficient function $\Gamma(z)$.

Q: *I bet I know the answer to this one! We know that a purely imaginary (i.e., reactive) load results in a purely reactive line impedance.*

Thus, a purely real (i.e., resistive) load will result in a purely resistive line impedance, right??

A: NOPE! The line impedance resulting from a real load is complex—it has both real and imaginary components!

Thus the line impedance, as well as reflection coefficient function, **cannot** be further simplified for the case where $Z_L = R_L$.

Q: *Why is that?*

A: Remember, a **lossless** transmission line has series inductance and shunt capacitance **only**. In other words, a length of lossless transmission line is a **purely reactive** device (it absorbs **no** energy!).

***** If we attach a **purely reactive** load at the end of the transmission line, we still have a **completely** reactive system (load and transmission line). Because this system has **no** resistive (i.e., real) component, the general expressions for line impedance, line voltage, etc. can be significantly **simplified**.

***** However, if we attach a **purely real** load to our reactive transmission line, we now have a **complex** system, with **both** real and imaginary (i.e., resistive and reactive) components. This **complex** case is exactly what our general expressions **already** describes—**no** further simplification is possible!

4. $Z_L = R_L + jX_L$

Now, let's look at the **general** case, where the **load** has both a **real** (resitive) and **imaginary** (reactive) component.

Q: *Haven't we **already** determined all the **general** expressions (e.g., $\Gamma_L, V(z), I(z), Z(z), \Gamma(z)$) for this general case? Is there **anything** else left to be determined?*

A: There is **one** last thing we need to discuss. It seems trivial, but its ramifications are **very** important!

For you see, the "general" case is **not**, in reality, quite so general. Although the reactive component of the load can be **either** positive or negative ($-\infty < X_L < \infty$), the resistive component of a passive load **must** be positive ($R_L > 0$)—there's **no** such thing as **negative** resistor!

This leads to one **very** important and useful result. Consider the **load reflection coefficient**:

$$\begin{aligned}\Gamma_L &= \frac{Z_L - Z_0}{Z_L + Z_0} \\ &= \frac{\left(R_L + jX_L\right) - Z_0}{\left(R_L + jX_L\right) + Z_0} \\ &= \frac{\left(R_L - Z_0\right) + jX_L}{\left(R_L + Z_0\right) + jX_L}\end{aligned}$$

Now let's look at the **magnitude** of this value:

$$
\begin{aligned}
\left|\Gamma_L\right|^2 &= \left|\frac{(R_L - Z_0) + jX_L}{(R_L + Z_0) + jX_L}\right|^2 \\
&= \frac{(R_L - Z_0)^2 + X_L^2}{(R_L + Z_0)^2 + X_L^2} \\
&= \frac{\left(R_L^2 - 2R_L Z_0 + Z_0^2\right) + X_L^2}{\left(R_L^2 + 2R_L Z_0 + Z_0^2\right) + X_L^2} \\
&= \frac{\left(R_L^2 + Z_0^2 + X_L^2\right) - 2R_L Z_0}{\left(R_L^2 + Z_0^2 + X_L^2\right) + 2R_L Z_0}
\end{aligned}
$$

It is apparent that since both R_L and Z_0 are **positive**, the **numerator** of the above expression must be **less** than (or equal to) the **denominator** of the above expression.

→ In other words, the magnitude of the load reflection coefficient is always **less** than or equal to one!

$$
\boxed{\left|\Gamma_L\right| \leq 1 \quad (\text{for } R_L \geq 0)}
$$

Moreover, we find that this means the reflection coefficient **function** likewise always has a magnitude **less** than or equal to one, for **all** values of position *z*.

$$
\left|\Gamma(z)\right| \leq 1 \quad (\text{for all } z)
$$

Which means, of course, that the **reflected** wave will always have a magnitude **less** than that of the **incident** wave magnitude:

$$\boxed{\left|V^{-}(z)\right| \leq \left|V^{+}(z)\right| \qquad \text{(for all } z\text{)}}$$

We will find out later that this result is consistent with **conservation of energy**—the reflected wave from a passive load **cannot** be larger than the wave incident on it.

The Propagation Constant β

Recall that the activity along a transmission line can be expressed in terms of two functions, functions that we have described as **wave** functions:

$$V^{+}(z) = V_0^{+}\, e^{-j\beta z}$$

$$V^{-}(z) = V_0^{-}\, e^{+j\beta z}$$

where β is a real constant with value:

$$\beta = \omega\sqrt{LC}$$

Q: *What **is** this constant β? What does it **physically** represent?*

A: Remember, a complex function can be expressed in terms of its **magnitude** and **phase**:

$$f(z) = |f(z)|\, e^{j\phi_f(z)}$$

Thus:

$$|V^{+}(z)| = |V_0^{+}| \qquad \phi^{+}(z) = -\beta z + \phi_0^{+}$$

$$|V^{-}(z)| = |V_0^{-}| \qquad \phi^{-}(z) = +\beta z + \phi_0^{-}$$

Therefore, $-\beta z + \phi_0^+$ represents the relative **phase** of wave $V^+(z)$; a **function** of transmission line **position** z. Since phase ϕ is expressed in **radians**, and z is distance (in meters), the value β must have **units** of:

$$\beta = \frac{\phi}{z} \qquad \frac{\text{radians}}{\text{meter}}$$

The **wavelength** λ of the propagating wave is defined as the **distance** $\Delta z_{2\pi}$ over which the relative phase changes by 2π **radians**. So:

$$2\pi = \phi(z + \Delta z_{2\pi}) - \phi(z) = \beta \, \Delta z_{2\pi} = \beta \lambda$$

or, rearranging:

$$\boxed{\beta = \frac{2\pi}{\lambda}}$$

Thus, the value β is thus essentially a **spatial frequency**, in the same way that ω is a temporal frequency:

$$\omega = \frac{2\pi}{T}$$

where T is the **time** required for the phase of the oscillating signal to change by a value of 2π radians, i.e.:

$$\omega T = 2\pi$$

Note that this time is the **period** of a sinewave, and related to its **frequency** in Hertz (cycles/second) as:

$$T = \frac{2\pi}{\omega} = \frac{1}{f}$$

Q: *So, just how* ***fast*** *does this wave propagate down a transmission line?*

We describe wave velocity in terms of its **phase velocity**—in other words, how **fast** does a specific value of absolute phase ϕ seem to **propagate** down the transmission line.

Since velocity is change in distance with respect to **time**, we need to first express our propagating wave in its real form:

$$\begin{aligned} v^+(z,t) &= Re\left\{V^+(z)\,e^{-j\omega t}\right\} \\ &= \left|V_0^+\right| \cos\left(\omega t - \beta z + \phi_0^+\right) \end{aligned}$$

Thus, the absolute phase is a function of **both** time and frequency:

$$\phi^+(z,t) = \omega t - \beta z + \phi_0^+$$

Now let's set this phase to some **arbitrary** value of ϕ_c radians.

$$\omega t - \beta z + \phi_0^+ = \phi_c$$

For **every** time t, there is **some** location z on a transmission line that has this phase value ϕ_c. That location is evidently:

$$z = \frac{\omega t + \phi_0^+ - \phi_c}{\beta}$$

Note as **time increases**, so too does the **location** z on the line where $\phi^+(z,t) = \phi_c$.

The **velocity** v_p at which this phase point moves down the line can be determined as:

$$v_p = \frac{dz}{dt} = \frac{d\left(\frac{\omega t + \phi_0^+ - \phi_c}{\beta}\right)}{dt} = \frac{\omega}{\beta}$$

This wave velocity is the **velocity of the propagating wave**!

Note that the value:

$$\frac{v_p}{\lambda} = \frac{\omega}{\beta}\frac{\beta}{2\pi} = \frac{\omega}{2\pi} = f$$

and thus we can conclude that:

$$v_p = f\lambda$$

as well as:

$$\beta = \frac{\omega}{v_p}$$

Q: *But these results were derived for the $V^+(z)$ wave; what about the **other** wave ($V^-(z)$)?*

A: The results are essentially the **same**, as each wave depends on the same value β.

The only **subtle difference** comes when we evaluate the phase velocity. For the wave $V^-(z)$, we find:

$$\phi^-(z,t) = \omega t + \beta z + \phi_0^-$$

Note the **plus sign** associated with βz!

We thus find that some arbitrary phase value will be located at location:

$$z = \frac{-\phi_0^- + \phi_c - \omega t}{\beta}$$

Note now that an **increasing time** will result in a **decreasing** value of **position** z. In other words this wave is propagating in the direction of decreasing position z—in the **opposite** direction of the $V^+(z)$ wave!

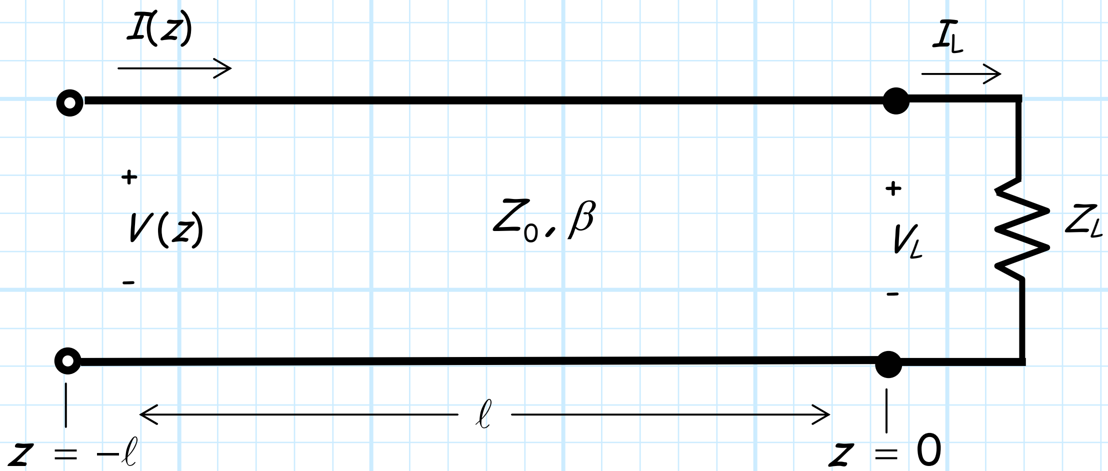
This is **further** verified by the derivative:

$$v_p = \frac{dz}{dt} = \frac{d\left(\dfrac{-\phi_0^- + \phi_c - \omega t}{\beta}\right)}{dt} = -\frac{\omega}{\beta}$$

Where the **minus sign** merely means that the wave propagates in the $-z$ direction. Otherwise, the **wavelength** and **velocity** of the two waves are **precisely** the same!

Transmission Line Input Impedance

Consider a **lossless** line, length ℓ, terminated with a **load** Z_L.



Let's determine the **input impedance** of this line!

Q: *Just what do you mean by* ***input*** *impedance?*

A: The input impedance is simply the line impedance seen at the **beginning** ($z = -\ell$) of the transmission line, i.e.:

$$Z_{in} = Z(z = -\ell) = \frac{V(z = -\ell)}{I(z = -\ell)}$$

Note Z_{in} equal to **neither** the load impedance Z_L **nor** the characteristic impedance Z_0!

$$Z_{in} \neq Z_L \quad \text{and} \quad Z_{in} \neq Z_0$$

To determine exactly what Z_{in} is, we first must determine the voltage and current at the **beginning** of the transmission line ($z = -\ell$).

$$V(z=-\ell) = V_0^+\left[e^{+j\beta\ell} + \Gamma_L\, e^{-j\beta\ell}\right]$$

$$I(z=-\ell) = \frac{V_0^+}{Z_0}\left[e^{+j\beta\ell} - \Gamma_L\, e^{-j\beta\ell}\right]$$

Therefore:

$$Z_{in} = \frac{V(z=-\ell)}{I(z=-\ell)} = Z_0\left(\frac{e^{+j\beta\ell} + \Gamma_L\, e^{-j\beta\ell}}{e^{+j\beta\ell} - \Gamma_L\, e^{-j\beta\ell}}\right)$$

We can explicitly write Z_{in} in terms of load Z_L using the previously determined relationship:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Combining these two expressions, we get:

$$\begin{aligned} Z_{in} &= Z_0\,\frac{(Z_L + Z_0)\,e^{+j\beta\ell} + (Z_L - Z_0)\,e^{-j\beta\ell}}{(Z_L + Z_0)\,e^{+j\beta\ell} - (Z_L - Z_0)\,e^{-j\beta\ell}} \\ &= Z_0\left(\frac{Z_L\left(e^{+j\beta\ell} + e^{-j\beta\ell}\right) + Z_0\left(e^{+j\beta\ell} - e^{-j\beta\ell}\right)}{Z_L\left(e^{+j\beta\ell} + e^{-j\beta\ell}\right) - Z_0\left(e^{+j\beta\ell} - e^{-j\beta\ell}\right)}\right) \end{aligned}$$

Now, recall **Euler's equations**:

$$\begin{aligned} e^{+j\beta\ell} &= \cos\beta\ell + j\sin\beta\ell \\ e^{-j\beta\ell} &= \cos\beta\ell - j\sin\beta\ell \end{aligned}$$

Using Euler's relationships, we can likewise write the input impedance without the **complex** exponentials:

$$Z_{in} = Z_0 \left(\frac{Z_L \cos\beta\ell + j\, Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j\, Z_L \sin\beta\ell} \right)$$

$$= Z_0 \left(\frac{Z_L + j\, Z_0 \tan\beta\ell}{Z_0 + j\, Z_L \tan\beta\ell} \right)$$

Note that depending on the values of β, Z_0 and ℓ, the input impedance can be **radically** different from the load impedance Z_L!

Q: *So is there a similar concept of **input reflection coefficient**?*

A: There sure is! As you might expect, it is simply the value of reflection coefficient function $\Gamma(z)$ evaluated at the beginning of the transmission line (i.e., at $z = -\ell$):

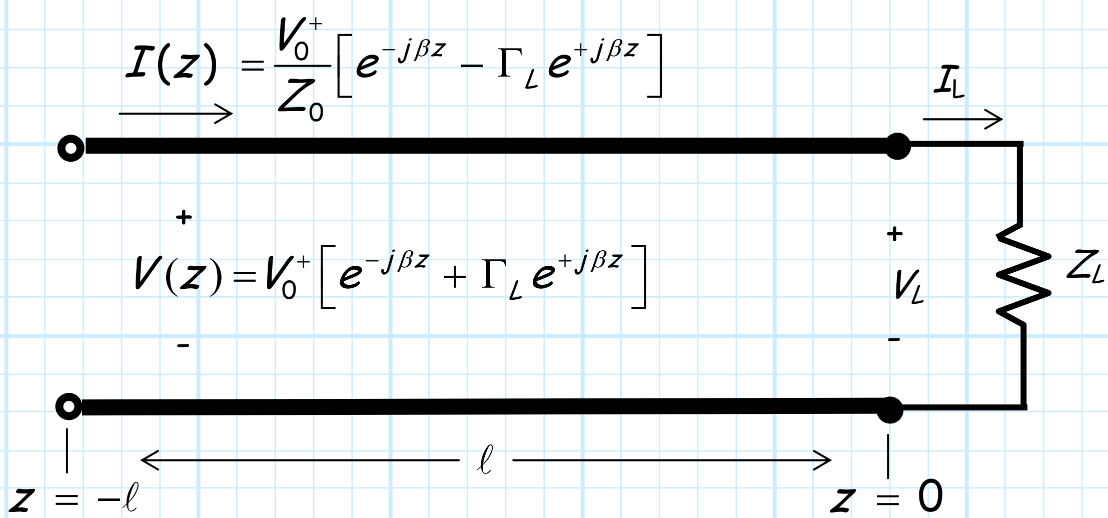
$$\Gamma_{in} \doteq \Gamma(z = -\ell) = \Gamma_0\, e^{-j2\beta\ell}$$

Note that the input impedance and input reflection coefficient are related in the same way as Z and Γ are at every other point on the transmission line:

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

Power Flow and Return Loss

We have discovered that **two waves propagate** along a transmission line, one in each direction ($V^+(z)$ and $V^-(z)$).



The result is that electromagnetic energy flows along the transmission line at a given rate (i.e., **power**).

Q: *How much power flows along a transmission line, and where does that power go?*

A: We can answer that question by determining the power **absorbed** by the **load**!

The **time average** power absorbed by an impedance Z_L is:

$$
\begin{aligned}
P_{abs} &= \frac{1}{2}\operatorname{Re}\left\{V_L I_L^*\right\} \\
&= \frac{1}{2}\operatorname{Re}\left\{V(z=0)I(z=0)^*\right\} \\
&= \frac{1}{2Z_0}\operatorname{Re}\left\{\left(V_0^+\left[e^{-j\beta 0}+\Gamma_L e^{+j\beta 0}\right]\right)\left(V_0^+\left[e^{-j\beta 0}-\Gamma_L e^{+j\beta 0}\right]\right)^*\right\} \\
&= \frac{\left|V_0^+\right|^2}{2Z_0}\operatorname{Re}\left\{1-\left(\Gamma_L^*-\Gamma_L\right)-\left|\Gamma_L\right|^2\right\} \\
&= \frac{\left|V_0^+\right|^2}{2Z_0}\left(1-\left|\Gamma_L\right|^2\right)
\end{aligned}
$$

The significance of this result can be seen by **rewriting** the expression as:

$$
\begin{aligned}
P_{abs} &= \frac{\left|V_0^+\right|^2}{2Z_0}\left(1-\left|\Gamma_L\right|^2\right) \\
&= \frac{\left|V_0^+\right|^2}{2Z_0}-\frac{\left|V_0^+\Gamma_L\right|^2}{2Z_0} \\
&= \frac{\left|V_0^+\right|^2}{2Z_0}-\frac{\left|V_0^-\right|^2}{2Z_0}
\end{aligned}
$$

The two terms in above expression have a very definite **physical meaning**. The first term is the time-averaged **power of the wave** propagating along the transmission line **toward the load**.

We say that this wave is **incident** on the load:

$$P_{inc} = P_{+} = \frac{\left|V_0^+\right|^2}{2Z_0}$$

Likewise, the second term of the P_{abs} equation describes the **power of the wave** moving in the other direction (**away from the load**). We refer to this as the wave **reflected** from the load:

$$P_{ref} = P_{-} = \frac{\left|V_0^-\right|^2}{2Z_0} = \frac{\left|\Gamma_L\right|^2\left|V_0^+\right|^2}{2Z_0} = \left|\Gamma_L\right|^2 P_{inc}$$

Thus, the power **absorbed** by the load is simply:

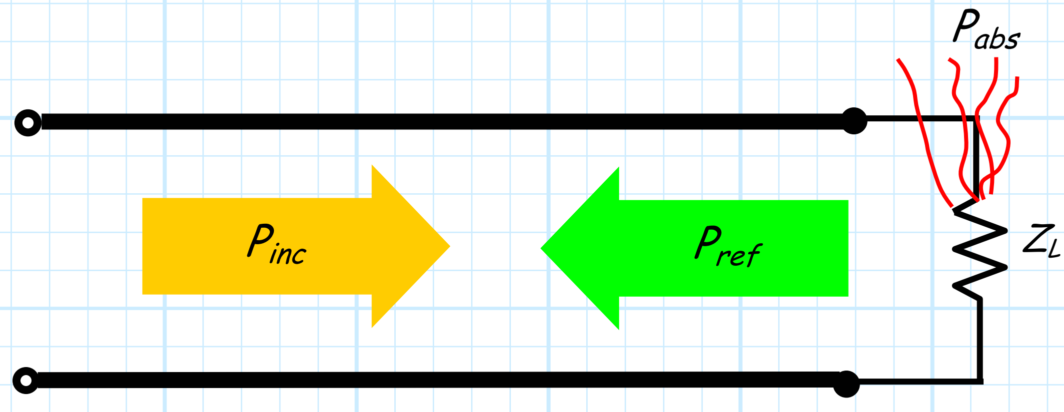
$$P_{abs} = P_{inc} - P_{ref}$$

or, rearranging, we find:

$$P_{inc} = P_{abs} + P_{ref}$$

This equation is simply an expression of the **conservation of energy** !

It says that power flowing **toward** the load (P_{inc}) is either **absorbed** by the load (P_{abs}) or **reflected** back from the load (P_{ref}).



Note that if $|\Gamma_L|^2 = 1$, then $P_{inc} = P_{ref}$, and therefore **no power** is absorbed by the **load**.

This of course **makes sense** !

The magnitude of the reflection coefficient ($|\Gamma_L|$) is equal to one **only** when the load impedance is **purely reactive** (i.e., purely imaginary).

Of course, a purely reactive element (e.g., capacitor or inductor) **cannot** absorb any power—**all** the power **must** be reflected!

Return Loss

The **ratio** of the reflected power to the incident power is known as **return loss**. Typically, return loss is expressed in **dB**:

$$R.L. = -10 \log_{10}\left[\frac{P_{ref}}{P_{inc}}\right] = -10 \log_{10} |\Gamma_L|^2$$

For **example**, if the return loss is **10dB**, then **10%** of the incident power is **reflected** at the load, with the remaining **90%** being **absorbed** by the load—we "lose" 10% of the incident power

Likewise, if the return loss is **30dB**, then **0.1 %** of the incident power is **reflected** at the load, with the remaining **99.9%** being **absorbed** by the load—we "lose" 0.1% of the incident power.

Thus, a **larger** numeric value for return loss **actually** indicates **less** lost power! An **ideal** return loss would be ∞ dB, whereas a return loss of 0 dB indicates that $|\Gamma_L| = 1$--the load is **reactive**!

VSWR

Consider again the **voltage** along a terminated transmission line, as a function of **position** z:

$$V(z) = V_0^+ \left[e^{-j\beta z} + \Gamma_L e^{+j\beta z} \right]$$

Recall this is a **complex** function, the magnitude of which expresses the **magnitude** of the **sinusoidal signal** at position z, while the phase of the complex value represents the relative **phase** of the sinusoidal signal.

Let's look at the **magnitude** only:

$$\begin{aligned} |V(z)| &= |V_0^+| \, |e^{-j\beta z} + \Gamma_L e^{+j\beta z}| \\ &= |V_0^+| |e^{-j\beta z}| |1 + \Gamma_L e^{+j2\beta z}| \\ &= |V_0^+| |1 + \Gamma_L e^{+j2\beta z}| \end{aligned}$$

ICBST the **largest** value of $|V(z)|$ occurs at the location z where:

$$\Gamma_L e^{+j2\beta z} = |\Gamma_L| + j0$$

while the **smallest** value of $|V(z)|$ occurs at the location z where:

$$\Gamma_L e^{+j2\beta z} = -|\Gamma_L| + j0$$

As a result we can conclude that:

$$|V(z)|_{max} = |V_0^+|\left(1+|\Gamma_L|\right)$$

$$|V(z)|_{min} = |V_0^+|\left(1-|\Gamma_L|\right)$$

The ratio of $|V(z)|_{max}$ to $|V(z)|_{min}$ is known as the **Voltage Standing Wave Ratio** (VSWR):

$$\boxed{\text{VSWR} \doteq \frac{|V(z)|_{max}}{|V(z)|_{min}} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} \qquad \therefore \qquad 1 \le VSWR \le \infty}$$

Note if $|\Gamma_L| = 0$ (i.e., $Z_L = Z_0$), then VSWR = 1. We find for this case:

$$|V(z)|_{\max} = |V(z)|_{\min} = |V_0^+|$$

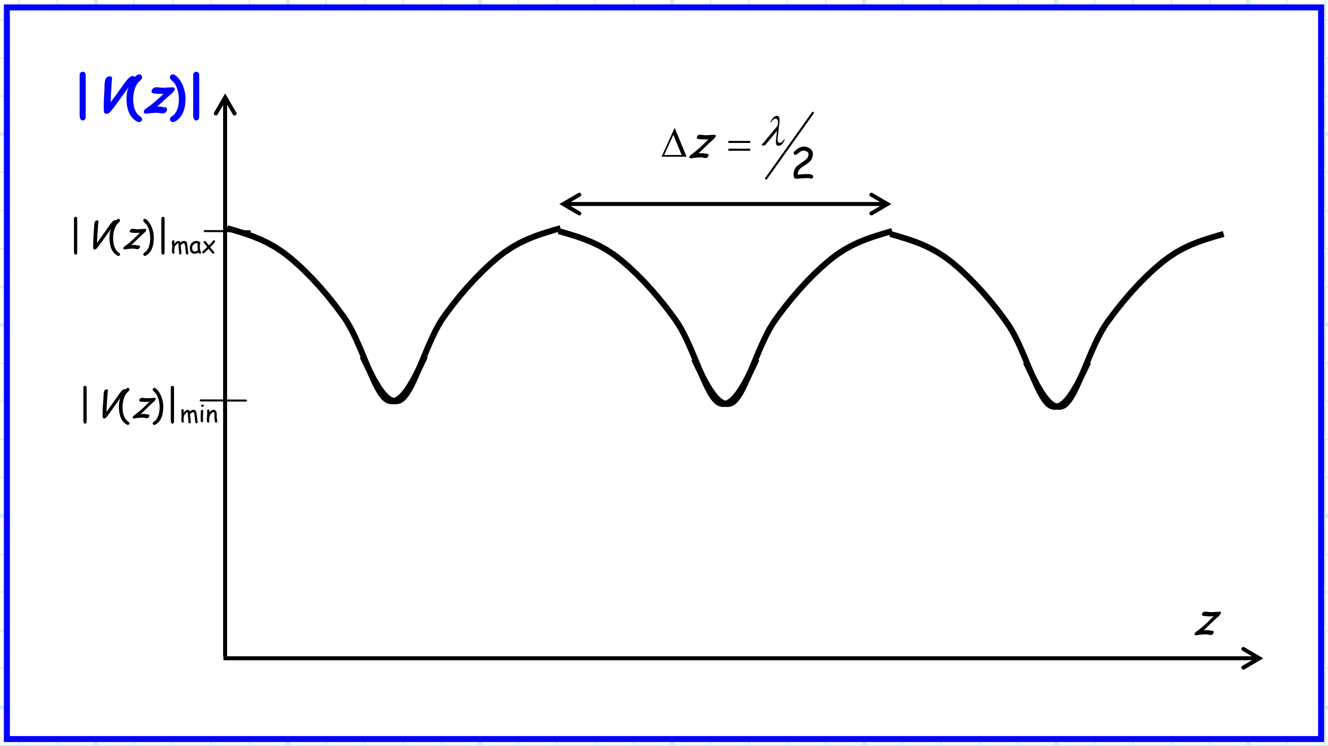
In other words, the voltage magnitude is a **constant** with respect to position *z*.

Conversely, if $|\Gamma_L| = 1$ (i.e., $Z_L = jX$), then VSWR = ∞. We find for **this** case:

$$|V(z)|_{\min} = 0 \qquad \text{and} \qquad |V(z)|_{\max} = 2|V_0^+|$$

In other words, the voltage magnitude varies **greatly** with respect to position *z*.

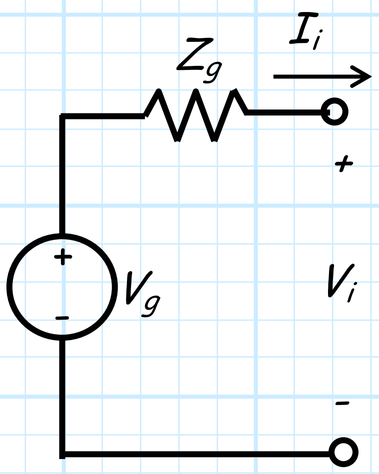
As with **return loss**, VSWR is dependent on the **magnitude** of Γ_L (i.e, $|\Gamma_L|$) **only** !

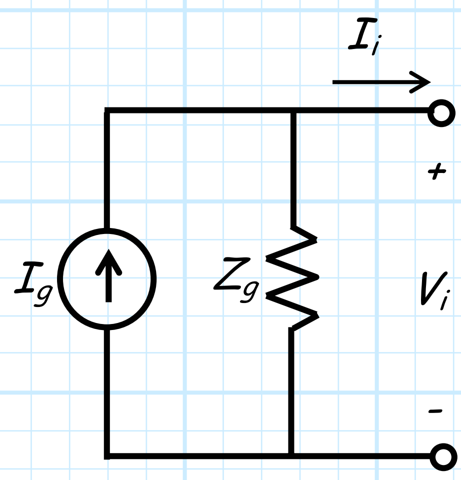


A Transmission Line Connecting Source & Load

We can think of a transmission line as a conduit that allows **power** to flow **from** an **output** of one device/network **to** an **input** of another.

To simplify our analysis, we can model the **input** of the device **receiving** the power with it input impedance (e.g., Z_L), while we can model the device **output delivering** the power with its Thevenin's or Norton's equivalent circuit.

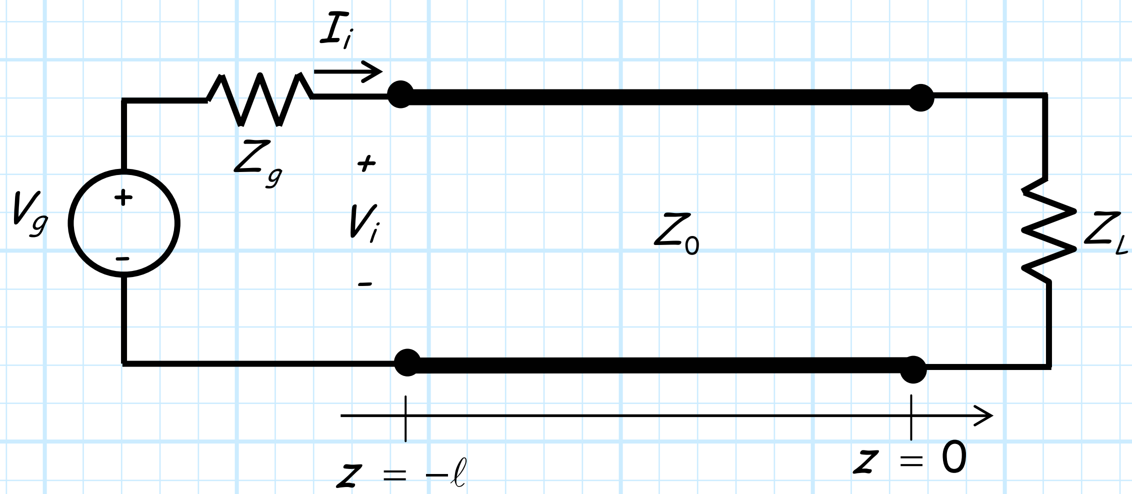




$$\boxed{V_g = V_i + Z_g I_i}$$

$$\boxed{I_g = \frac{V_i}{Z_g} + I_i}$$

Typically, the power source is modeled with its **Thevenin's** equivalent; however, we will find that the **Norton's** equivalent circuit is useful if we express the remainder of the circuit in terms of its **admittance** values (e.g., $Y_0, Y_L, Y(z)$).



Recall from the telegrapher's equations that the current and voltage along the transmission line are:

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z}$$

At $z = 0$, we enforced the **boundary condition** resulting from Ohm's Law:

$$Z_L = \frac{V_L}{I_L} = \frac{V(z=0)}{I(z=0)} = \frac{\left(V_0^+ + V_0^-\right)}{\left(\frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}\right)}$$

Which resulted in:

$$\frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \doteq \Gamma_L$$

So therefore:

$$V(z) = V_0^+ \left[e^{-j\beta z} + \Gamma_L \, e^{+j\beta z} \right]$$

$$I(z) = \frac{V_0^+}{Z_0} \left[e^{-j\beta z} - \Gamma_L \, e^{+j\beta z} \right]$$

We are left with the **question**: just what is the **value** of complex constant V_0^+ ?!?

This constant depends on the signal **source**! To determine its exact **value**, we must now apply **boundary conditions** at $z = -\ell$.

We know that at the **beginning** of the transmission line:

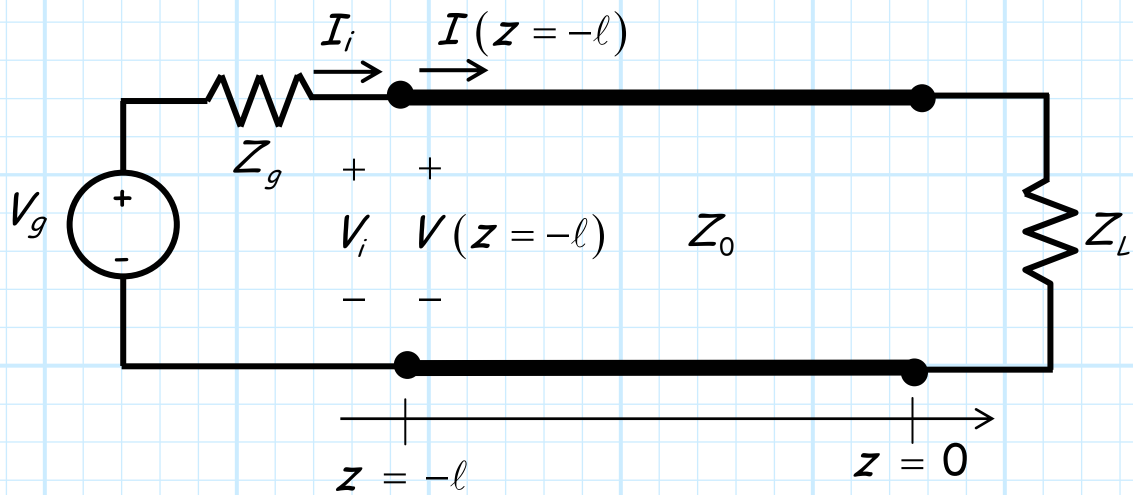
$$V(z = -\ell) = V_0^+ \left[e^{+j\beta \ell} + \Gamma_L \, e^{-j\beta \ell} \right]$$

$$I(z = -\ell) = \frac{V_0^+}{Z_0} \left[e^{+j\beta \ell} - \Gamma_L \, e^{-j\beta \ell} \right]$$

Likewise, we know that the **source** must satisfy:

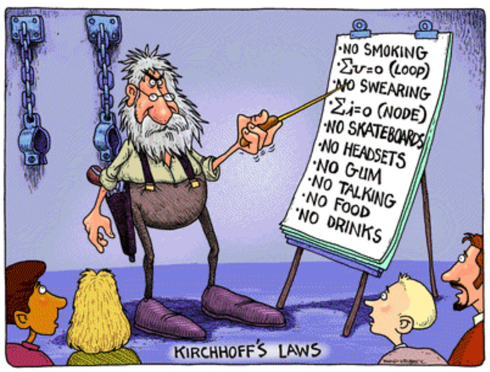
$$V_g = V_i + Z_g \, I_i$$

To relate these **three** expressions, we need to apply **boundary conditions** at $z = -\ell$:



From **KVL** we find:

$$V_i = V(z=-\ell)$$



And from **KCL**:

$$I_i = I(z=-\ell)$$

Combining these equations, we find:

$$V_g = V_i + Z_g I_i$$

$$V_g = V_0^+\left[e^{+j\beta\ell} + \Gamma_L e^{-j\beta\ell}\right] + Z_g \frac{V_0^+}{Z_0}\left[e^{+j\beta\ell} - \Gamma_L e^{-j\beta\ell}\right]$$

One equation → **one** unknown (V_0^+)!!

Solving, we find the value of V_0^+:

$$\boxed{V_0^+ = V_g\, e^{-j\beta\ell} \frac{Z_0}{Z_0\left(1+\Gamma_{in}\right)+Z_g\left(1-\Gamma_{in}\right)}}$$

where:

$$\Gamma_{in} = \Gamma\left(z=-\ell\right) = \Gamma_L\, e^{-j2\beta\ell}$$

There is one **very important** point that must be made about the result:

$$V_0^+ = V_g\, e^{-j\beta\ell} \frac{Z_0}{Z_0\left(1+\Gamma_{in}\right)+Z_g\left(1-\Gamma_{in}\right)}$$

And that is—the wave $V_0^+(z)$ **incident** on the load Z_L is actually dependent **on** the value of load Z_L !!!!!

Remember:

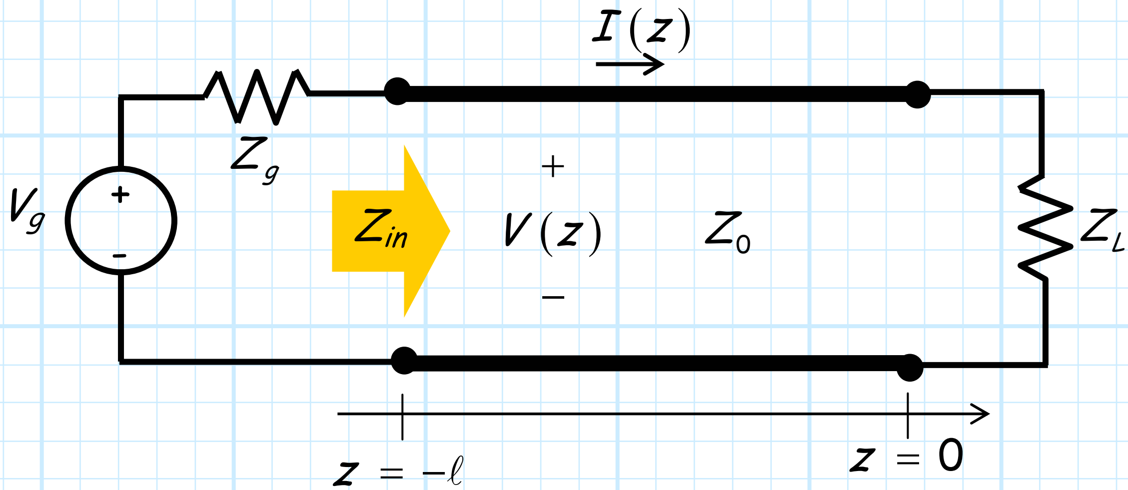
$$\Gamma_{in} = \Gamma\left(z=-\ell\right) = \Gamma_L\, e^{-j2\beta\ell}$$

We tend to think of the incident wave $V_0^+(z)$ being "**caused**" by the source, and it is certainly true that $V_0^+(z)$ **depends** on the source—after all, $V_0^+(z) = 0$ if $V_g = 0$. However, we find from the equation above that it **likewise** depends on the value of the load!

Thus we **cannot**—in general—consider the incident wave to be the "**cause**" and the reflected wave the "**effect**". Instead, each wave must obtain the proper **amplitude** (e.g., V_0^+, V_0^-) so that the boundary conditions are satisfied at **both** the beginning and end of the transmission line.

Delivered Power

Q: *If the purpose of a transmission line is to transfer **power** from a source to a load, then exactly how much power is **delivered** to Z_L for the circuit shown below ??*



A: We of course **could** determine V_0^+ and V_0^-, and then determine the power absorbed by the load (P_{abs}) as:

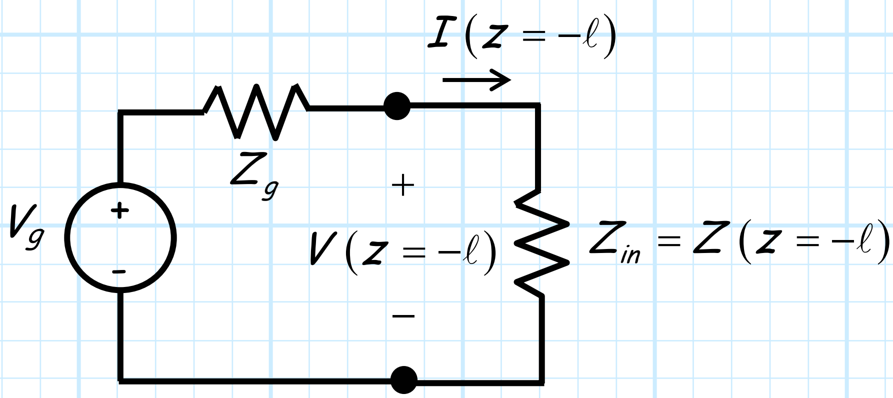
$$P_{abs} = \frac{1}{2}\text{Re}\left\{V(z=0)\, I^*(z=0)\right\}$$

However, if the transmission line is **lossless**, then we know that the power delivered to the load must be **equal** to the power "delivered" to the **input** (P_{in}) of the transmission line:

$$P_{abs} = P_{in} = \frac{1}{2}\text{Re}\left\{V(z=-\ell)\, I^*(z=-\ell)\right\}$$

However, we can determine this power **without** having to solve for V_0^+ and V_0^- (i.e., $V(z)$ and $I(z)$). We can simply use our knowledge of **circuit theory**!

We can **transform** load Z_L to the beginning of the transmission line, so that we can replace the transmission line with its **input impedance** Z_{in}:



Note by **voltage division** we can determine:

$$V(z=-\ell)=V_g\frac{Z_{in}}{Z_g+Z_{in}}$$

And from **Ohm's Law** we conclude:

$$I(z=-\ell)=\frac{V_g}{Z_g+Z_{in}}$$

And thus, the **power** P_{in} delivered to Z_{in} (and thus the **power** P_{abs} delivered to the load Z_L) is:

$$P_{abs} = P_{in} = \frac{1}{2}\text{Re}\left\{V(z=-\ell)\, I^*(z=-\ell)\right\}$$

$$= \frac{1}{2}\text{Re}\left\{V_g \frac{Z_{in}}{Z_g + Z_{in}} \frac{V_g^*}{\left(Z_g + Z_{in}\right)^*}\right\}$$

$$= \frac{1}{2}\frac{\left|V_g\right|^2}{\left|Z_g + Z_{in}\right|^2}\text{Re}\left\{Z_{in}\right\}$$

$$= \frac{1}{2}\left|V_g\right|^2 \frac{\left|Z_{in}\right|^2}{\left|Z_g + Z_{in}\right|^2}\text{Re}\left\{Y_{in}\right\}$$

Note that we could **also** determine P_{abs} from our **earlier** expression:

$$P_{abs} = \frac{\left|V_0^+\right|^2}{2\,Z_0}\left(1 - \left|\Gamma_L\right|^2\right)$$

But we would of course have to **first** determine V_0^+ (!):

$$V_0^+ = V_g\, e^{-j\beta\ell}\, \frac{Z_0}{Z_0\left(1+\Gamma_{in}\right) + Z_g\left(1-\Gamma_{in}\right)}$$

Special Cases of Source and Load Impedance

Let's look at **specific cases** of Z_g and Z_L, and determine how they affect V_0^+ and P_{abs}.

$$\boxed{Z_g = Z_0}$$

For this case, we find that V_0^+ **simplifies** greatly:

$$
\begin{aligned}
V_0^+ &= V_g \, e^{-j\beta\ell} \, \frac{Z_0}{Z_0\left(1+\Gamma_{in}\right)+Z_g\left(1-\Gamma_{in}\right)} \\
&= V_g \, e^{-j\beta\ell} \, \frac{Z_0}{Z_0\left(1+\Gamma_{in}\right)+Z_0\left(1-\Gamma_{in}\right)} \\
&= V_g \, e^{-j\beta\ell} \, \frac{1}{1+\Gamma_{in}+1-\Gamma_{in}} \\
&= \frac{1}{2} V_g \, e^{-j\beta\ell}
\end{aligned}
$$

Look at what **this** says!

It says that the incident wave in this case is **independent** of the load attached at the other end!

Thus, for the **one** case $Z_g = Z_0$, we in fact can consider $V^+(z)$ as being the source wave, and then the reflected wave $V^-(z)$ as being the result of this stimulus.

Remember, the complex value V_0^+ is the value of the incident wave evaluated at the end of the transmission line ($V_0^+ = V^+(z=0)$). We can likewise determine the value of the incident wave at the **beginning** of the transmission line (i.e., $V^+(z=-\ell)$). For this case, where $Z_g = Z_0$, we find that this value can be very simply stated (!):

$$
\begin{aligned}
V^+(z=-\ell) &= V_0^+ e^{-j\beta(z=-\ell)} \\
&= \left(\frac{1}{2} V_g\, e^{-j\beta\ell}\right) e^{+j\beta\ell} \\
&= \frac{V_g}{2}
\end{aligned}
$$

Likewise, we find that the delivered power for this case can be simply stated as:

$$
\begin{aligned}
P_{abs} &= \frac{\left|V_0^+\right|^2}{2\, Z_0}\left(1-\left|\Gamma_L\right|^2\right) \\
&= \frac{\left|V_g\right|^2}{8\, Z_0}\left(1-\left|\Gamma_L\right|^2\right)
\end{aligned}
$$

$$\boxed{Z_{in} = Z_g^*}$$

For this case, we find Z_L takes on whatever value required to make $Z_{in} = Z_g^*$. This is a **very** important case!

First, using the fact that:

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} = \frac{Z_g^* - Z_0}{Z_g^* + Z_0}$$

We can show that (trust me!):

$$V_0^+ = V_g\, e^{-j\beta\ell}\, \frac{Z_g^* + Z_0}{4\,\mathrm{Re}\{Z_g\}}$$

Not a particularly interesting result, but let's look at the absorbed power.

$$
\begin{aligned}
P_{abs} &= \frac{1}{2}\frac{\left|V_g\right|^2}{\left|Z_g + Z_{in}\right|^2}\,\text{Re}\left\{Z_{in}\right\} \\
&= \frac{1}{2}\frac{\left|V_g\right|^2}{\left|Z_g + Z_g^*\right|^2}\,\text{Re}\left\{Z_g^*\right\} \\
&= \frac{1}{2}\frac{\left|V_g\right|^2}{\left|2\,\text{Re}\left\{Z_g^*\right\}\right|^2}\,\text{Re}\left\{Z_g^*\right\} \\
&= \frac{1}{2}\left|V_g\right|^2\frac{1}{4\,\text{Re}\left\{Z_g^*\right\}} = P_{avl}
\end{aligned}
$$

Although this result does not look particularly interesting **either**, we find the result is **very** important!

It can be shown that—for a **given** V_g and Z_g —the value of input impedance Z_{in} that will absorb the l**argest possible** amount of power is the value $Z_{in} = Z_g^*$.

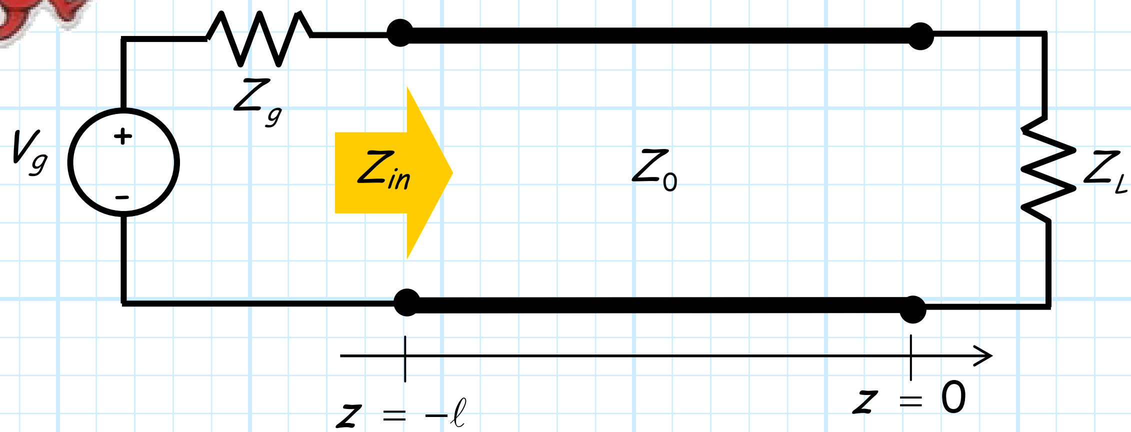
This case is known as the **conjugate match**, and is essentially the goal of every transmission line problem—to deliver the largest possible power to Z_{in}, and thus to Z_L as well!

This maximum delivered power is known as the **available power** (P_{avl})of the source.

There are **two** very important things to understand about this result!

Very Important Thing #1

Consider again the terminated transmission line:



Recall that if $Z_L = Z_0$, the **reflected** wave will be **zero**, and the absorbed power will be:

$$P_{abs} = \frac{\left|V_g\right|^2}{2} \frac{Z_0}{\left|Z_0 + Z_g\right|^2}$$

But note if $Z_L = Z_0$, the input impedance $Z_{in} = Z_0$—but then $Z_{in} \neq Z_g^*$ (generally)! In other words, $Z_L = Z_0$ does **not** (generally) result in a **conjugate match**, and thus setting $Z_L = Z_0$ does **not** result in maximum power absorption!

Q: *Huh!? This makes* ***no*** *sense! A load value of $Z_L = Z_0$ will* ***minimize*** *the reflected wave ($P^- = 0$)—**all** of the incident power will be absorbed. Any other value of $Z_L = Z_0$ will result in* ***some*** *of the incident wave being reflected—how in the world could this* ***increase*** *absorbed power?*

*After all, just **look** at the expression for absorbed power:*

$$P_{abs} = \frac{\left|V_0^+\right|^2}{2\,Z_0}\left(1-\left|\Gamma_L\right|^2\right)$$

***Clearly**, this value is maximized when $\Gamma_L = 0$ (i.e., when $Z_L = Z_0$)!!!*

A: You are forgetting one very important fact! Although it **is** true that the load impedance Z_L affects the **reflected** wave power P^-, the value of Z_L —as we have shown in this handout—**likewise** helps determine the value of the **incident** wave (i.e., the value of P^+) as well.

Thus, the value of Z_L that minimizes P^- will **not** generally maximize P^+, **nor** will the value of Z_L that maximizes P^+ likewise minimize P^-.

Instead, the value of Z_L that maximizes the **absorbed** power is, by definition, the value that maximizes the **difference** $P^+ - P^-$.

We find that this value of Z_L is the value that makes Z_{in} as "close" as possible to the **ideal** case of $Z_{in} = Z_g^*$.

Q: *Yes, but what about the case where $Z_g = Z_0$? For **that** case, we determined that the incident wave **is** independent of Z_L. Thus, it would seem that at least for that case, the*

***delivered** power would be maximized when the **reflected** power was minimized (i.e., $Z_L = Z_0$).*

A: True! But think about what the **input** impedance would be in that case—$Z_{in} = Z_0$. Oh by the way, that provides a **conjugate match** ($Z_{in} = Z_0 = Z_g^*$)!

Thus, in some ways, the case $Z_g = Z_0 = Z_L$ (i.e., **both** source and load impedances are numerically equal to Z_0) is **ideal**. A **conjugate match** occurs, the incident wave is **independent** of Z_L, there is **no** reflected wave, and all the math **simplifies** quite nicely:

$$V_0^+ = \frac{1}{2} V_g \, e^{-j\beta\ell} \qquad\qquad P_{abs} = \frac{|V_g|^2}{8\,Z_0}$$

Very Important Thing #2

Note the conjugate match criteria **says**:

Given V_g and Z_g, maximum power transfer occurs when $Z_{in} = Z_g^*$.

It does **NOT** say:

Given V_g and Z_{in}, maximum power transfer occurs when $Z_g^* = Z_{in}$.

This last statement is in fact **false!**

A **factual** statement is this:

Given V_g and Z_{in}, maximum power transfer occurs when $Z_g = 0$.

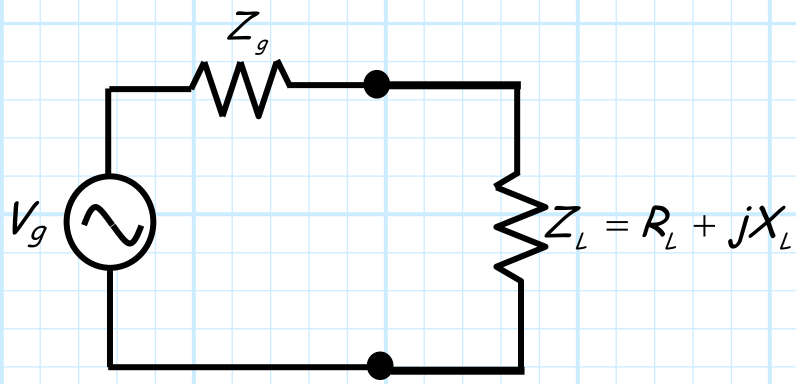
A fact that is **evident** when observing the expression for **available power**:

$$P_{available} = \frac{1}{2}\left|V_g\right|^2 \frac{1}{4\,\text{Re}\left\{Z_g^*\right\}} = \frac{\left|V_g\right|^2}{8R_g}$$

In other words, given a choice, use a source with the **smallest** possible **output resistance** (given that V_g remains **constant**). This will maximize the **available power** from your source!

Matching Networks

Consider again the problem where a **passive load** is attached to an **active source**:



The load will **absorb power**—power that is **delivered** to it by the **source**.

$$
\begin{aligned}
P_L &= \frac{1}{2} Re\left\{V_L I_L^*\right\} \\
&= \frac{1}{2} Re\left\{\left(V_g \frac{Z_L}{Z_g + Z_L}\right)\left(\frac{V_g}{Z_g + Z_L}\right)^*\right\} \\
&= \frac{1}{2}\left|V_g\right|^2 \frac{Re\left\{Z_L\right\}}{\left|Z_g + Z_L\right|^2} \\
&= \frac{1}{2}\left|V_g\right|^2 \frac{R_L}{\left|Z_g + Z_L\right|^2}
\end{aligned}
$$

Recall that the power delivered to the load will be **maximized** (for a given V_g and Z_g) **if** the load impedance is equal to the **complex conjugate** of the source impedance ($Z_L = Z_g^*$).

We call this maximum power the **available power** P_{avl} **of the source**—it is, after all, the **largest** amount of power that the source can **ever** deliver!

$$P_L^{max} \doteq P_{avl} = \frac{|V_g|^2}{8\,R_g}$$

* Note the available power of the **source** is dependent on **source** parameters **only** (i.e., V_g and R_g). This makes sense! Do **you** see why?

* Thus, we can say that to "take full advantage" of all the available power of the source, we must to make the load impedance the complex conjugate of the source impedance.

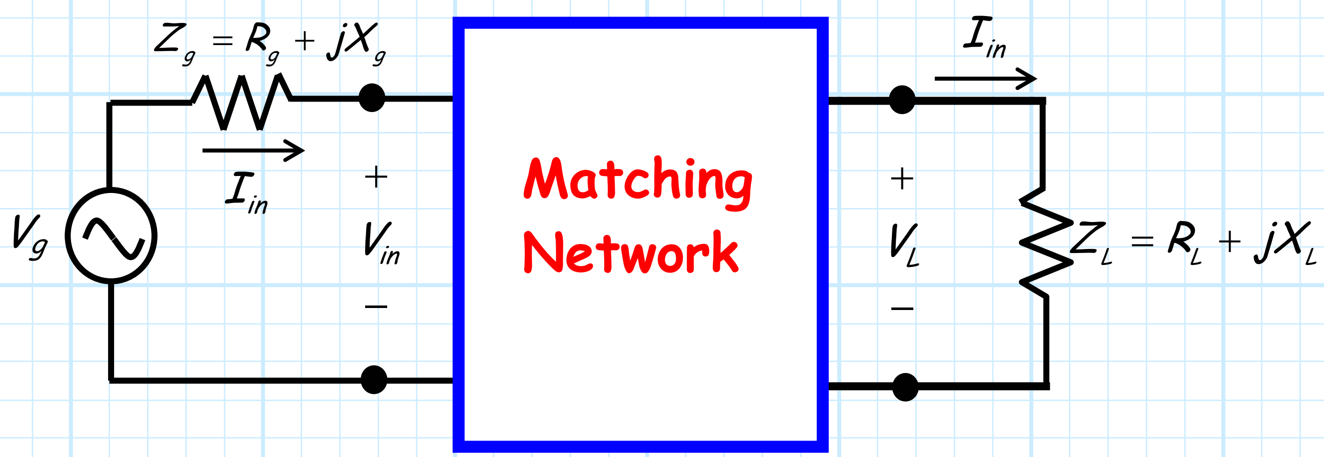
* Otherwise, the power delivered to the load will be less than power made available by the source! In other "words":

$$P_L \leq P_{avl}$$

Q: *But, you said that the load impedance typically models the input impedance of some useful device. We **don't** typically get to "select" or adjust this impedance—it **is** what it **is**. Must we then simply **accept** the fact that the delivered power will be less than the available power?*

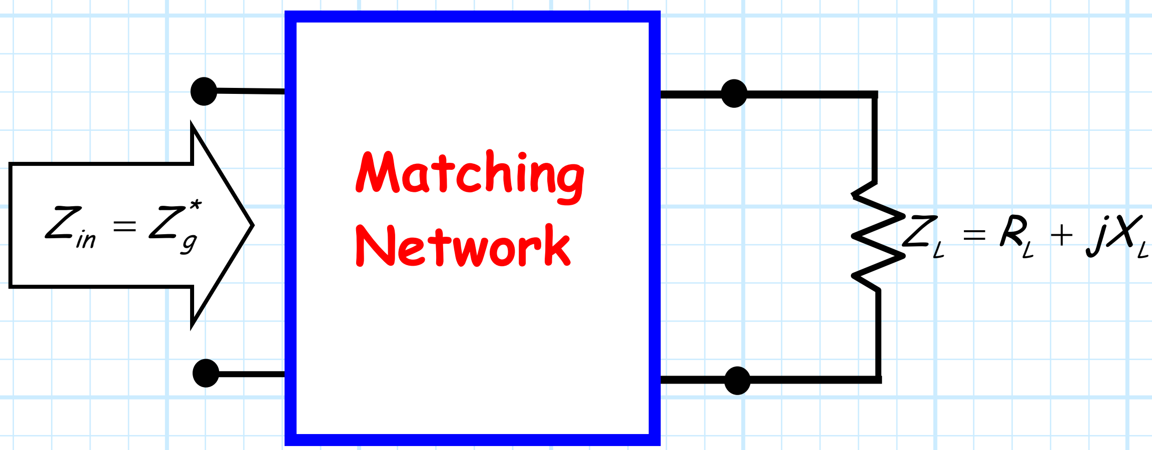
A: NO! We can in fact modify our circuit such that all available source power is delivered to the load—**without** in any way **altering** the impedance value of that **load**!

To accomplish this, we must insert a **matching network** between the source and the load:



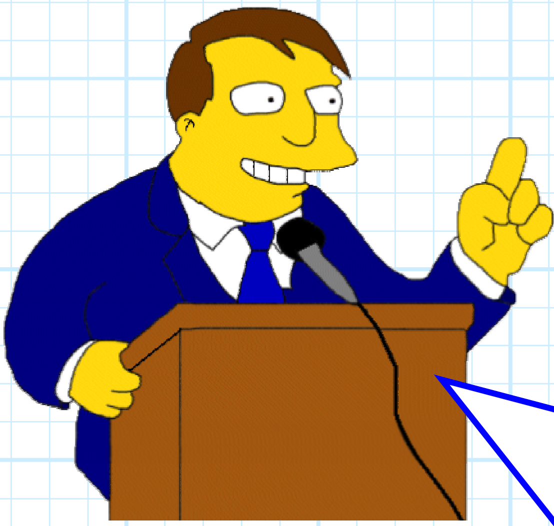
The sole purpose of this matching network is to "**transform**" the load impedance into an input impedance that **is conjugate matched** to the source! I.E.:

$$Z_{in} = \frac{V_{in}}{I_{in}} = Z_g^*$$



Because of this, **all** available source power is delivered to the input of the matching network (i.e., delivered to Z_{in}):

$$P_{in} = P_{avl}$$



Q: *Wait just one second! The matching network ensures that **all** available power is delivered to the **input** of the matching network, but that does **not** mean (necessarily) that this power will be delivered to the **load** Z_L. The power delivered to the load **could** still be **much less** than the available power!*

A: True! To ensure that the **available power** delivered to the input of the matching network is **entirely** delivered to the **load**, we must construct our matching network such that it **cannot absorb any power**—the matching network must be **lossless**!

We must construct our matching network entirely with **reactive elements**!

Examples of reactive elements include inductors, capacitors, transformers, as well as lengths of **lossless transmission lines**.

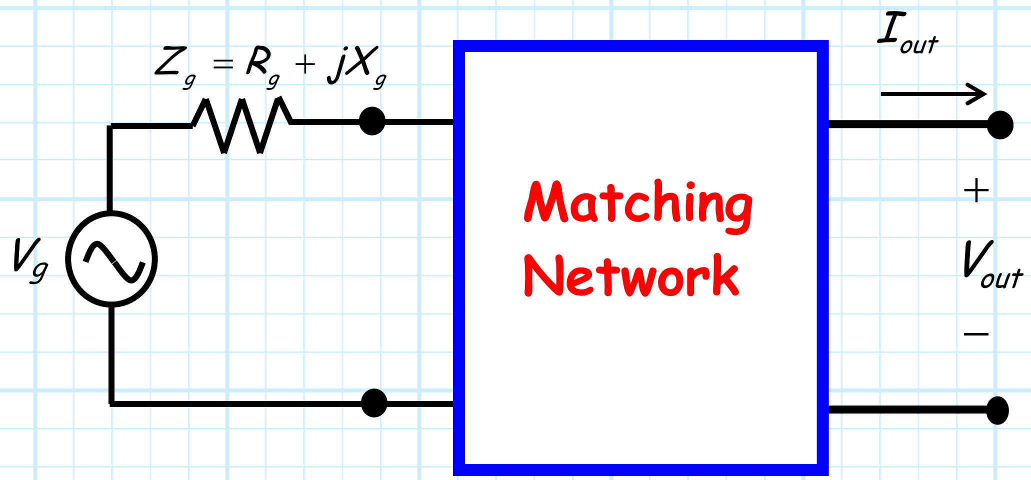
Thus, constructing a proper lossless matching network will lead to the **happy** condition where:

$$P_L = P_{in} = P_{avl}$$

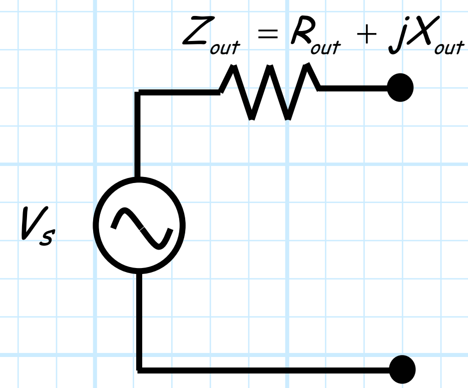
* Note that the design and construction of this lossless network will depend on **both** the value of source impedance Z_g and load impedance Z_L.

* However, the matching network does **not physically alter** the values of either of these two quantities—the source and load are left physically unchanged!

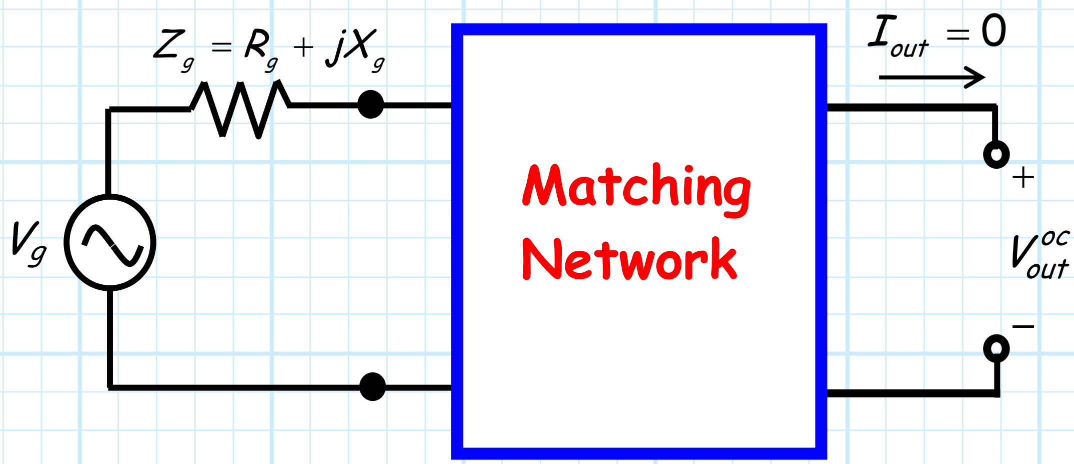
Now, let's consider the matching network from a **different perspective**. Instead of defining it in terms of its **input impedance** when attached the **load**, let's describe it in terms of its **output impedance** when attached to the **source**:



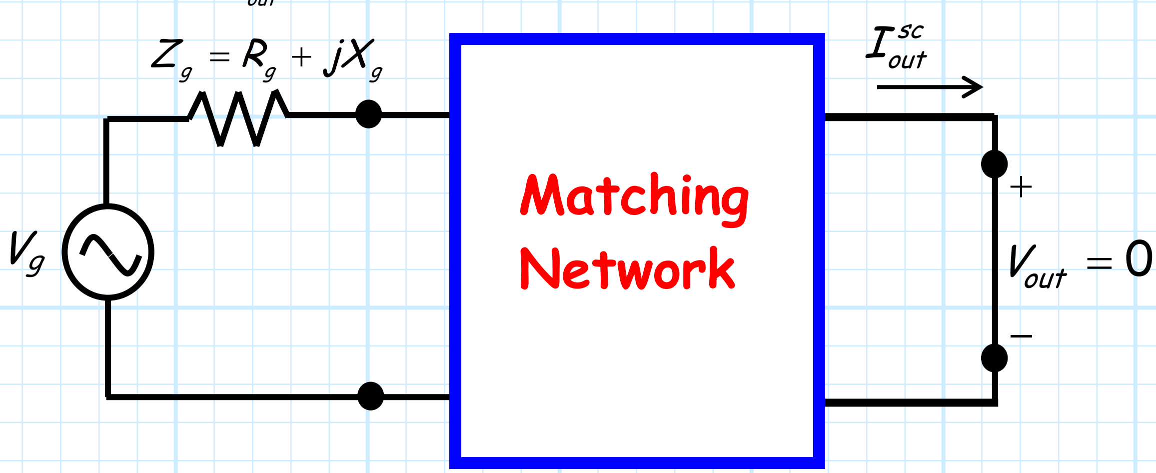
This "new" source (i.e., the original source with the matching network attached) can be expressed in terms of its **Thevenin's equivalent circuit**:



This equivalent circuit can be determined by first evaluating (or measuring) the **open-circuit** output **voltage** V_{out}^{oc}:



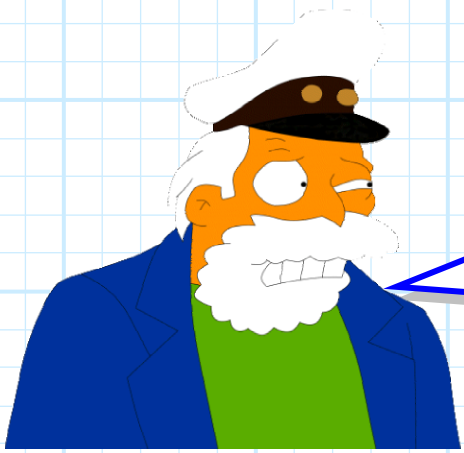
And likewise evaluating (or measuring) the **short-circuit** output **current** I_{out}^{sc}:



From these two values (V_{out}^{oc} and I_{out}^{sc}) we can determine the **Thevenin's equivalent source**:

$$V_s = V_{out}^{oc} \qquad\qquad Z_{out} = \frac{V_{out}^{oc}}{I_{out}^{oc}}$$

Note that in general that $V_s \neq V_g$ and $Z_{out} \neq Z_g$—the matching network "transforms" **both** the values of both the impedance **and** the voltage source.



Q: *Arrrgg! Doesn't that mean that the **available power** of this "transformed" source will be **different** from the original?*

A: Nope. **If** the matching network is **lossless,** the available power of this equivalent source is **identical** to the available power of the original source—the lossless matching network does **not** alter the available power!

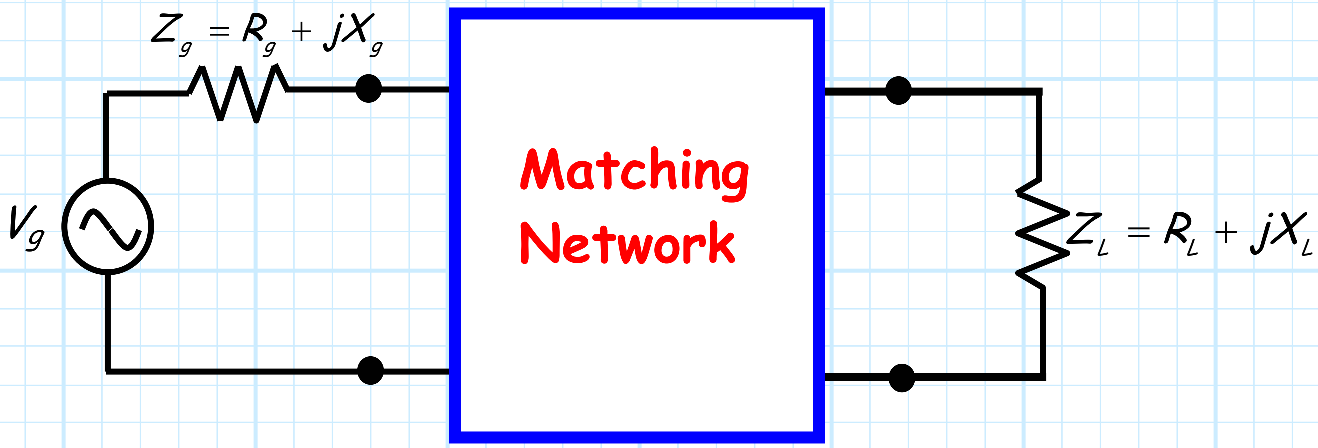
$$P_{avl} = \frac{|V_g|^2}{8\,R_g} = \frac{|V_s|^2}{8\,R_{out}}$$

Now, for a **properly** designed, **lossless** matching network, it turns out that (as **you** might have expected!) the output impedance Z_{out} is equal to the **complex conjugate** of the load impedance. I.E.:

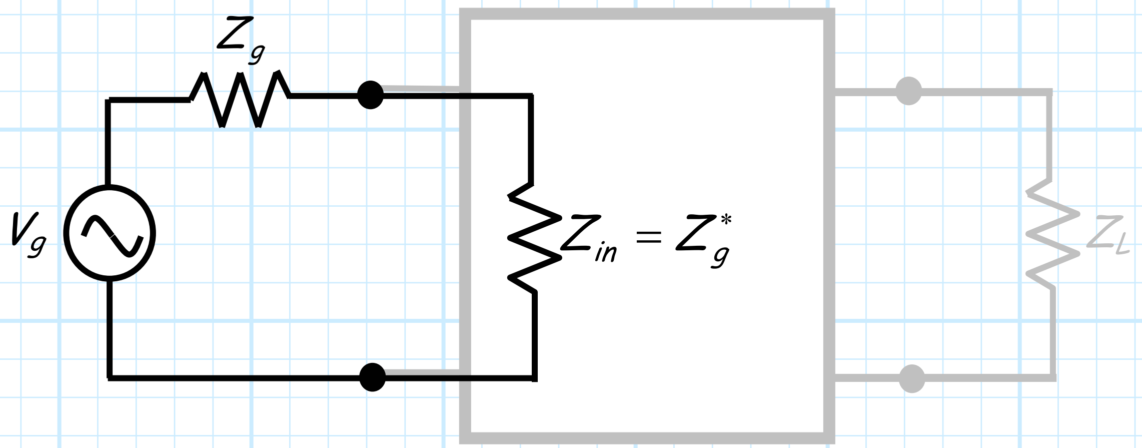
$$Z_{out} = Z_L^*$$

The source and load are again matched!

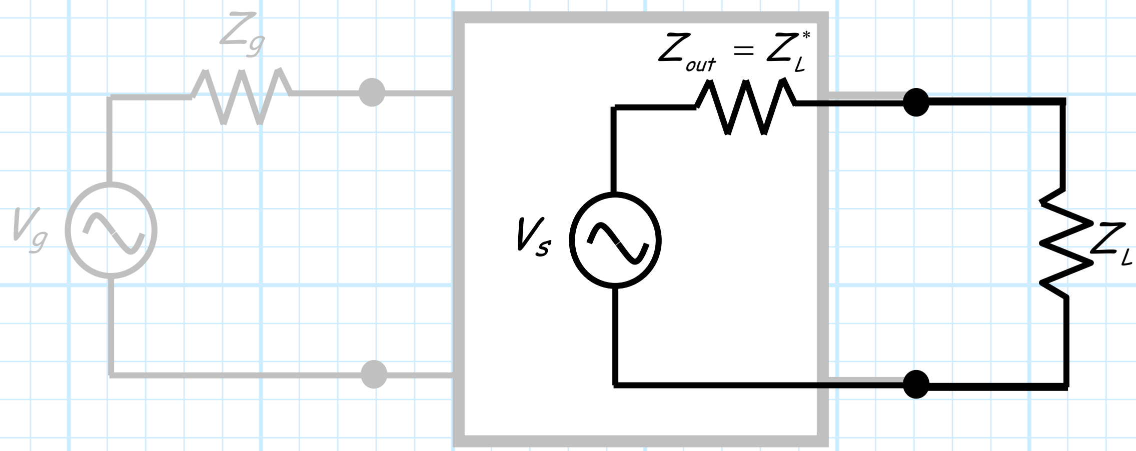
Thus, we can look at the matching network in two equivalent ways:



1. As a network attached to a load, one that "transforms" its impedance to Z_{in}—a value matched to the source impedance Z_g:

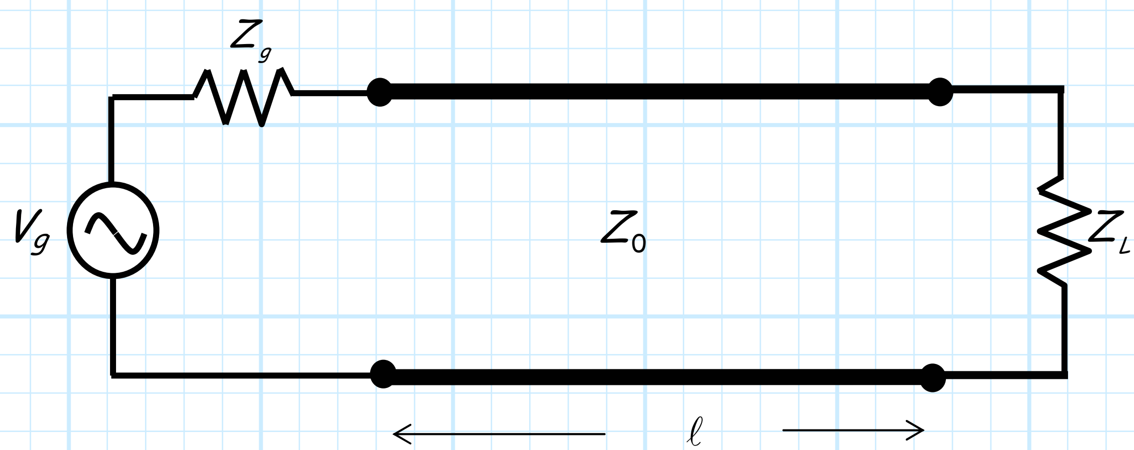


2. Or, as network attached to a source, one that "transforms" its impedance to Z_{out} —a value matched to the load impedance Z_L:



Either way, the source and load impedance are conjugate matched—**all** the available power is delivered to the load!

Recall that a primary purpose of a transmission line is to allow the transfer of power from a source to a load.



Recall that the **efficacy** of this power transfer depends on:

1. the source impedance Z_g.

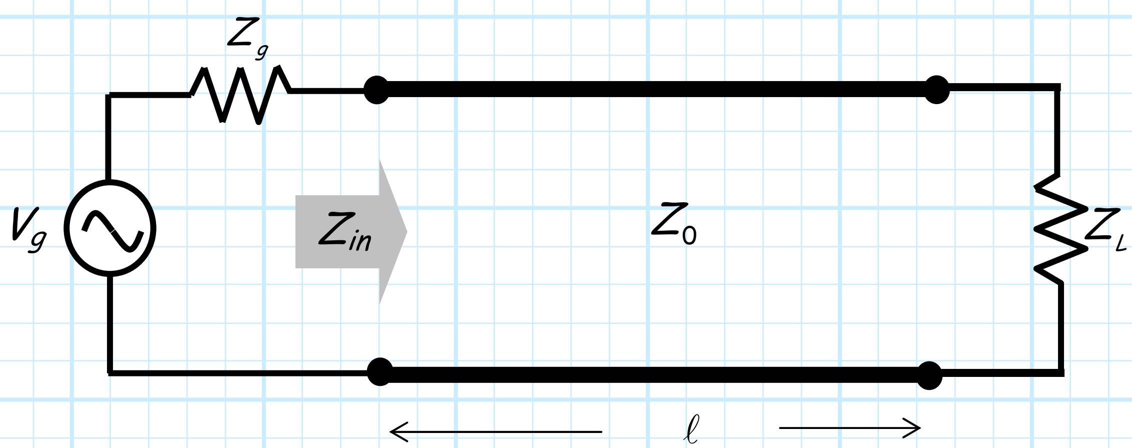
2. load impedance Z_L.

3. the transmission line characteristic impedance Z_0.

4. the transmission line length ℓ.

Matching Networks and Transmission Lines

Recall that a primary purpose of a transmission line is to allow the transfer of **power** from a source to a load.



Q: *So, say we directly connect an **arbitrary** source to an **arbitrary** load via a length of transmission line. Will the power delivered to the load be equal to the **available power** of the source?*

A: Not likely! Remember we determined earlier that the efficacy of power transfer depends on:

1. the source impedance Z_g.

2. load impedance Z_L.

3. the transmission line characteristic impedance Z_0.

4. the transmission line length ℓ.

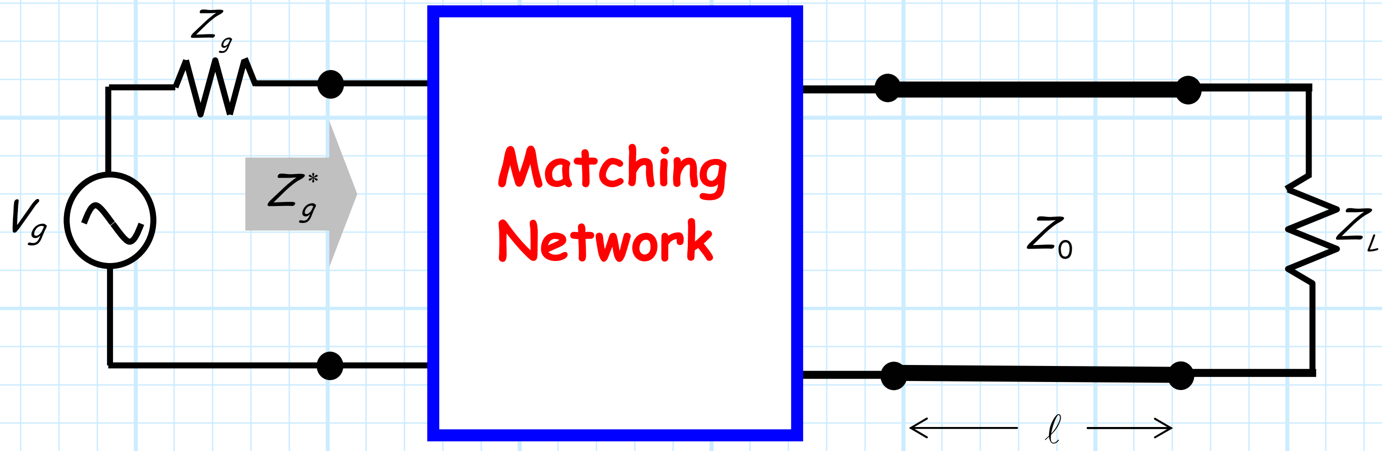
Recall that **maximum** power transfer occurred only when these four parameters resulted in the **input impedance** of the transmission line being equal to the **complex conjugate** of the **source impedance** (i.e., $Z_{in}^* = Z_g$).

It is of course **unlikely** that the very **specific** conditions of a **conjugate match** will occur if we simply connect a length of transmission line between an **arbitrary** source and load, and thus the power delivered to the load will generally be **less** than the **available power** of the source.

Q: *Is there any way to use a* ***matching network*** *to fix this problem? Can the power delivered to the load be increased to* ***equal*** *the available power of the source if there is a transmission line connecting them?*

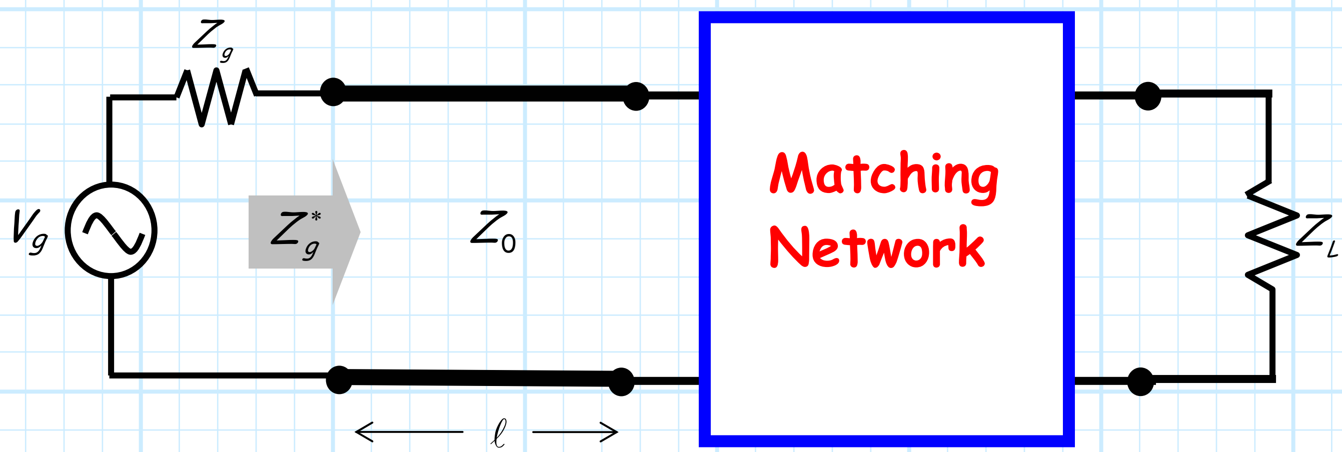
A: There sure is! We **can** likewise construct a matching network for the case where the source and load are connected by a **transmission line**.

For example, we can construct a network to transform the **input impedance** of the transmission line into the complex conjugate of **the source impedance**.

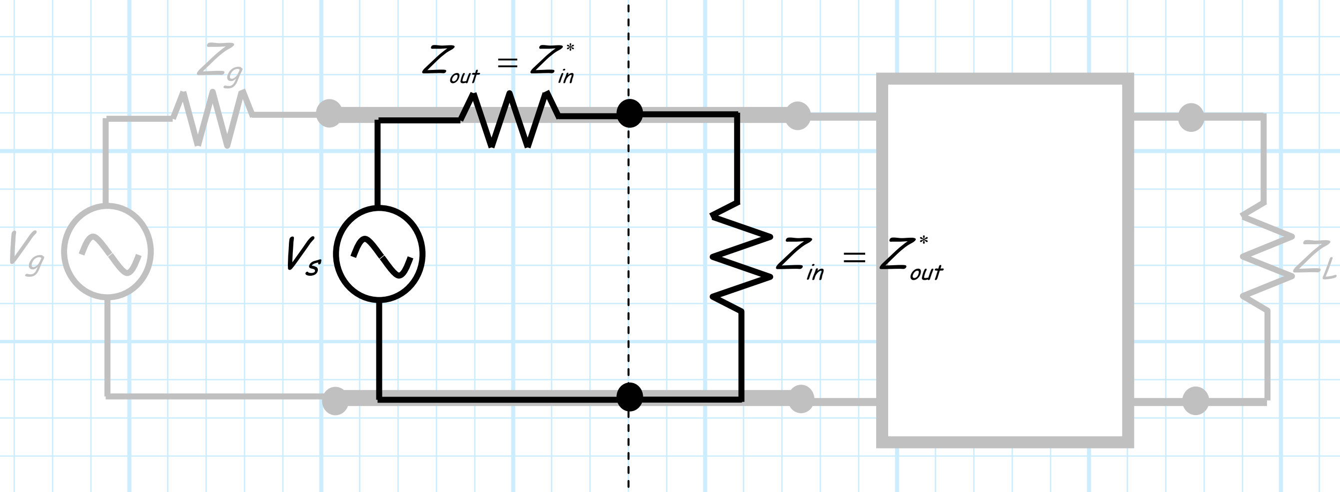


Q: *But, do we **have** to place the matching network between the source and the transmission line?*

A: Nope! We could **also** place a (different) matching network between the transmission line and the load.



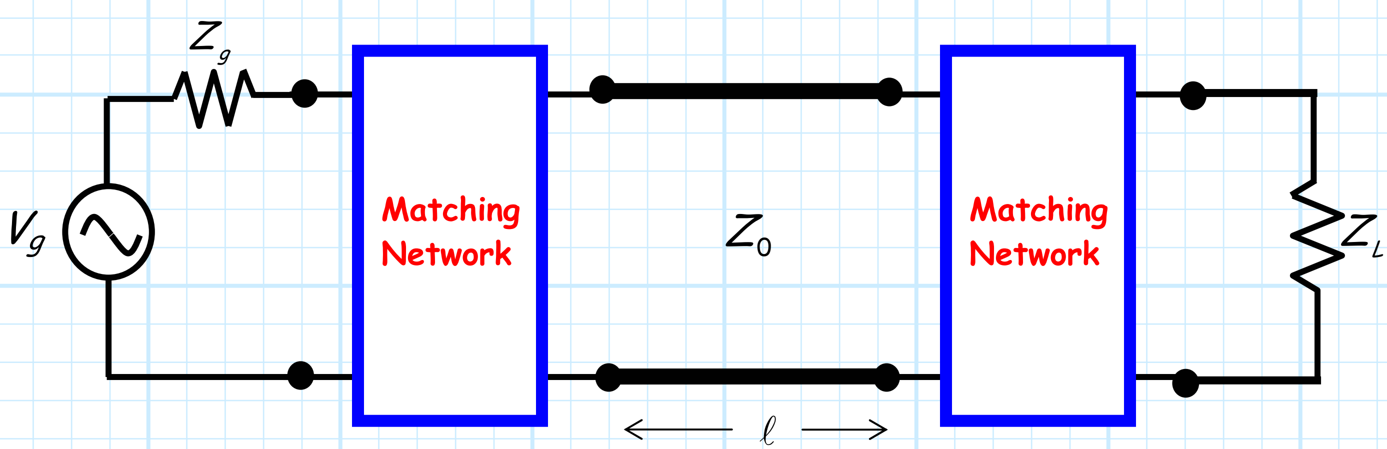
In either case, we find that at **any** and **all** points along this matched circuit, the output impedance of the equivalent **source** (i.e., looking left) will be equal to the **complex conjugate** of the **input** impedance (i.e., looking right).



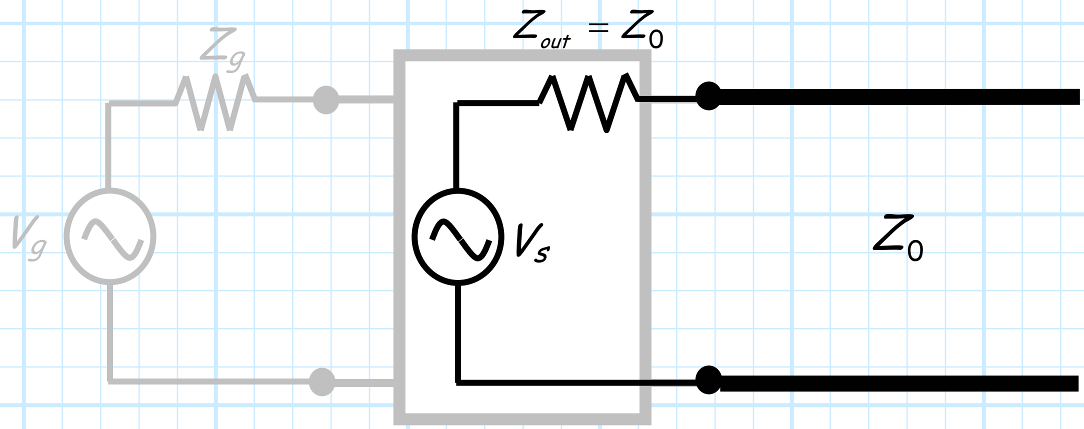
Q: *So **which** method should we chose? Do engineers typically place the matching network between the source and the transmission line, **or** place it between the transmission line and the load?*

A: Actually, the typical solution is to do **both**!

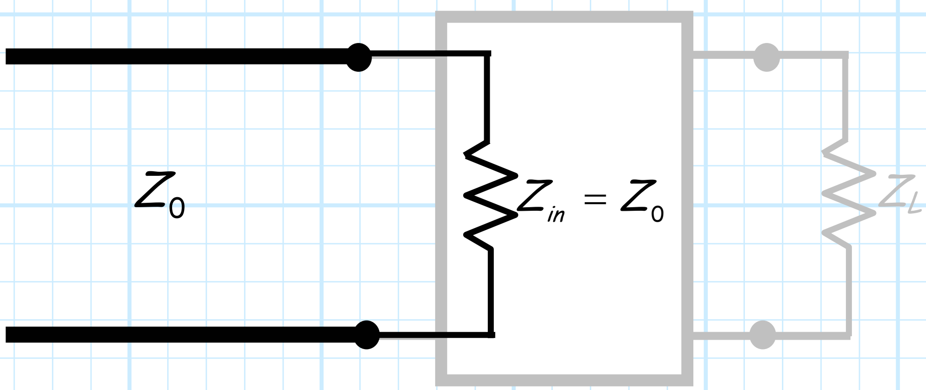
We find that often there is a matching network between the a source and the transmission line, **and** between the line and the load.



The first network matches the **source** to the **transmission line**—in other words, it transforms the **output impedance** of the equivalent source to a value numerically equal to **characteristic impedance** Z_0:



The second network matches the **load** to the **transmission line**—in other words it transforms the **load impedance** to a value numerically equal to **characteristic impedance** Z_0:



Q: *Yikes! Why would we want to build* ***two*** *separate matching networks, instead of just* ***one****?*

A: By using two separate matching networks, we can **decouple** the design problem. Recall again that the design of a **single** matching network solution would depend on four separate parameters:

1. the source impedance Z_g.

2. load impedance Z_L.

3. the transmission line characteristic impedance Z_0.

4. the transmission line length ℓ.

Alternatively, the design of the network matching the **source** and **transmission line** depends on **only**:

1. the source impedance Z_g.

2. the transmission line characteristic impedance Z_0.

Whereas, the design of the network matching the **load** and **transmission line** depends on **only**:

1. the source impedance Z_L.

2. the transmission line characteristic impedance Z_0.

Note that **neither** design depends on the transmission line **length** ℓ!

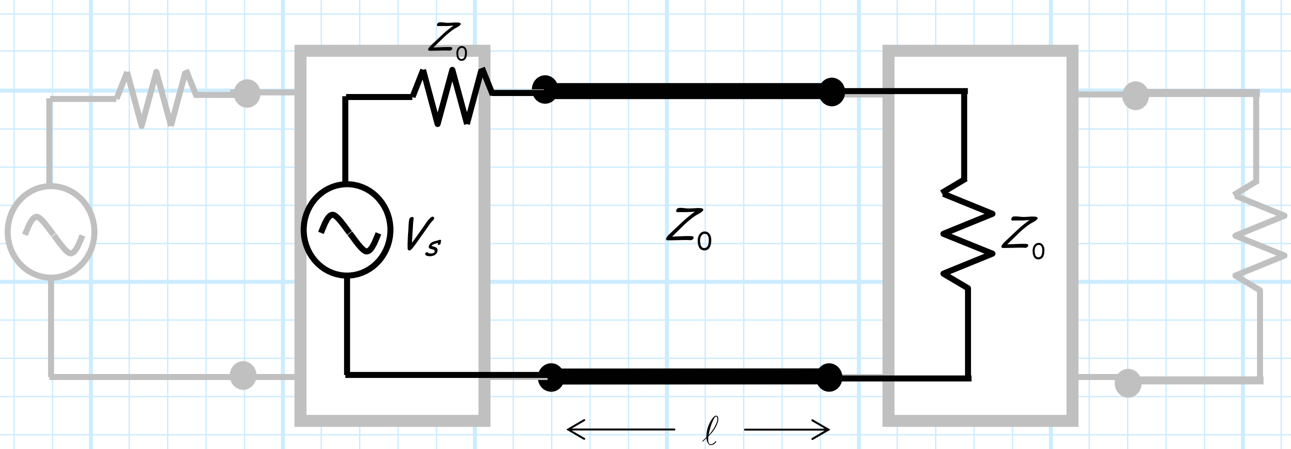
Q: *How is that possible?*

A: Remember the case where $Z_g = Z_0 = Z_L$. For that **special** case, we found that a conjugate match was the result—**regardless** of the transmission line length.

Thus, by matching the source to line impedance Z_0 **and** likewise matching the load to the line impedance, a conjugate match is **assured**—but the **length** of the transmission line does **not** matter!

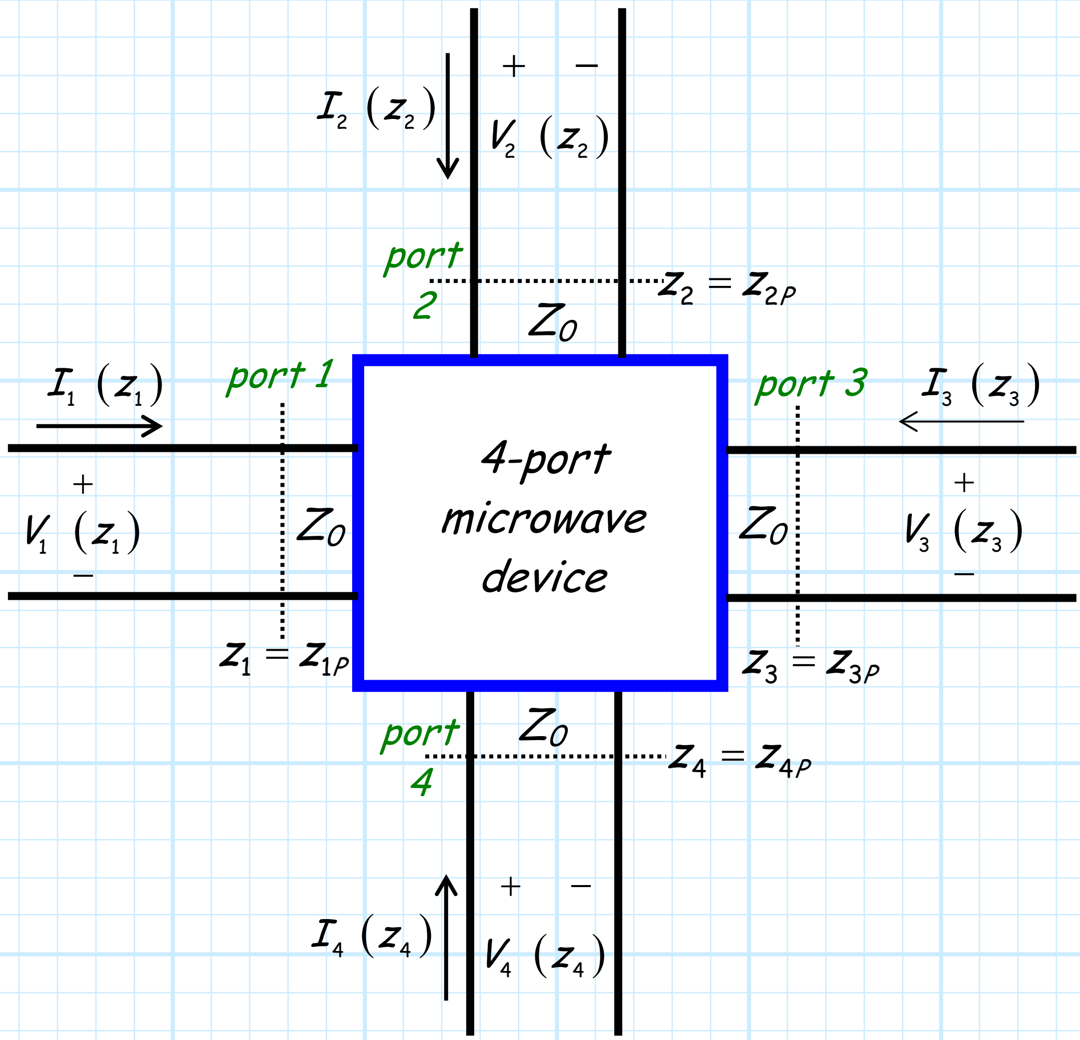
In fact, the typically problem for microwave engineers is to match a load (e.g., device input impedance) to a **standard** transmission line impedance (typically $Z_0 = 50\Omega$); **or** to independently match a source (e.g., device output impedance) to a **standard** line impedance.

A **conjugate match** is thus obtained by connecting the two with a transmission line of **any length**!



The Impedance Matrix

Consider the **4-port** microwave device shown below:



Note in this example, there are four **identical** transmission lines connected to the same "box". Inside this box there may be a very **simple** linear device/circuit, **or** it might contain a very large and **complex** linear microwave system.

→ Either way, the "box" can be fully characterized by its **impedance matrix**!

First, note that each transmission line has a specific **location** that effectively defines the **input** to the device (i.e., z_{1P}, z_{2P}, z_{3P}, z_{4P}). These often arbitrary positions are known as the **port** locations, or port **planes** of the device.

Thus, the **voltage** and **current** at port n is:

$$V_n(z_n = z_{nP}) \qquad\qquad I_n(z_n = z_{nP})$$

We can **simplify** this cumbersome notation by simply **defining** port n current and voltage as I_n and V_n:

$$V_n = V_n(z_n = z_{nP}) \qquad\qquad I_n = I_n(z_n = z_{nP})$$

For **example**, the current at port **3** would be $I_3 = I_3(z_3 = z_{3P})$.

Now, say there exists a non-zero current at **port 1** (i.e., $I_1 \neq 0$), while the current at all **other** ports are known to be **zero** (i.e., $I_2 = I_3 = I_4 = 0$).

Say we measure/determine the **current** at port **1** (i.e., determine I_1), and we then measure/determine the **voltage** at the port **2** plane (i.e., determine V_2).

The complex ratio between V_2 and I_1 is know as the **trans-impedance parameter** Z_{21}:

$$Z_{21} = \frac{V_2}{I_1}$$

Likewise, the trans-impedance parameters Z_{31} and Z_{41} are:

$$Z_{31} = \frac{V_3}{I_1} \qquad \text{and} \qquad Z_{41} = \frac{V_4}{I_1}$$

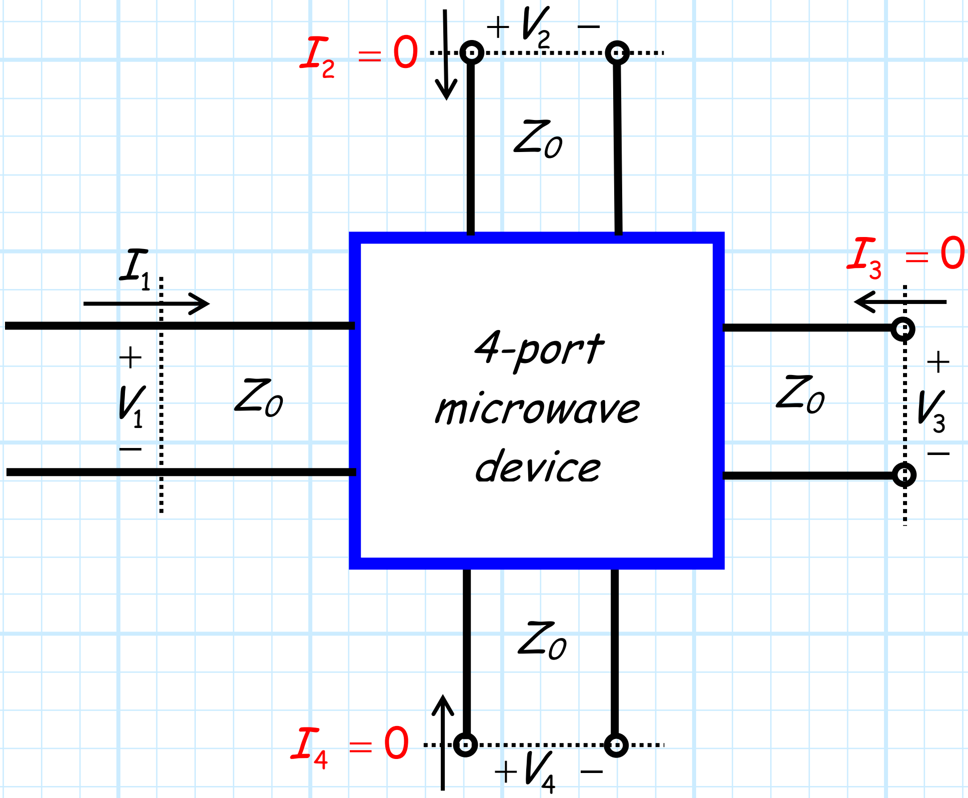
We of course could **also** define, say, trans-impedance parameter Z_{34} as the ratio between the complex values I_4 (the current into port 4) and V_3 (the voltage at port 3), given that the current at all other ports (1, 2, and 3) are zero.

Thus, more **generally**, the ratio of the current into port n and the voltage at port m is:

$$\boxed{Z_{mn} = \frac{V_m}{I_n} \qquad \text{(given that } I_k = 0 \text{ for all } k \neq n\text{)}}$$

Q: *But how do we ensure that all but **one** port current is zero ?*

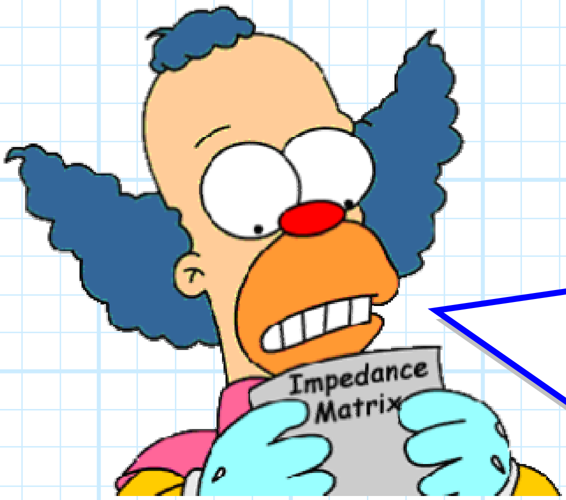
A: Place an **open** circuit **at** those ports!



Placing an **open** at a port (and it must be **at** the port!) **enforces** the condition that $I = 0$.

Now, we can thus **equivalently** state the definition of trans-impedance as:

$$Z_{mn} = \frac{V_m}{I_n} \qquad \text{(given that all ports } k \neq n \text{ are } \textbf{open}\text{)}$$



Q: *As impossible as it sounds, this handout is even more **boring** and **pointless** than any of your previous efforts. **Why** are we studying this? After all, what is the likelihood that a device will have an **open** circuit on **all** but one of its ports?!*

A: OK, say that **none** of our ports are **open-circuited**, such that we have currents **simultaneously** on **each** of the **four** ports of our device.

Since the device is **linear**, the voltage at any **one** port due to **all** the port currents is simply the coherent **sum** of the voltage at that port due to **each** of the currents!

For example, the voltage at port 3 can be determined by:

$$V_3 = Z_{34}\, I_4 + Z_{33}\, I_3 + Z_{32}\, I_2 + Z_{31}\, I_1$$

More generally, the voltage at port m of an N-port device is:

$$\boxed{V_m = \sum_{n=1}^{N} Z_{mn}\, I_n}$$

This expression can be written in **matrix** form as:

$$\bar{\mathbf{V}} = \bar{\bar{\mathbf{Z}}}\,\bar{\mathbf{I}}$$

Where $\bar{\mathbf{I}}$ is the **vector**:

$$\bar{\mathbf{I}} = \left[I_1, I_2, I_3, \cdots, I_N\right]^T$$

and $\bar{\mathbf{V}}$ is the vector:

$$\bar{\mathbf{V}} = \left[V_1, V_2, V_3, \ldots, V_N\right]^T$$

And the matrix $\bar{\bar{\mathbf{Z}}}$ is called the **impedance matrix**:

$$\bar{\bar{\mathbf{Z}}} = \begin{bmatrix} Z_{11} & \cdots & Z_{1n} \\ \vdots & \ddots & \vdots \\ Z_{m1} & \cdots & Z_{mn} \end{bmatrix}$$

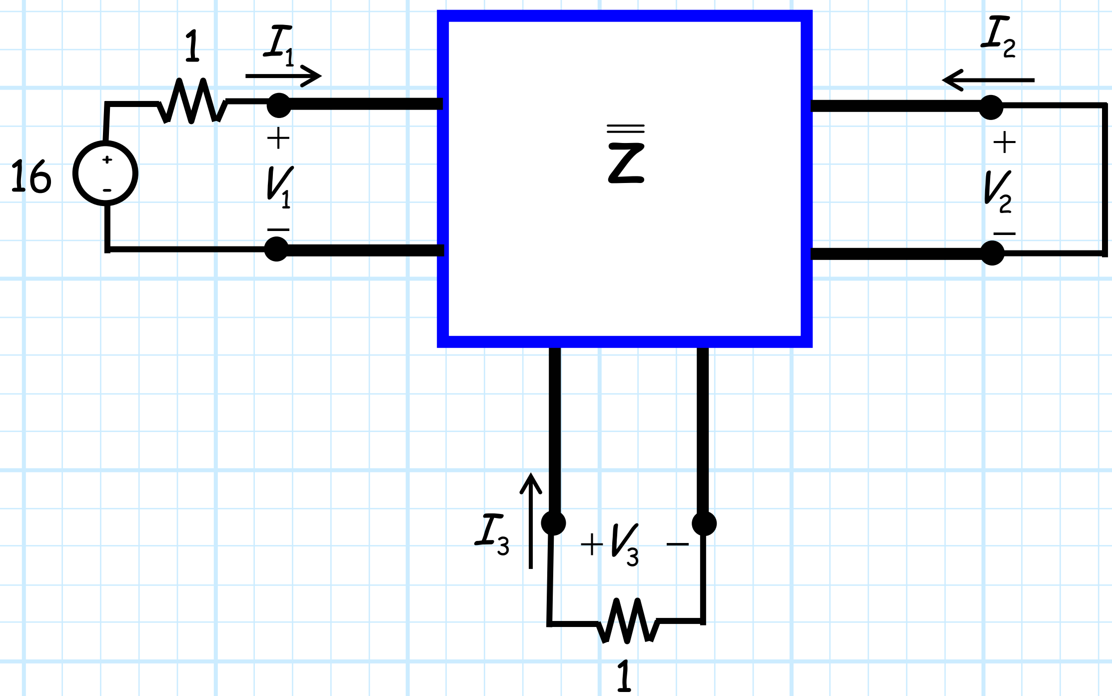
The impedance matrix is a N by N matrix that **completely characterizes** a linear, N-port device. Effectively, the impedance matrix describes a multi-port device the way that Z_L describes a single-port device (e.g., a load)!

But **beware**! The values of the impedance matrix for a particular device or network, just like Z_L, are **frequency dependent**! Thus, it may be more instructive to **explicitly** write:

$$\overline{\overline{\mathbf{Z}}}(\omega)=\begin{bmatrix} Z_{11}(\omega) & \ldots & Z_{1n}(\omega) \\ \vdots & \ddots & \vdots \\ Z_{m1}(\omega) & \cdots & Z_{mn}(\omega) \end{bmatrix}$$

Example: Using the Impedance Matrix

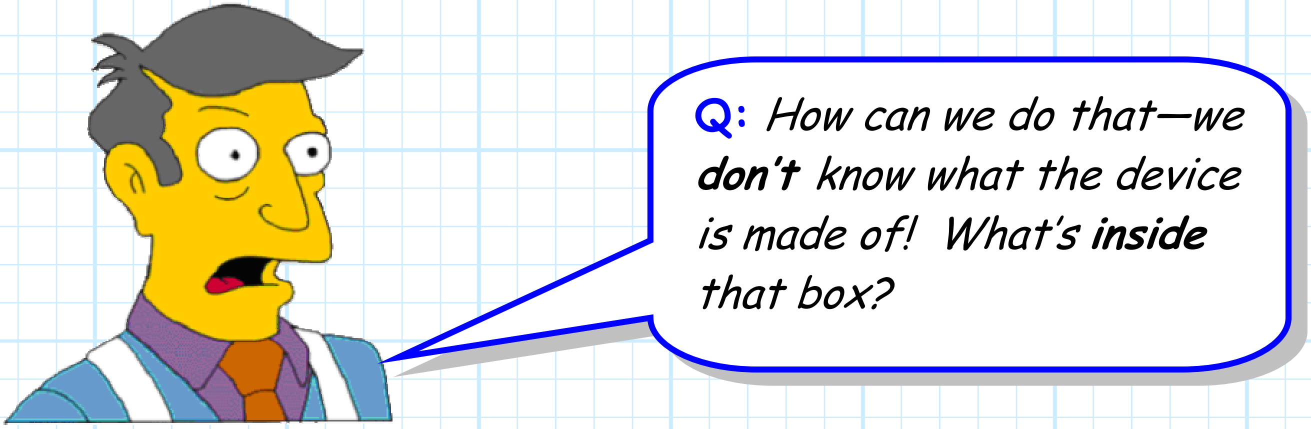
Consider the following circuit:



Where the 3-port **device** is characterized by the **impedance matrix**:

$$\bar{\bar{\mathbf{Z}}} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 4 \\ 2 & 4 & 1 \end{bmatrix}$$

Let's now determine all port **voltages** V_1, V_2, V_3 and all **currents** I_1, I_2, I_3.



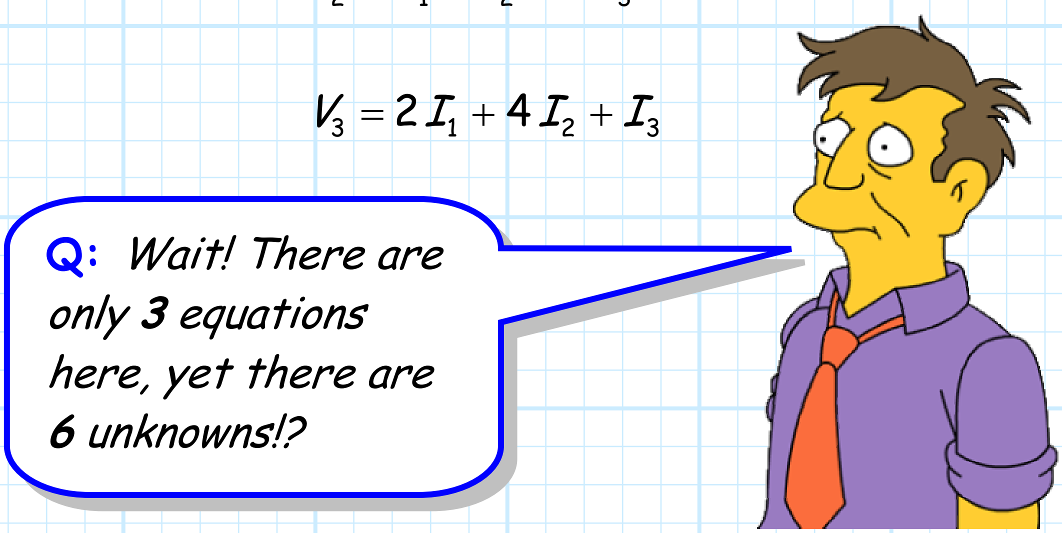
A: We **don't** need to know what's inside that box! We know its impedance matrix, and that **completely** characterizes the device (or, at least, characterizes it at **one** frequency).

Thus, we have enough information to solve this problem. From the impedance matrix we know:

$$V_1 = 2\,I_1 + I_2 + 2\,I_3$$

$$V_2 = I_1 + I_2 + 4\,I_3$$

$$V_3 = 2\,I_1 + 4\,I_2 + I_3$$



A: True! The impedance matrix describes the device in the box, but it does **not** describe the devices **attached** to it. We require **more** equations to describe them.

1. The **source** at port 1 is described by the equation:

$$V_1 = 16.0 - (1)\, I_1$$

2. The **short** circuit on port 2 means that:

$$V_2 = 0$$

3. While the **load** on port 3 leads to:

$$V_3 = -(1)\, I_3 \quad \text{(note the minus sign!)}$$

Now we have **6** equations and **6** unknowns! Combining equations, we find:

$$\begin{aligned} V_1 = 16 - I_1 &= 2\, I_1 + I_2 + 2\, I_3 \\ \therefore \quad 16 &= 3\, I_1 + I_2 + 2\, I_3 \end{aligned}$$

$$\begin{aligned} V_2 = 0 &= I_1 + I_2 + 4\, I_3 \\ \therefore 0 &= I_1 + I_2 + 4\, I_3 \end{aligned}$$

$$\begin{aligned} V_3 = -I_3 &= 2\, I_1 + 4\, I_2 + I_3 \\ \therefore \quad 0 &= 2\, I_1 + 4\, I_2 + 2 I_3 \end{aligned}$$

Solving, we find (I'll let **you** do the algebraic details!):

$$I_1 = 7.0 \qquad I_2 = -3.0 \qquad I_3 = -1.0$$

$$V_1 = 9.0 \qquad V_2 = 0.0 \qquad V_3 = 1.0$$

The Scattering Matrix

At **"low"** frequencies, we can completely characterize a **linear** device or network using an **impedance** matrix, which relates the currents and voltages at **each** device terminal to the currents and voltages at **all** other terminals.

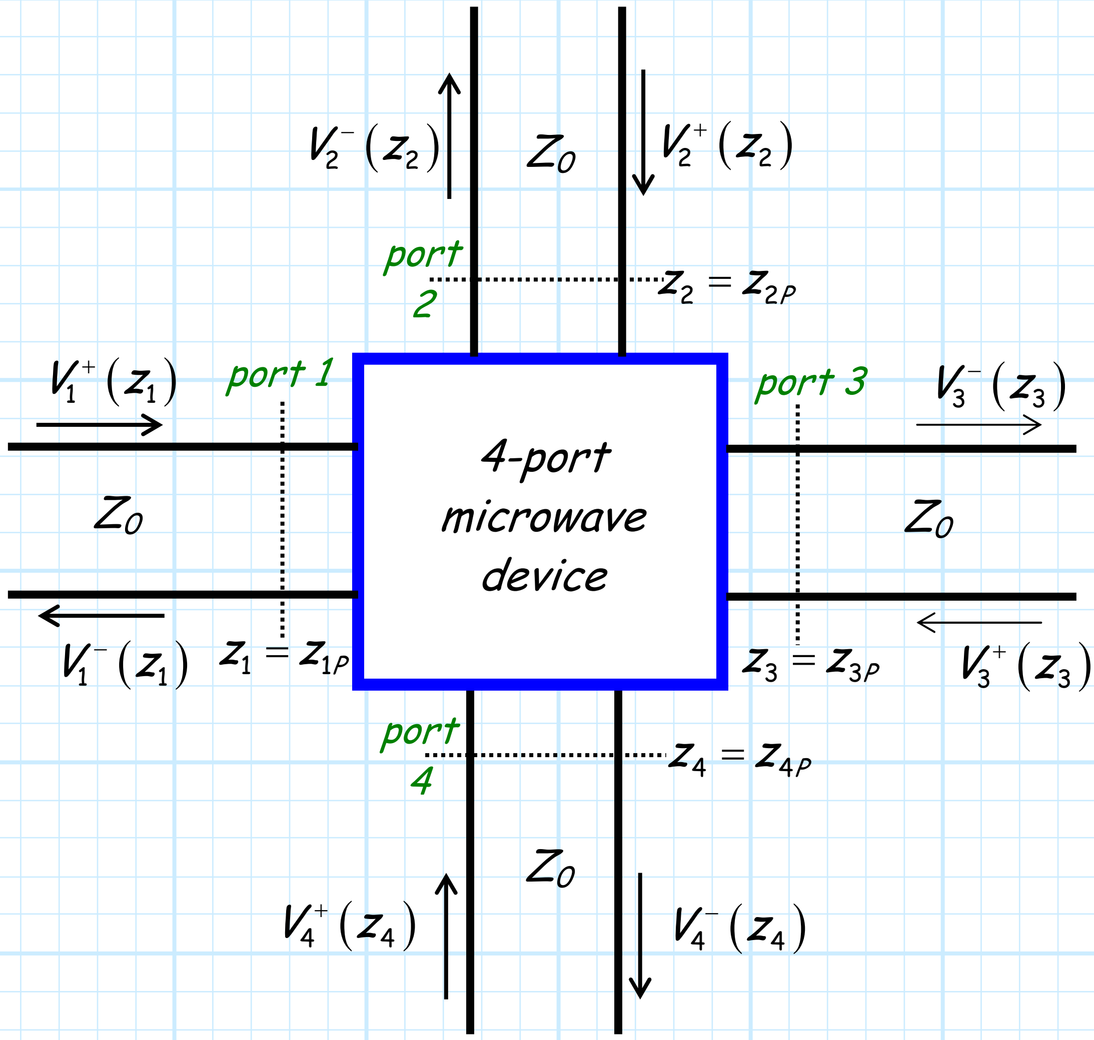
But, at microwave frequencies, it is **difficult** to measure **total** currents and voltages!

* Instead, we can measure the **magnitude** and **phase** of each of the two transmission line **waves** $V^+(z)$ and $V^-(z)$.

* In other words, we can determine the relationship between the incident and reflected wave at **each** device terminal to the incident and reflected waves at **all** other terminals.

These relationships are completely represented by the **scattering matrix**. It **completely** describes the behavior of a linear, multi-port device at a **given frequency** ω.

Consider the **4-port** microwave device shown below:



Note in this example, there are four **identical** transmission lines connected to the same "box". Inside this box there may be a very **simple** linear device/circuit, **or** it might contain a very large and **complex** linear microwave system.

→ Either way, the "box" can be fully characterized by its **scattering parameters**!

First, note that each transmission line has a specific **location** that effectively defines the **input** to the device (i.e., z_{1P}, z_{2P}, z_{3P}, z_{4P}). These often arbitrary positions are known as the **port** locations, or port **planes** of the device.

Say there exists an **incident** wave on **port 1** (i.e., $V_1^+(z_1) \neq 0$), while the incident waves on all other ports are known to be **zero** (i.e., $V_2^+(z_2) = V_3^+(z_3) = V_4^+(z_4) = 0$).

Say we measure/determine the voltage of the wave flowing **into port 1**, at the port 1 **plane** (i.e., determine $V_1^+(z_1 = z_{1P})$).
Say we then measure/determine the voltage of the wave flowing **out** of **port 2**, at the port 2 plane (i.e., determine $V_2^-(z_2 = z_{2P})$).

The complex ratio between $V_1^+(z_1 = z_{1P})$ and $V_2^-(z_2 = z_{2P})$ is know as the **scattering parameter** S_{21}:

$$S_{21} = \frac{V_2^-(z = z_2)}{V_1^+(z = z_1)} = \frac{V_{02}^- \, e^{+j\beta z_{2P}}}{V_{01}^+ \, e^{-j\beta z_{1P}}} = \frac{V_{02}^-}{V_{01}^+} e^{+j\beta(z_{2P}+z_{1P})}$$

Likewise, the scattering parameters S_{31} and S_{41} are:

$$S_{31} = \frac{V_3^-(z_3 = z_{3P})}{V_1^+(z_1 = z_{1P})} \qquad \text{and} \qquad S_{41} = \frac{V_4^-(z_4 = z_{4P})}{V_1^+(z_1 = z_{1P})}$$

We of course could **also** define, say, scattering parameter S_{34} as the ratio between the complex values $V_4^+(z_4 = z_{4P})$ (the wave **into** port 4) and $V_3^-(z_3 = z_{3P})$ (the wave **out of** port 3), given that the input to all other ports (1,2, and 3) are zero.

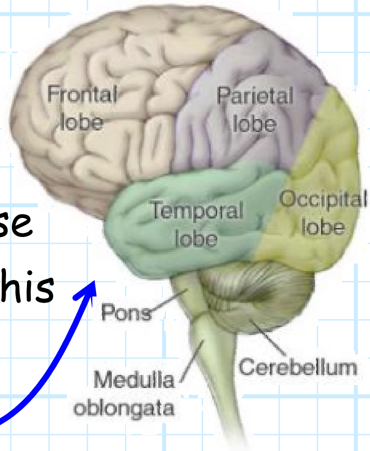
Thus, more **generally**, the ratio of the wave incident on port n to the wave emerging from port m is:

$$S_{mn} = \frac{V_m^-(z_m = z_{mP})}{V_n^+(z_n = z_{nP})} \qquad \text{(given that} \quad V_k^+\left(z_k\right) = 0 \ \text{ for all } k \neq n\text{)}$$

Note that frequently the port positions are assigned a **zero** value (e.g., $z_{1P} = 0$, $z_{2P} = 0$). This of course **simplifies** the scattering parameter calculation:

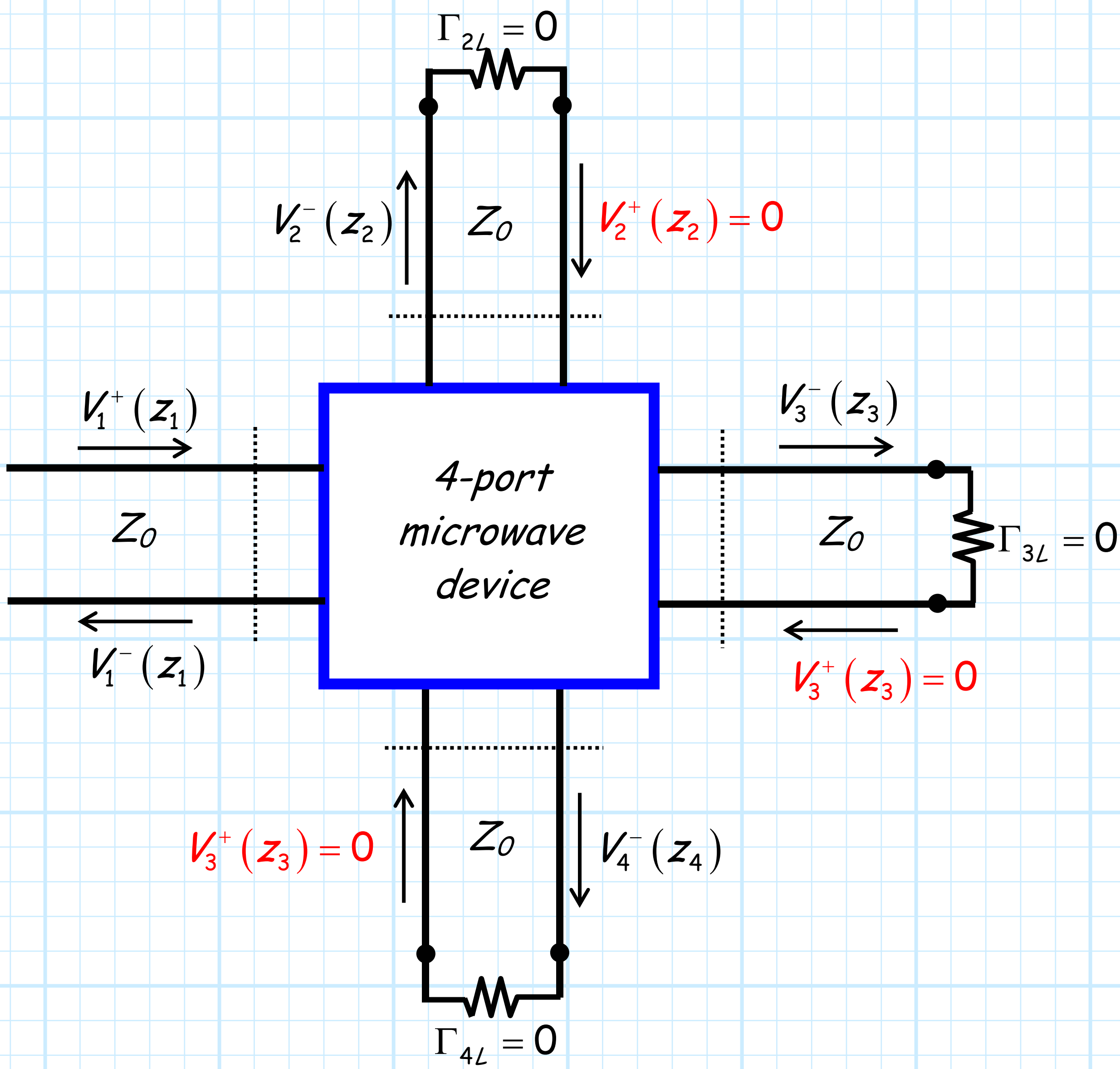
$$S_{mn} = \frac{V_m^-(z_m = 0)}{V_n^+(z_n = 0)} = \frac{V_{0m}^- \, e^{+j\beta 0}}{V_{0n}^+ \, e^{-j\beta 0}} = \frac{V_{0m}^-}{V_{0n}^+}$$

We will **generally assume** that the port locations are defined as $z_{nP} = 0$, and thus use the **above** notation. But **remember** where this expression came from!



Q: *But how do we ensure that **only one** incident wave is non-zero ?*

A: Terminate all other ports with a **matched load**!



Note that **if** the ports are terminated in a **matched load** (i.e., $Z_L = Z_0$), then $\Gamma_{nL} = 0$ and therefore:

$$V_n^+(z_n) = 0$$

In other words, terminating a port ensures that there will be **no signal** incident on that port!

Q: *Just between you and me, I think you've messed this up!* ***In all*** *previous handouts you said that if* $\Gamma_L = 0$, the wave in the **minus** direction would be zero:

$$V^-(z) = 0 \quad \text{if} \quad \Gamma_L = 0$$

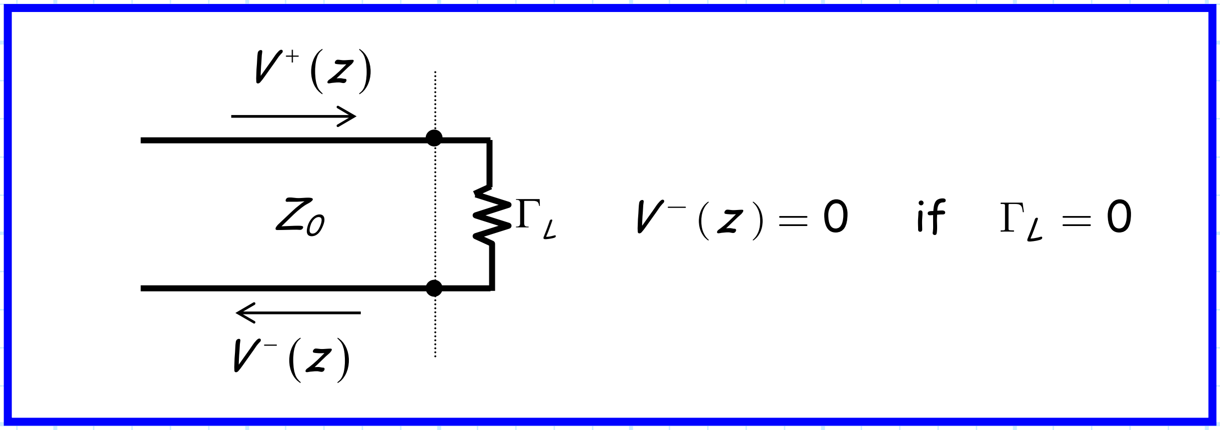
but just ***now*** *you said that the wave in the* ***positive*** *direction would be zero:*

$$V^+(z) = 0 \quad \text{if} \quad \Gamma_L = 0$$

Of course, there is ***no way*** *that* ***both*** *statements can be correct!*

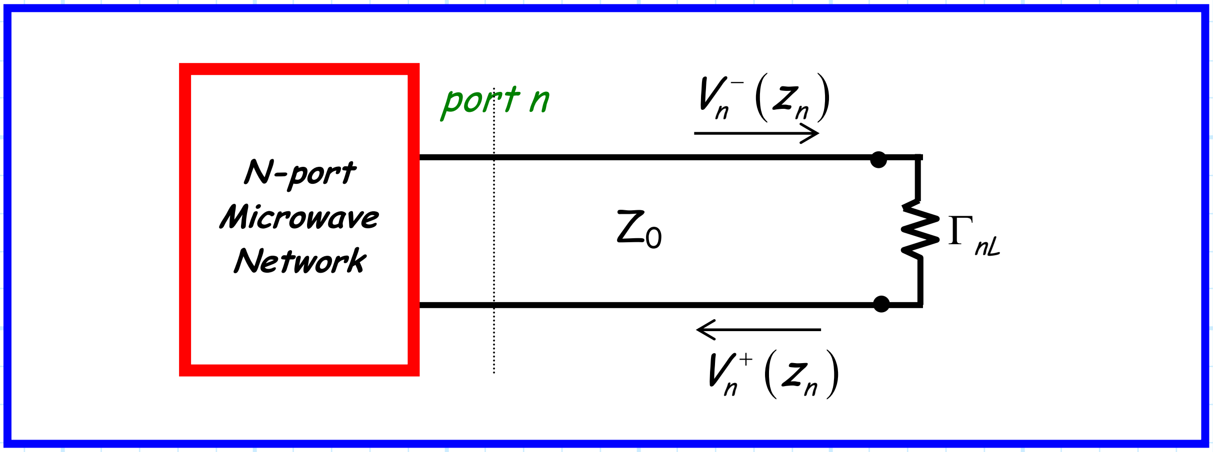
A: Actually, **both** statements are correct! You must be careful to understand the **physical definitions** of the plus and minus directions—in other words, the propagation directions of waves $V_n^+(z_n)$ and $V_n^-(z_n)$!

For example, we **originally** analyzed this case:



In this original case, the wave **incident** on the load is $V^+(z)$ (**plus** direction), while the **reflected** wave is $V^-(z)$ (**minus** direction).

Contrast this with the case we are **now** considering:

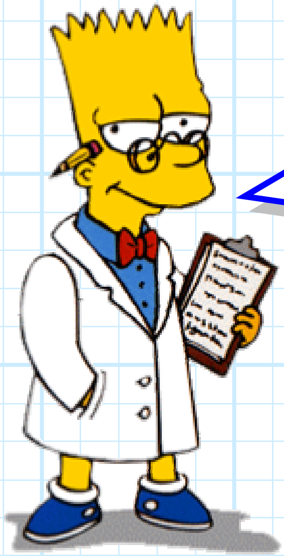


For this current case, the situation is **reversed**. The wave incident on the load is **now** denoted as $V_n^-(z_n)$ (coming **out** of port n), while the wave reflected off the load is **now** denoted as $V_n^+(z_n)$ (going **into** port n).

As a result, $V_n^+(z_n) = 0$ when $\Gamma_{nL} = 0$!

Perhaps we could more **generally** state that:

$$V^{reflected}(z = z_L) = \Gamma_L \, V^{incident}(z = z_L)$$



*For each case, **you** must be able to correctly identify the mathematical statement describing the wave **incident** on, and **reflected** from, some passive load.*

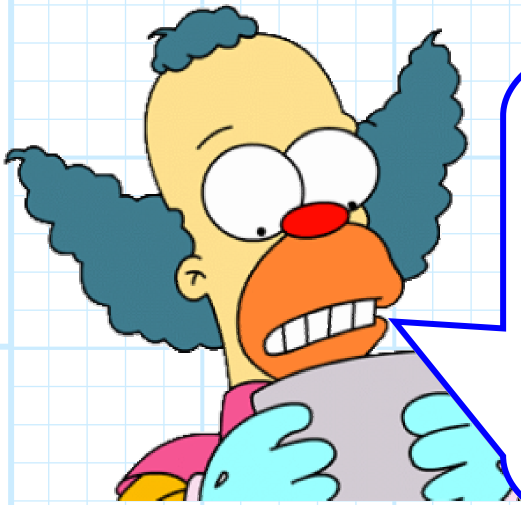
*Like most equations in engineering, the **variable names** can **change**, but the **physics** described by the mathematics will **not**!*

Now, **back** to our discussion of **S-parameters**. We found that **if** $z_{nP} = 0$ for all ports n, the scattering parameters could be directly written in terms of wave **amplitudes** V_{0n}^{+} and V_{0m}^{-}.

$$S_{mn} = \frac{V_{0m}^{-}}{V_{0n}^{+}} \qquad \text{(given that} \quad V_k^{+}(z_k) = 0 \ \text{ for all } k \neq n\text{)}$$

Which we can now **equivalently** state as:

$$S_{mn} = \frac{V_{0m}^{-}}{V_{0n}^{+}} \qquad \text{(given that all ports, except port } n\text{, are } \textbf{matched}\text{)}$$



Q: *As impossible as it sounds, this handout is even more* ***boring*** *and* ***pointless*** *than any of your previous efforts. Why are we studying this? After all, what is the likelihood that a microwave network will have only* ***one*** *incident wave—that* ***all*** *of the ports will be matched?!*

A: OK, say that our ports are **not** matched, such that we have waves **simultaneously** incident on **each** of the **four** ports of our device.

Since the device is **linear**, the output at any port due to **all** the incident waves is simply the coherent **sum** of the output at that port due to **each** input wave!

For example, the output wave at port 3 can be determined by (assuming $z_{nP} = 0$):

$$V_{03}^{-} = S_{34}\, V_{04}^{+} + S_{33}\, V_{03}^{+} + S_{32}\, V_{02}^{+} + S_{31}\, V_{01}^{+}$$

More generally, the output wave voltage at port m of an N-port device is:

$$\boxed{V_{0m}^{-} = \sum_{n=1}^{N} S_{mn}\, V_{0n}^{+} \qquad (z_{nP} = 0)}$$

This expression can be written in **matrix** form as:

$$\bar{\mathbf{V}}^- = \bar{\bar{\mathbf{S}}}\,\bar{\mathbf{V}}^+$$

Where $\bar{\mathbf{V}}^-$ is the **vector**:

$$\bar{\mathbf{V}}^- = \left[V_{01}^-, V_{02}^-, V_{03}^-, \ldots, V_{0N}^-\right]^T$$

and $\bar{\mathbf{V}}^+$ is the vector:

$$\bar{\mathbf{V}}^+ = \left[V_{01}^+, V_{02}^+, V_{03}^+, \ldots, V_{0N}^+\right]^T$$

Therefore $\bar{\bar{\mathbf{S}}}$ is the **scattering matrix**:

$$\bar{\bar{\mathbf{S}}} = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{m1} & \cdots & S_{mn} \end{bmatrix}$$

The scattering matrix is a N by N matrix that **completely characterizes** a linear, N-port device. Effectively, the scattering matrix describes a multi-port device the way that Γ_L describes a single-port device (e.g., a load)!

But **beware**! The values of the scattering matrix for a particular device or network, just like Γ_L, are **frequency dependent**! Thus, it may be more instructive to **explicitly** write:

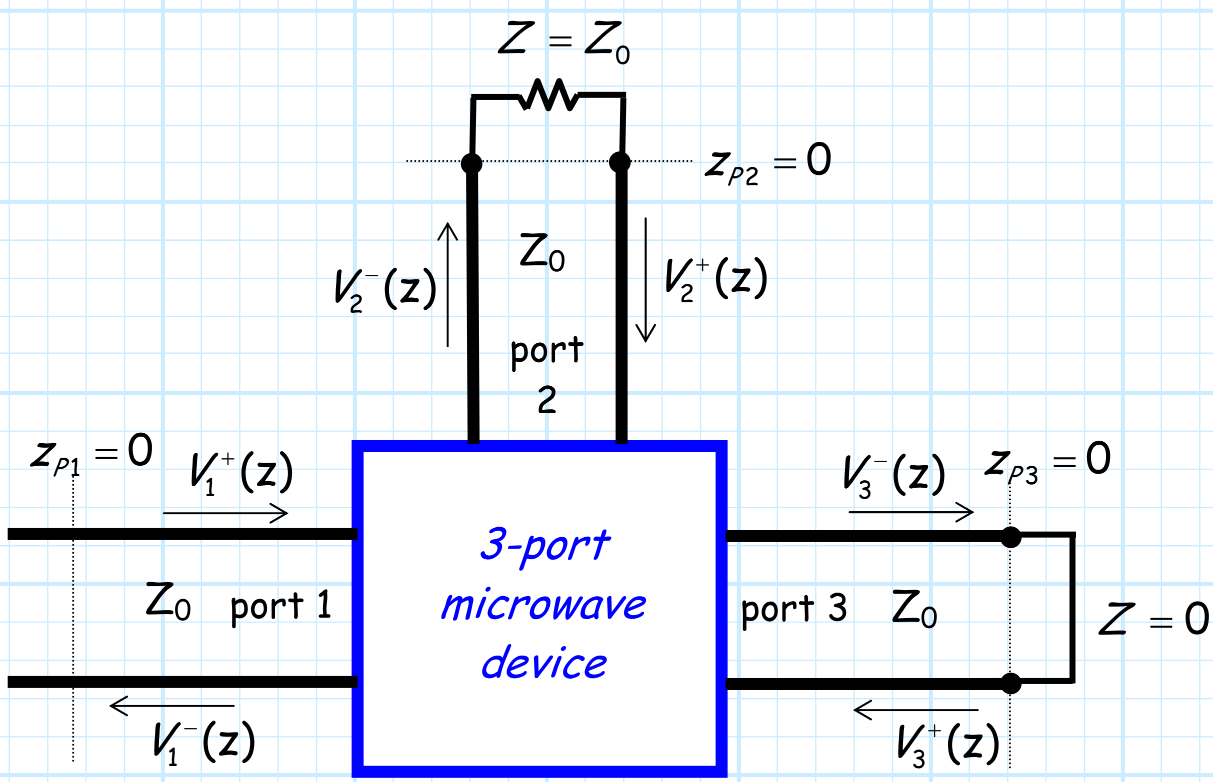
$$\bar{\bar{\mathbf{S}}}(\omega) = \begin{bmatrix} S_{11}(\omega) & \cdots & S_{1n}(\omega) \\ \vdots & \ddots & \vdots \\ S_{m1}(\omega) & \cdots & S_{mn}(\omega) \end{bmatrix}$$

Example: The Scattering Matrix

Say we have a 3-port network that is completely characterized at some frequency ω by the **scattering matrix**:

$$\bar{\bar{S}} = \begin{bmatrix} 0.0 & 0.2 & 0.5 \\ 0.5 & 0.0 & 0.2 \\ 0.5 & 0.5 & 0.0 \end{bmatrix}$$

A **matched load** is attached to port 2, while a **short circuit** has been placed at port 3:



Because of the **matched** load at port 2 (i.e., $\Gamma_L = 0$), we know that:

$$\frac{V_2^+(z_2=0)}{V_2^-(z_2=0)} = \frac{V_{02}^+}{V_{02}^-} = 0$$

and therefore:

$$V_{02}^+ = 0$$

*You've made a terrible mistake! Fortunately, **I** was here to correct it for you—since $\Gamma_L = 0$, the constant V_{02}^- (**not** V_{02}^+) is equal to zero.*

NO!! Remember, the signal $V_2^-(z)$ is **incident** on the matched load, and $V_2^+(z)$ is the **reflected** wave from the load (i.e., $V_2^+(z)$ is incident on **port 2**). Therefore, $V_{02}^+ = 0$ **is correct**!

Likewise, because of the **short** circuit at port 3 ($\Gamma_L = -1$):

$$\frac{V_3^+(z_3=0)}{V_3^-(z_3=0)} = \frac{V_{03}^+}{V_{03}^-} = -1$$

and therefore:

$$V_{03}^+ = -V_{03}^-$$

Problem:

a) Find the **reflection** coefficient at port 1, i.e.:

$$\Gamma_1 \doteq \frac{V_{01}^-}{V_{01}^+}$$

b) Find the **transmission** coefficient from port 1 to port 2, i.e.,

$$T_{21} \doteq \frac{V_{02}^-}{V_{01}^+}$$

I am amused by the trivial problems that ***you*** *apparently find so difficult.* ***I*** *know that:*

$$\Gamma_1 = \frac{V_{01}^-}{V_{01}^+} = S_{11} = 0.0$$

and

$$T_{21} = \frac{V_{02}^-}{V_{01}^+} = S_{21} = 0.5$$

NO!!! The above statement is **not correct**!

Remember, $V_{01}^- / V_{01}^+ = S_{11}$ **only** if ports 2 and 3 are terminated in **matched** loads! In this problem port 3 is terminated with a **short circuit**.

Therefore:

$$\Gamma_1 = \frac{V_{01}^-}{V_{01}^+} \neq S_{11}$$

and similarly:

$$T_{21} = \frac{V_{02}^-}{V_{01}^+} \neq S_{21}$$

To determine the values T_{21} and Γ_1, we must start with the **three** equations provided by the **scattering matrix**:

$$V_{01}^- = \qquad\qquad 0.2\,V_{02}^+ + 0.5\,V_{03}^+$$

$$V_{02}^- = 0.5\,V_{01}^+ \qquad\qquad + 0.2\,V_{03}^+$$

$$V_{03}^- = 0.5\,V_{01}^+ + 0.5\,V_{02}^+$$

and the **two** equations provided by the **attached loads**:

$$V_{02}^+ = 0$$

$$V_{03}^+ = -V_{03}^-$$

We can divide all of these equations by V_{01}^{+}, resulting in:

$$\Gamma_1 = \frac{V_{01}^{-}}{V_{01}^{+}} = \qquad 0.2\,\frac{V_{02}^{+}}{V_{01}^{+}} + 0.5\,\frac{V_{03}^{+}}{V_{01}^{+}}$$

$$T_{21} = \frac{V_{02}^{-}}{V_{01}^{+}} = 0.5 \qquad + 0.2\,\frac{V_{03}^{+}}{V_{01}^{+}}$$

$$\frac{V_{03}^{-}}{V_{01}^{+}} = 0.5 + 0.5\,\frac{V_{02}^{+}}{V_{01}^{+}}$$

$$\frac{V_{02}^{+}}{V_{01}^{+}} = 0$$

$$\frac{V_{03}^{+}}{V_{01}^{+}} = -\frac{V_{03}^{-}}{V_{01}^{+}}$$

Look what we have—**5** equations and **5** unknowns! Inserting equations 4 and 5 into equations 1 through 3, we get:

$$\Gamma_1 = \frac{V_{01}^{-}}{V_{01}^{+}} = -0.5\,\frac{V_{03}^{+}}{V_{01}^{+}}$$

$$T_{21} = \frac{V_{02}^{-}}{V_{01}^{+}} = 0.5 - 0.2\,\frac{V_{03}^{+}}{V_{01}^{+}}$$

$$\frac{V_{03}^{-}}{V_{01}^{+}} = 0.5$$

Solving, we find:

$$\Gamma_1 = -0.5(0.5) = -0.25$$

$$T_{21} = 0.5 - 0.2(0.5) = 0.4$$

Example: Scattering Parameters

Consider a **two-port device** with a scattering matrix (at some specific frequency ω_0):

$$\bar{\bar{\mathbf{S}}}(\omega = \omega_0) = \begin{bmatrix} 0.1 & j0.7 \\ j0.7 & -0.2 \end{bmatrix}$$

and $Z_0 = 50\Omega$.

Say that the transmission line connected to **port 2** of this device is terminated in a **matched** load, and that the wave **incident** on **port 1** is:

$$V_1^+(z_1) = -j2\, e^{-j\beta z_1}$$

where $z_{1P} = z_{2P} = 0$.

Determine:

1. the port voltages $V_1(z_1 = z_{1P})$ and $V_2(z_2 = z_{2P})$.

2. the port currents $I_1(z_1 = z_{1P})$ and $I_2(z_2 = z_{2P})$.

3. the net power flowing into port 1

1. Since the **incident** wave on port 1 is:

$$V_1^+(z_1) = -j2\, e^{-j\beta z_1}$$

we can conclude (since $z_{1P} = 0$):

$$\begin{aligned} V_1^+(z_1 = z_{1P}) &= -j2\, e^{-j\beta z_{1P}} \\ &= -j2\, e^{-j\beta(0)} \\ &= -j2 \end{aligned}$$

and since port 2 is **matched** (and **only** because its matched!), we find:

$$\begin{aligned} V_1^-(z_1 = z_{1P}) &= S_{11}\, V_1^+(z_1 = z_{1P}) \\ &= 0.1(-j2) \\ &= -j0.2 \end{aligned}$$

The voltage at port 1 is thus:

$$\begin{aligned} V_1(z_1 = z_{1P}) &= V_1^+(z_1 = z_{1P}) + V_1^-(z_1 = z_{1P}) \\ &= -j2.0 - j0.2 \\ &= -j2.2 \\ &= 2.2\, e^{-j\pi/2} \end{aligned}$$

Likewise, since port 2 is **matched**:

$$V_2^+(z_2 = z_{2P}) = 0$$

And also:

$$\begin{aligned} V_2^-(z_2 = z_{2P}) &= S_{21}\, V_1^+(z_1 = z_{1P}) \\ &= j0.7\,(-j2) \\ &= 1.4 \end{aligned}$$

Therefore:

$$\begin{aligned} V_2(z_2 = z_{2P}) &= V_2^+(z_2 = z_{2P}) + V_2^-(z_2 = z_{2P}) \\ &= 0 + 1.4 \\ &= 1.4 \\ &= 1.4\, e^{-j0} \end{aligned}$$

2. The port **currents** can be easily determined from the results of the previous section.

$$\begin{aligned} I_1(z_1 = z_{1P}) &= I_1^+(z_1 = z_{1P}) - I_1^-(z_1 = z_{1P}) \\ &= \frac{V_1^+(z_1 = z_{1P})}{Z_0} - \frac{V_1^-(z_1 = z_{1P})}{Z_0} \\ &= -j\,\frac{2.0}{50} + j\,\frac{0.2}{50} \\ &= -j\,\frac{1.8}{50} \\ &= -j0.036 \\ &= 0.036\, e^{-j\pi/2} \end{aligned}$$

and:

$$
\begin{aligned}
I_2(z_2 = z_{2P}) &= I_2^+(z_2 = z_{2P}) - I_2^-(z_2 = z_{2P}) \\
&= \frac{V_2^+(z_2 = z_{2P})}{Z_0} - \frac{V_2^-(z_2 = z_{2P})}{Z_0} \\
&= \frac{0}{50} - \frac{1.4}{50} \\
&= -0.028 \\
&= 0.028\, e^{+j\pi}
\end{aligned}
$$

3. The **net power** flowing into port 1 is:

$$
\begin{aligned}
\Delta P_1 &= P_1^+ - P_1^- \\
&= \frac{\left|V_{01}^+\right|^2}{2Z_0} - \frac{\left|V_{01}^-\right|^2}{2Z_0} \\
&= \frac{(2)^2 - (0.2)^2}{2(50)} \\
&= 0.0396 \quad \textit{Watts}
\end{aligned}
$$

Matched, Lossless, Reciprocal Devices

Often, we describe a device or network as **matched**, **lossless**, or **reciprocal**.

Q: *What do these three terms mean??*

A: Let's explain each of them **one at a time**!

Matched

A matched device is another way of saying that the **input impedance** at each port is numerically **equal to** Z_0 when **all** other ports are terminated in **matched loads**. As a result, the input **reflection coefficient** of each port is **zero**—no signal will come out of a port when a signal is incident on **that** port (and **only** that port !).

In other words, we want:

$$V_{0m}^- = S_{mm} V_{0m}^+ = 0 \quad \text{for all } m$$

a result that occurs when:

$$\boxed{S_{mm} = 0 \ \text{ for all } m \text{ if } \textbf{matched.}}$$

We find therefore that a matched device will exhibit a scattering matrix where all **diagonal elements** are **zero**.

Therefore:

$$\bar{\bar{\mathbf{S}}} = \begin{bmatrix} 0 & 0.1 & j0.2 \\ 0.1 & 0 & 0.3 \\ j0.2 & 0.3 & 0 \end{bmatrix}$$

is an example of a scattering matrix for a **matched,** three port device.

Lossless

For a lossless device, all of the power that delivered to each device port must eventually find its way **out**!

In other words, power is not **absorbed** by the network—no power to be **converted to heat**!

Recall the **power incident** on some port m is related to the amplitude of the **incident wave** (V_{0m}^{+}) as:

$$P_m^{+} = \frac{\left|V_{0m}^{+}\right|^2}{2Z_0}$$

While power of the **wave exiting** the port is:

$$P_m^{-} = \frac{\left|V_{0m}^{-}\right|^2}{2Z_0}$$

Thus, the power **delivered** to that port is the **difference** of the two:

$$\Delta P_m = P_m^+ - P_m^- = \frac{\left|V_{0m}^+\right|^2}{2Z_0} - \frac{\left|V_{0m}^-\right|^2}{2Z_0}$$

Thus, the **total power incident** on an N-port device is:

$$P^+ = \sum_{m=1}^{N} P_m^+ = \frac{1}{2Z_0}\sum_{m=1}^{N}\left|V_{0m}^+\right|^2$$

Note that:

$$\sum_{m=1}^{N}\left|V_{0m}^+\right|^2 = \overline{\mathbf{V}^+}^H\,\overline{\mathbf{V}^+}$$

where operator H indicates the **conjugate transpose** (i.e., Hermetian transpose) operation, so that $\overline{\mathbf{V}^+}^H\,\overline{\mathbf{V}^+}$ is the **inner product** (i.e., dot product, or scalar product) of complex vector $\overline{\mathbf{V}^+}$ with itself.

Thus, we can write the **total power incident** on the device as:

$$P^+ = \frac{1}{2Z_0}\sum_{m=1}^{N}\left|V_{0m}^+\right|^2 = \frac{\overline{\mathbf{V}^+}^H\,\overline{\mathbf{V}^+}}{2Z_0}$$

Similarly, we can express the **total power** of the **waves exiting** our M-port network to be:

$$P^- = \frac{1}{2Z_0}\sum_{m=1}^{N}\left|V_{0m}^-\right|^2 = \frac{\overline{\mathbf{V}^-}^H\,\overline{\mathbf{V}^-}}{2Z_0}$$

Now, recalling that the incident and exiting wave amplitudes are **related** by the **scattering matrix** of the device:

$$\bar{\mathbf{V}}^- = \bar{\bar{\mathbf{S}}}\,\bar{\mathbf{V}}^+$$

Thus we find:

$$P^- = \frac{\overline{\mathbf{V}^-}^H\,\overline{\mathbf{V}^-}}{2Z_0} = \frac{\overline{\mathbf{V}^+}^H\,\bar{\bar{\mathbf{S}}}^H\,\bar{\bar{\mathbf{S}}}\,\overline{\mathbf{V}^+}}{2Z_0}$$

Now, the **total power delivered** to the network is:

$$\Delta P = \sum_{m=1}^{M} \Delta P = P^+ - P^-$$

Or explicitly:

$$\begin{aligned}\Delta P &= P^+ - P^- \\ &= \frac{\overline{\mathbf{V}^+}^H\,\overline{\mathbf{V}^+}}{2Z_0} - \frac{\overline{\mathbf{V}^+}^H\,\bar{\bar{\mathbf{S}}}^H\,\bar{\bar{\mathbf{S}}}\,\overline{\mathbf{V}^+}}{2Z_0} \\ &= \frac{1}{2Z_0}\,\overline{\mathbf{V}^+}^H\left(\bar{\bar{\mathbf{I}}} - \bar{\bar{\mathbf{S}}}^H\,\bar{\bar{\mathbf{S}}}\right)\overline{\mathbf{V}^+}\end{aligned}$$

where $\bar{\bar{\mathbf{I}}}$ is the **identity matrix**.

Q: *Is there actually some* ***point*** *to this long, rambling, complex presentation?*

A: Absolutely! If our M-port device is lossless then the total power exiting the device must **always** be equal to the total power incident on it.

If network is **lossless,** then $P^{+} = P^{-}$.

Or stated another way, the total **power delivered** to the device (i.e., the power absorbed by the device) must always be **zero** if the device is lossless!

If network is **lossless,** then $\Delta P = 0$

Thus, we can conclude from our math that for a **lossless device**:

$$\Delta P = \frac{1}{2Z_0}\ \overline{\mathbf{V}^{+}}^{H}\left(\bar{\bar{\mathbf{I}}} - \bar{\bar{\mathbf{S}}}^{H}\bar{\bar{\mathbf{S}}}\right)\overline{\mathbf{V}^{+}} = 0 \qquad \text{for all } \overline{\mathbf{V}^{+}}$$

This is true **only** if:

$$\bar{\bar{\mathbf{I}}} - \bar{\bar{\mathbf{S}}}^{H}\bar{\bar{\mathbf{S}}} = 0 \qquad \Rightarrow \qquad \bar{\bar{\mathbf{S}}}^{H}\bar{\bar{\mathbf{S}}} = \bar{\bar{\mathbf{I}}}$$

Thus, we can conclude that the **scattering matrix** of a **lossless** device has the **characteristic**:

If a network is **lossless**, then $\bar{\bar{\mathbf{S}}}^{H}\bar{\bar{\mathbf{S}}} = \bar{\bar{\mathbf{I}}}$

Q: *Huh? What exactly is this supposed to* ***tell*** *us?*

A: A matrix that satisfies $\bar{\bar{\mathbf{S}}}^{H}\bar{\bar{\mathbf{S}}} = \bar{\bar{\mathbf{I}}}$ is a special kind of matrix known as a **unitary matrix**.

If a network is **lossless**, then its scattering matrix $\bar{\bar{\mathbf{S}}}$ is **unitary**.

Q: *How do I **recognize** a unitary matrix if I **see** one?*

A: The **columns** of a unitary matrix form an **orthonormal set**!

$$\bar{\bar{\mathbf{S}}} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{22} \\ S_{31} & S_{33} & S_{33} & S_{33} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix}$$

matrix columns

In other words, each **column** of the scattering matrix will have a **magnitude equal to one**:

$$\sum_{m=1}^{N} \left|S_{mn}\right|^2 = 1 \quad \text{for all } n$$

while the inner product (i.e., dot product) of **dissimilar columns** must be **zero**.

$$\sum_{n=1}^{N} S_{ni} S_{nj}^* = S_{1i} S_{1j}^* + S_{2i} S_{2j}^* + \cdots + S_{Ni} S_{Nj}^* = 0 \quad \text{for all } i \neq j$$

In other words, dissimilar columns are **orthogonal**.

Consider, for example, a lossless **three-port** device. Say a signal is incident on port 1, and that **all** other ports are **terminated**. The power **incident** on port 1 is therefore:

$$P_1^+ = \frac{\left|V_{01}^+\right|^2}{2Z_0}$$

while the power **exiting** the device at each port is:

$$P_m^- = \frac{\left|V_{0m}^-\right|^2}{2Z_0} = \frac{\left|S_{m1}V_{01}^-\right|^2}{2Z_0} = \left|S_{m1}\right|^2 P_1^+$$

The **total** power exiting the device is therefore:

$$\begin{aligned} P^- &= P_1^- + P_2^- + P_3^- \\ &= \left|S_{11}\right|^2 P_1^+ + \left|S_{21}\right|^2 P_1^+ + \left|S_{31}\right|^2 P_1^+ \\ &= \left(\left|S_{11}\right|^2 + \left|S_{21}\right|^2 + \left|S_{31}\right|^2\right) P_1^+ \end{aligned}$$

Since this device is **lossless**, then the incident power (**only** on port 1) is **equal** to exiting power (i.e, $P^- = P_1^+$). This is true **only** if:

$$\left|S_{11}\right|^2 + \left|S_{21}\right|^2 + \left|S_{31}\right|^2 = 1$$

Of course, this will likewise be true if the incident wave is placed on **any** of the **other** ports of this lossless device:

$$\begin{aligned} \left|S_{12}\right|^2 + \left|S_{22}\right|^2 + \left|S_{32}\right|^2 + \left|S_{42}\right|^2 &= 1 \\ \left|S_{13}\right|^2 + \left|S_{23}\right|^2 + \left|S_{33}\right|^2 + \left|S_{43}\right|^2 &= 1 \\ \left|S_{14}\right|^2 + \left|S_{24}\right|^2 + \left|S_{34}\right|^2 + \left|S_{44}\right|^2 &= 1 \end{aligned}$$

We can state in general then that:

$$\sum_{m=1}^{3} \left| S_{mn} \right|^2 = 1 \quad \text{for all } n$$

In other words, the columns of the scattering matrix must have **unit magnitude** (a requirement of all **unitary** matrices). It is apparent that this must be true for energy to be conserved.

An **example** of a (unitary) scattering matrix for a **lossless** device is:

$$\bar{\bar{\mathbf{S}}} = \begin{bmatrix} 0 & \frac{1}{2} & j\frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 0 & 0 & j\frac{\sqrt{3}}{2} \\ j\frac{\sqrt{3}}{2} & 0 & 0 & \frac{1}{2} \\ 0 & j\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Reciprocal

Reciprocity results when we build a **passive** (i.e., unpowered) device with **simple** materials.

For a reciprocal network, we find that the elements of the scattering matrix are **related** as:

$$S_{mn} = S_{nm}$$

For example, a reciprocal device will have $S_{21} = S_{12}$ or $S_{32} = S_{23}$. We can write reciprocity in matrix form as:

$$\boxed{\bar{\bar{\mathbf{S}}}^T = \bar{\bar{\mathbf{S}}} \quad \text{if } \textbf{reciprocal}}$$

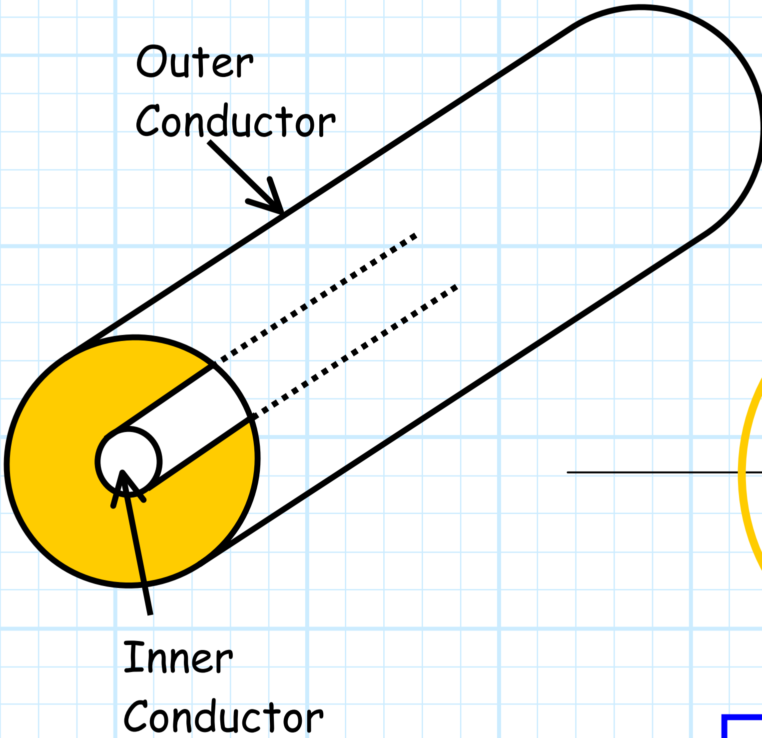
where T indicates (non-conjugate) transpose.

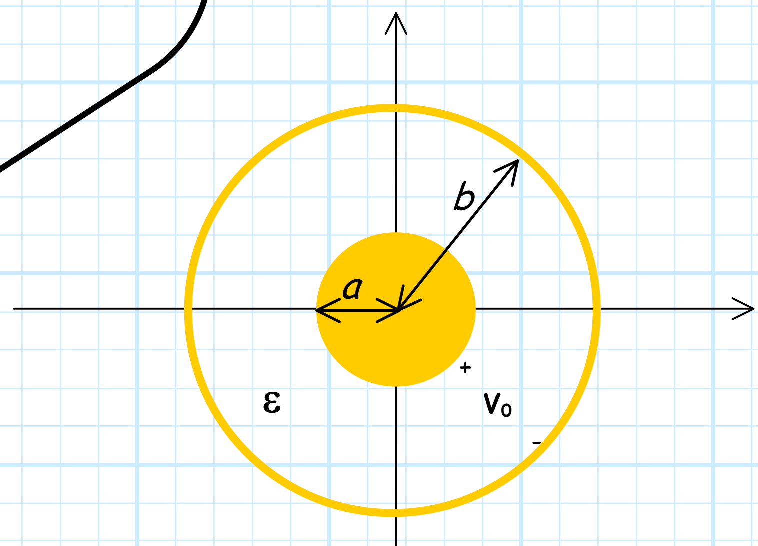
An **example** of a scattering matrix describing a **reciprocal**, but **lossy** and **non-matched** device is:

$$\bar{\bar{\mathbf{S}}} = \begin{bmatrix} 0.10 & -0.40 & -j0.20 & 0.05 \\ -0.40 & j0.20 & 0 & j0.10 \\ -j0.20 & 0 & 0.10 - j0.30 & -0.12 \\ 0.05 & j0.10 & -0.12 & 0 \end{bmatrix}$$

Coaxial Transmission Lines

The most **common** type of transmission line!



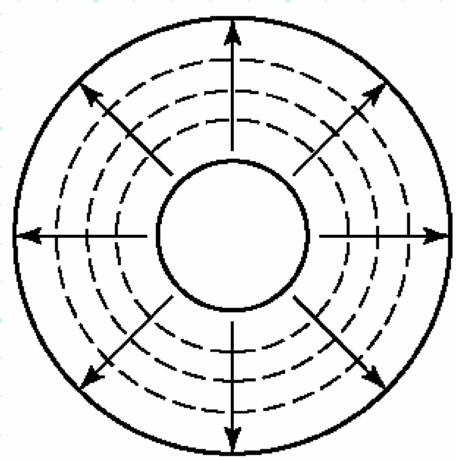


Coax Cross-Section

The **electric** field (⟶)points in the direction $\hat{a}_\rho$.

The **magnetic** field (- - - -)points in the direction $\hat{a}_\phi$.

E. M. Power flows in the direction $\hat{a}_z$.



➡ A TEM wave!

Recall from EECS 220 that the **capacitance** per/unit length of a coaxial transmission line is:

$$C = \frac{2\pi\varepsilon}{\ln[b/a]} \qquad \left[\frac{\text{farads}}{\text{meter}}\right]$$

And that the **inductance** per unit length is :

$$L = \frac{\mu_0}{2\pi}\ln\left[\frac{b}{a}\right] \qquad \left[\frac{\text{Henries}}{\text{m}}\right]$$

Where of course the **characteristic impedance** is:

$$Z_o = \sqrt{\frac{L}{C}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon}}\ln\left[\frac{b}{a}\right]$$

and:

$$\beta = \omega\sqrt{LC} = \omega\sqrt{\mu_0\,\varepsilon}$$

Therefore the **propagation velocity** of each TEM wave within a coaxial transmission line is:

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu_0\,\varepsilon}} = \frac{1}{\sqrt{\mu_0\,\varepsilon_0}}\frac{1}{\sqrt{\varepsilon_r}} = c\frac{1}{\sqrt{\varepsilon_r}}$$

where $\varepsilon_r = \varepsilon/\varepsilon_0$ is the relative dielectric constant, and c is the "speed of light" ($c = 3\times10^8\ m/s$).

Note then that we can likewise express β in terms ε_r:

$$\beta = \omega\sqrt{\mu_0\,\varepsilon} = \omega\sqrt{\mu_0\,\varepsilon_0}\sqrt{\varepsilon_r} = \frac{\omega}{c}\sqrt{\varepsilon_r}$$

Now, the **size** of the coaxial line (a and b) determines **more** than simply Z_0 and β (L and C) of the transmission line. Additionally, the line radius determines the **weight** and bulk of the line, as well as its **power handling** capabilities.

Unfortunately, these two characteristics **conflict** with each other!

1. Obviously, to **minimize** the weight and bulk of a coaxial transmission line, we should make a and b as **small** as possible.

2. However, for a given line voltage, reducing a and b causes the **electric field** within the coaxial line to **increase** (recall the units of electric field are V/m).

A higher electric field causes **two** problems: first, it results in greater **line attenuation** (larger α); second, it can result in **dielectric breakdown**.

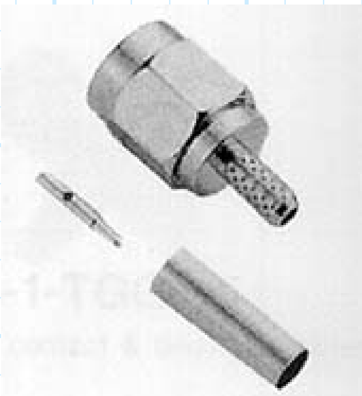
Dielectric breakdown results when the electric field within the transmission line becomes so large that the dielectric material is **ionized**. Suddenly, the dielectric becomes a **conductor**, and the value G gets **very** large!

This generally results in the **destruction** of the coax line, and thus must be **avoided**. Thus, **large** coaxial lines are required when extremely **low-loss** is required (i.e., line length ℓ is large), **or** the delivered **power** is large.

Otherwise, we try to keep our coax lines as **small** as possible!

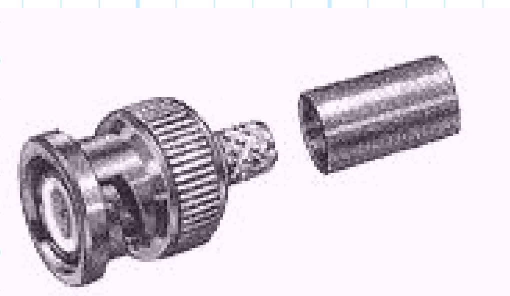
Coaxial Connectors

There are many types of **connectors** that are used to connect coaxial lines to RF/microwave devices. They include:



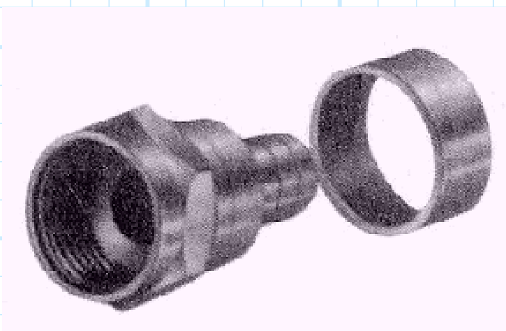
SMA

The workhorse **microwave** connector. Small size, but works well to > 20 GHz. By microwave standards, moderately priced.



BNC

The workhorse **RF** connector. Relatively small and cheap, and easy to connect. Don't use this connector past 2 GHz!

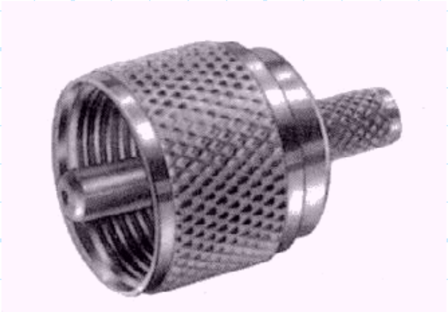


F

A poorman's BNC. The RF connector used on most consumer products such as **TVs**. Cheap, but difficult to connect and not reliable.

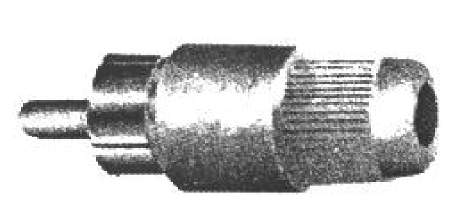
N

The **original** microwave connector. Good performance (up to 18GHz), and moderate cost, but large (about 2 cm in diameter) ! However, can handle greater **power** than SMA.



UHF

The poorman's **N.** About the same size, although **reduced** reliability and performance.



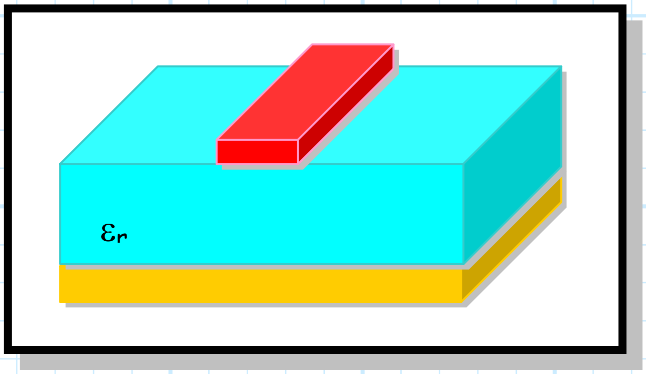
RCA

Not really an RF connector. Used primarily in consumer application for video and audio signals (i.e., <20 MHz). Cheap and easy to connect.

APC-7 and APC-3.5

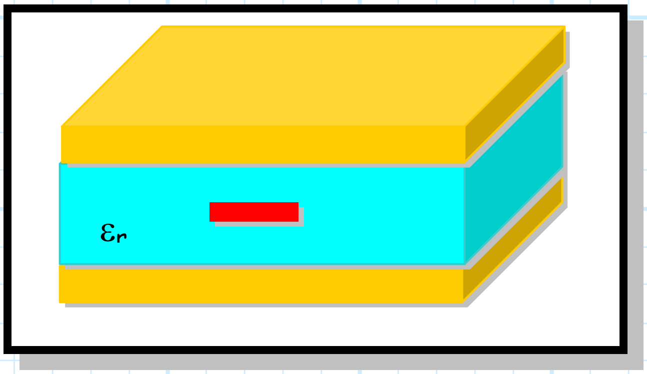
The top of the line connector. Best performance, but cost **big $$$**. Used primarily in test equipment (e.g., network analyzers). 3.5 can work to nearly 40 GHz.

Printed Circuit Board Transmission Lines



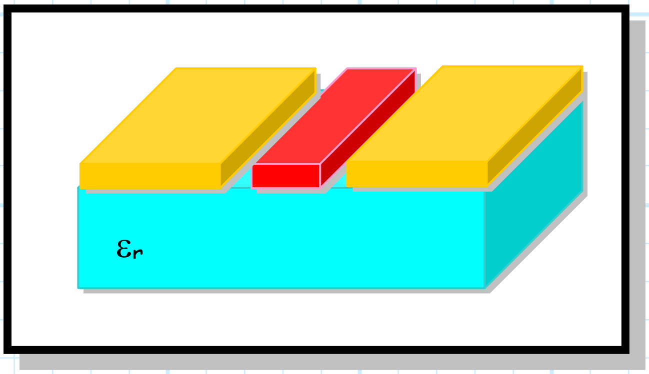
Microstrip

Probably most **popular** PCB transmission line. Easy fabrication and connection, yet is **slightly** dispersive, lossy, and difficult to analyze.



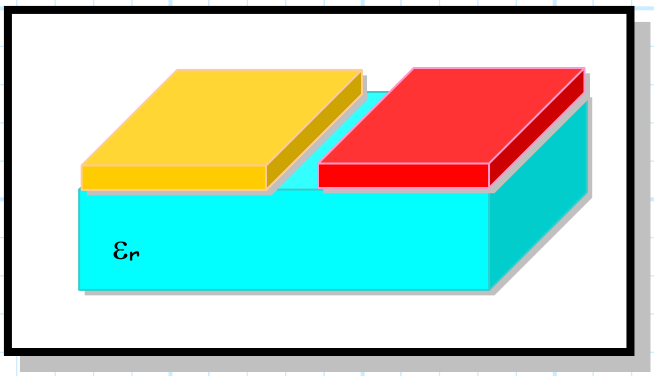
Stripline

Better than microstrip in that it is **not** dispersive, and is more easily analyzed. However, fabrication and connection is more difficult.



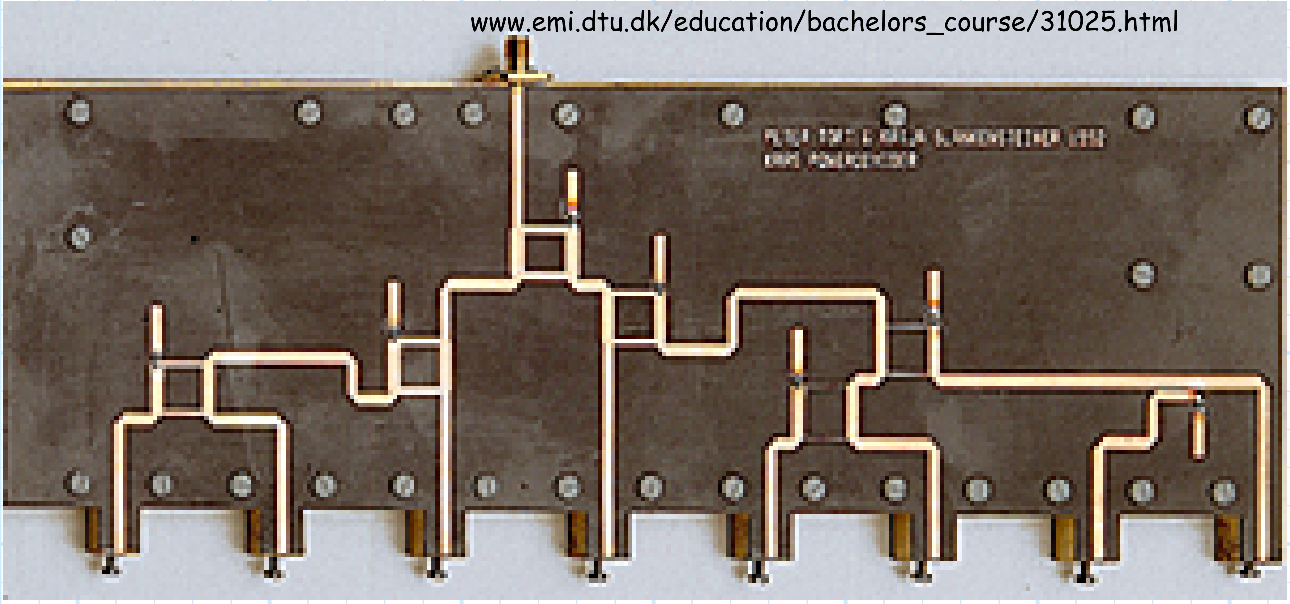
Coplanar Waveguide

The **newest** technology. Perhaps easiest to fabricate and connect components, as **both** ground and conductor are on one side of the board.

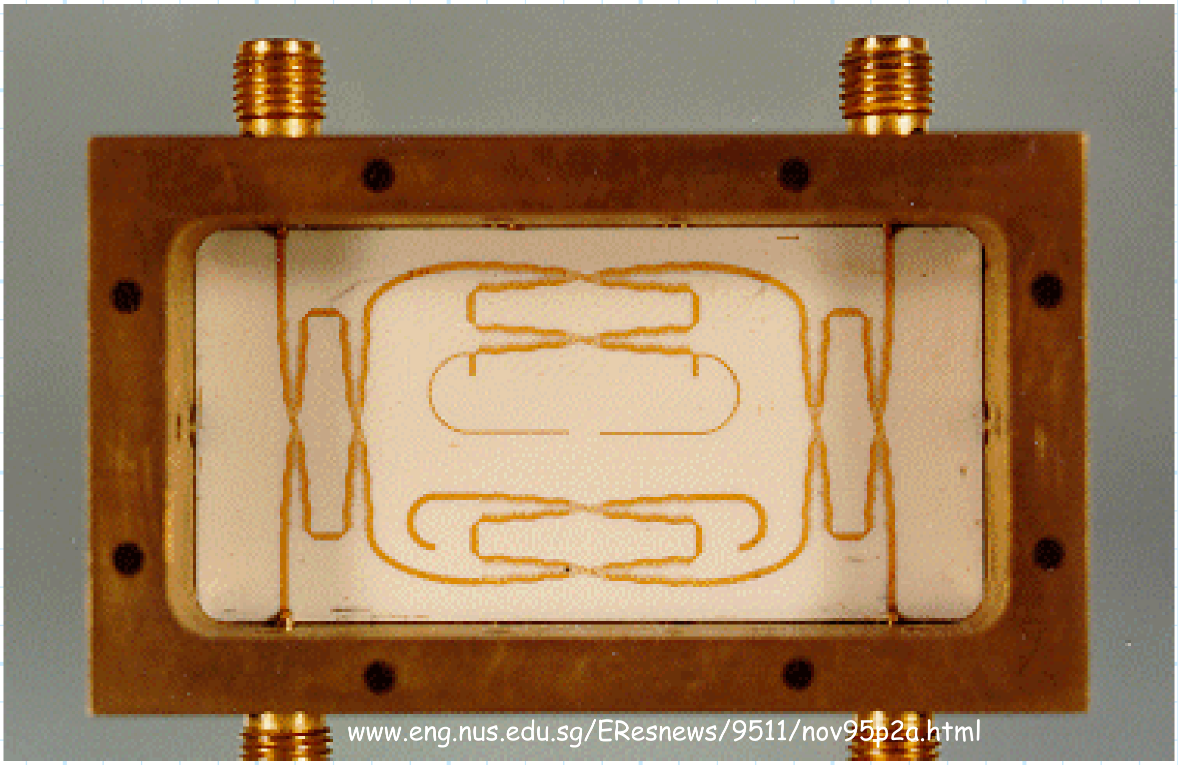


Slotline

Essentially, a dual wire tranmission line. Best for **"balanced"** applications. Not used much.



An antenna array feed, constructed using **microstrip** transmission lines and circuits.



A wideband **microstrip** coupler.