III Antenna Fundamentals

Now we will discuss what occurs between the transmitter and receiver. Recall this region is called the **channel**, and we couple an electromagnetic wave to/from the channel using an **antenna**.

A. Wave Propagation

We must first **review** the basics of electromagnetic propagation in free-space.

HO: EM Wave Propagation in Free-Space

HO: The Poynting Vector

Electromagnetic Wave Propagation

Maxwell's equations were cobbled together from a **variety** of results from different scientists (e.g. Ampere, Faraday), whose work mainly was done using either **static** or **slowly** time-varying sources and fields.

Maxwell brought these results together to form a **complete** theory of electromagnetics—a theory that then predicted a most **startling** result!

To see this result, consider first the **free-space** Maxwell's Equations in a **source-free** region (e.g., a vacuum). In other words, the fields in a region **far away** from the current and charges that created them:

$$\nabla \times \mathbf{B}(\bar{r},t) = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}(\bar{r},t)}{\partial t}$$

$$\nabla \times \mathbf{E}(\bar{r},t) = -\frac{\partial \mathbf{B}(\bar{r},t)}{\partial t}$$

$$\nabla \cdot \mathbf{E}(\bar{r},t) = 0$$

$$\nabla \cdot \mathbf{B}(\bar{r},t) = 0$$

Say we take the curl of Faraday's Law:

$$\nabla \mathrm{x} \nabla \mathrm{x} \mathbf{E}(\bar{r},t) = -\frac{\partial \nabla \mathrm{x} \mathbf{B}(\bar{r},t)}{\partial t}$$

Inserting Ampere's Law into this, we get:

$$\begin{aligned}\nabla \mathrm{x} \nabla \mathrm{x} \mathbf{E}(\bar{r},t) &= -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}(\bar{r},t)}{\partial t}\right) \\ &= -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}(\bar{r},t)}{\partial t^2}\end{aligned}$$

Recalling that if $\nabla \cdot \mathbf{E}(\bar{r}) = 0$ then $\nabla \mathrm{x} \nabla \mathrm{x} \mathbf{E}(\bar{r}) \doteq \nabla^2 \mathbf{E}(\bar{r})$, we can write the following differential equation, one which describes the **behavior** on an electric field in a **vacuum**:

$$\nabla^2 \mathbf{E}(\bar{r},t) + \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}(\bar{r},t)}{\partial t^2} = 0$$

This result is none as the vector wave equation, and is very similar to the transmission line wave equations we studied at the beginning of this class.

This result means that electric field $\mathbf{E}(\bar{r},t)$ cannot be any arbitrary function of position $\bar{r}$ and time t. Instead, an electric field $\mathbf{E}(\bar{r},t)$ is physically possible only if it satisfies the differential equation above!

Q: *So, what are some **solutions** to this equation?*

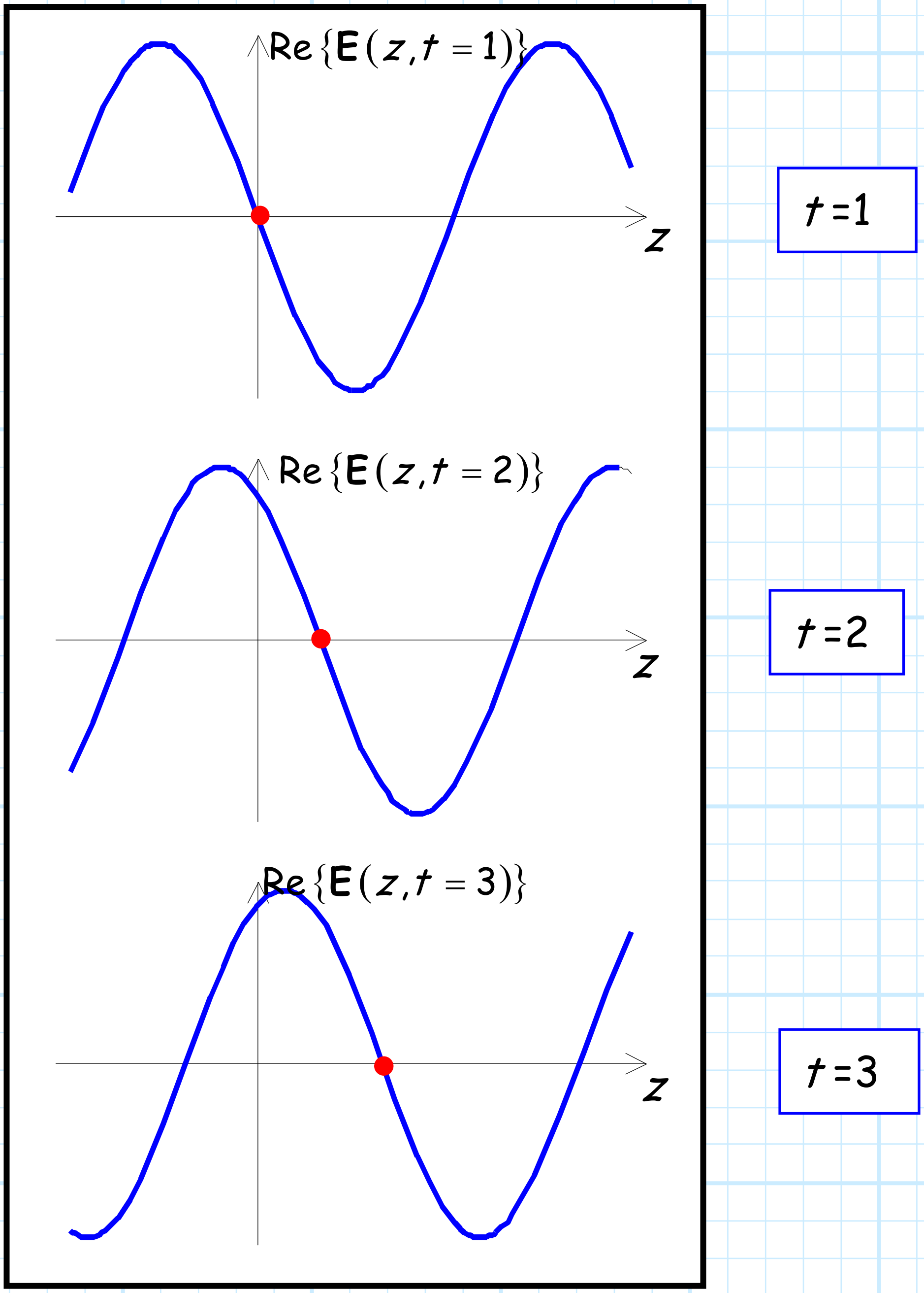
A: The **simplest** solution is the **plane-wave solution**. It is:

$$\mathbf{E}(\bar{r},t) = \left(E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}}\right) e^{j\omega\left(t - z\sqrt{\mu_0\varepsilon_0}\right)}$$

For this solution, the electric field is varying with time in a **sinusoidal** manner (that eigen function thing!), with an angular **frequency** of ω radians/sec. Note this field is a function of spatial coordinate z only, but the direction of the electric field is orthogonal to the z-axis.

Q: *What does this equation **tell** us about $\mathbf{E}(\bar{r},t)$? What is this electric field **doing**??*

A: Lets **plot** $Re\{\mathbf{E}(\bar{r},t)\}$ as a function of position z, for different times t, and find out!



Here the red dot indicates **plane of constant phase**, for this case a phase of **zero** radians, i.e., $\phi = \omega\left(t - z\sqrt{\mu_0\varepsilon_0}\right) = 0$. Note that this dot appears to be **moving forward** along the *z*- axis as a function of time.

The electric field is moving !

Q: *How **fast** is it moving?*

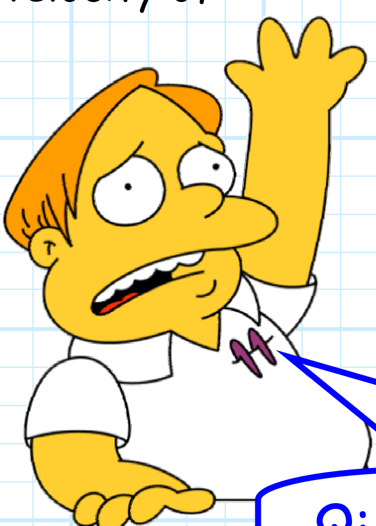
A: Lets see how fast the **red dot** (i.e., the plane of constant phase) is moving! Rearranging $\omega\left(t - z\sqrt{\mu_0\varepsilon_0}\right) = 0$, we get the position z of the dot as a function of time t:

$$z = \frac{t}{\sqrt{\mu_0\varepsilon_0}}$$

Its **velocity** is just the **time derivative** of its position:

$$v_p = \frac{dz}{dt} = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

Hey we can calculate this! The **electric field** is moving at a velocity of:

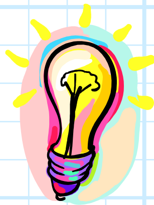


$$\begin{aligned} v_p &= \frac{1}{\sqrt{\mu_0\varepsilon_0}} \\ &= \frac{1}{\sqrt{\left(4\pi\text{x}10^{-7}\right)\left(8.854\text{x}10^{-12}\right)}} \\ &= \mathbf{3x10^8} \quad \left[\frac{\text{meters}}{\text{second}}\right] \end{aligned}$$

Q: *Hey wait a minute! 3 x 10^8 meters/second—that's the **speed of light**!?!*

A: True! We find that the magnetic field will likewise move in the **same** direction and with the **same** velocity as the electric field.

We call the combination of the two fields a **propagating** (i.e., moving) **electromagnetic wave**.



Light is a propagating electromagnetic wave!

This was a **stunning** result in Maxwell's time. No one had linked light with the phenomena of electricity and magnetism. Among other things, it meant that "light" could be made with much greater wavelengths (i.e., **lower frequencies**) than the light visible to us humans.

Henrich Hertz first succeeded in creating and measuring this low frequency "light". Since then, humans have put this low-frequency light to **great** use. We often refer to it as a "**radio waves**"—a propagating electromagnetic wave with a frequency in the range of 1 MHz to 20 GHz. We use it for all "**wireless**" technologies !

Given the results above, we can rewrite our **plane-wave** solution as:

$$\begin{aligned}\mathbf{E}(\bar{r},t) &= \left(E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}}\right)e^{j\omega\left(t - z\sqrt{\mu_0\varepsilon_0}\right)} \\ &= \left(E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}}\right)e^{j\omega\left(t - z/c\right)} \\ &= \left(E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}}\right)e^{-j\left(\omega/c\right)z}e^{j\omega t}\end{aligned}$$

Now, making the **definition**:

$$k_0 = \frac{\omega}{c} = \frac{2\pi}{\lambda} \qquad \left[radians\,/\,meter\right]$$

We get:

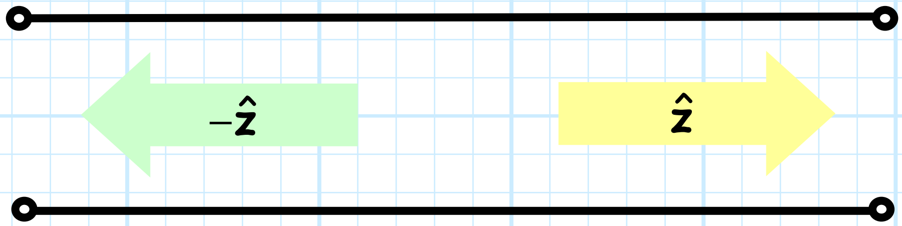
$$\mathbf{E}(\bar{r},t) = \left(E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}}\right)e^{-jk_0 z}e^{j\omega t}$$

Q: *This **plane-wave** solution reminds me somewhat of the solution to the **telegrapher's equations**, with k_0 **analogous** to β. Is this just a coincidence?*

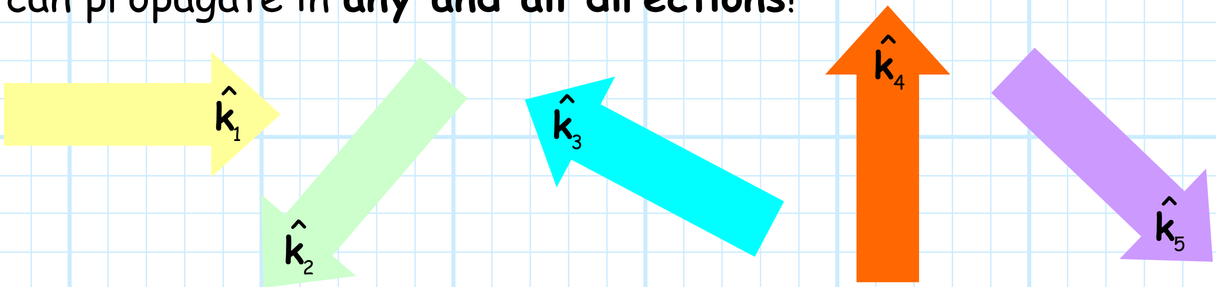
A: Nope! Since we have voltages and currents along our transmission line, we **must** also have electric fields and magnetic fields. In fact, the voltage and current wave solutions for a transmission line can likewise be expressed as propagating electric and magnetic (i.e., **electromagnetic**) fields.

But, there is one super-huge **difference** between the transmission line solutions and the plane wave solution presented here!

The propagating wave along at transmission line is **constrained** to one of two directions—the plus z direction or the minus z direction.

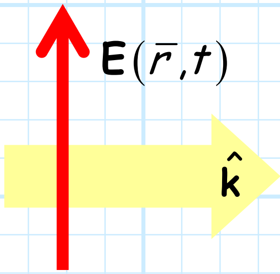


In contrast, **nothing** constrains a plane wave in free space—it can propagate in **any and all directions**!



Although the **plane-wave** solution shown above propagates in the $\hat{z}$ direction, the solution would be equally valid in the $-\hat{y}$ direction or $\hat{x}$ direction, or any **arbitrary direction** $\hat{k}$.

The only constraint is that the direction of the **electric field** vector be **orthogonal** to the **direction** of wave propagation, i.e.:



$$\hat{k} \cdot \mathbf{E}(\bar{r},t) = 0$$

Q: *Are there any **other** solutions to this vector wave equation?*

A: Plenty! Since the wave equation is a **linear** differential equation, **superposition** holds.

In other words, a weighted sum of solutions is **also** a solution. This means that we can (and often do) have **multiple** waves propagating simultaneously in all **different** directions.

Moreover, there are many **other** solutions besides the plane-wave solution. The most relevant of these, perhaps, is the **spherical wave**:

$$\mathbf{E}(\bar{r},t) = \left(E_\theta(\theta,\phi)\,\hat{\boldsymbol{\theta}} + E_\phi(\theta,\phi)\,\hat{\boldsymbol{\varphi}}\right)\frac{e^{-jk_0 r}}{r}e^{j\omega t}$$

Note the spherical wave is (most easily) expressed using the **spherical coordinate system** (i.e., coordinates r,θ,ϕ and base vectors $\hat{\mathbf{r}},\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\varphi}}$).

The spherical wave **propagates outward from the origin** (i.e., in the direction $\hat{\mathbf{k}} = \hat{\mathbf{r}}$). In other words, a **sphere** of constant phase (as opposed to a plane of constant phase) propagates outward from the origin. Thus, this sphere of constant phase "expands" as a function of time—sort of like a balloon being filled with air!

We likewise see from the expression above that the **direction** of the electric field is likewise **orthogonal** to the direction of wave propagation.

$$\hat{\mathbf{k}}\cdot\mathbf{E}(\bar{r},t) = \hat{\mathbf{r}}\cdot\mathbf{E}(\bar{r},t) = 0$$

The Poynting Vector

Recall that plane waves and spherical waves are electro-**magnetic** waves .

In other words, they consist of **both** electric and **magnetic** fields!

Q: *You provided us with the **electric** field representations of plane and spherical waves, is there some way to use these to determine the corresponding **magnetic** field?*

A: You bet! Just apply **Faraday's Law**:

$$\nabla \times \mathbf{E}(\bar{r},t) = -\mu_0 \frac{\partial \mathbf{H}(\bar{r},t)}{\partial t}$$

If the electric field of a **plane wave** is:

$$\mathbf{E}(\bar{r},t) = \left(E_x\ \hat{\mathbf{x}} + E_y\ \hat{\mathbf{y}}\right) e^{-jk_0 z} e^{j\omega t}$$

Then we find the magnetic field **must** be:

$$\begin{aligned}\mathbf{H}(\bar{r},t) &= \mu_0 \frac{k_0}{\omega}\left(-E_y\ \hat{\mathbf{x}} + E_x\ \hat{\mathbf{y}}\right) e^{-jk_0 z} e^{j\omega t} \\ &= \sqrt{\frac{\mu_0}{\varepsilon_0}}\left(-E_y\ \hat{\mathbf{x}} + E_x\ \hat{\mathbf{y}}\right) e^{-jk_0 z} e^{j\omega t}\end{aligned}$$

Now, making the definition:

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377 \quad Ohms$$

We find the corresponding **magnetic** field for our **plane wave** solution is thus:

$$\mathbf{H}(\bar{r},t) = \eta_0 \left(-E_y\ \hat{\mathbf{x}} + E_x\ \hat{\mathbf{y}}\right) e^{-jk_0 z} e^{j\omega t}$$

The value η_0 is know as the **wave impedance**, or sometimes called the characteristic impedance of free space.

Q: *Why is η_0 referred to as an* ***impedance****? Does it really have units of* ***Ohms****?*

A: Consider the **magnitude** of both $\mathbf{E}(\bar{r},t)$ and $\mathbf{H}(\bar{r},t)$:

$$\begin{aligned}
\left|\mathbf{E}(\bar{r},t)\right|^2 &= \mathbf{E}(\bar{r},t) \cdot \mathbf{E}^*(\bar{r},t) \\
&= e^{-jk_0 z} e^{j\omega t} \left(E_x\ \hat{\mathbf{x}} + E_y\ \hat{\mathbf{y}}\right) \cdot \left(E_x^*\ \hat{\mathbf{x}} + E_y^*\ \hat{\mathbf{y}}\right) e^{+jk_0 z} e^{-j\omega t} \\
&= e^{-j(k_0 z - k_0 z)} e^{j(\omega t - \omega t)} \left(E_x E_x^*\ \hat{\mathbf{x}} \cdot \hat{\mathbf{x}} + E_y E_x^*\ \hat{\mathbf{y}} \cdot \hat{\mathbf{x}}\right. \\
&\quad \left. + E_x E_y^*\ \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} + E_y E_y^*\ \hat{\mathbf{y}} \cdot \hat{\mathbf{y}}\right) \\
&= \left|E_x\right|^2 + \left|E_y\right|^2
\end{aligned}$$

Therefore:

$$\left|\mathbf{E}(\bar{r},t)\right| = \sqrt{\left|E_x\right|^2 + \left|E_y\right|^2} \quad V/m$$

Using the same procedure for the magnetic field, we find:

$$\left|\mathbf{H}(\bar{r},t)\right| = \frac{1}{\eta_0}\sqrt{\left|E_x\right|^2 + \left|E_y\right|^2} \quad A/m$$

Note that the magnitude of both the electric field and magnetic field of a **plane wave** are **constants** with respect to space and time!

Now, let's take the **ratio** of these two values:

$$\frac{\left|\mathbf{E}(\bar{r},t)\right|}{\left|\mathbf{H}(\bar{r},t)\right|} = \frac{\sqrt{\left|E_x\right|^2 + \left|E_y\right|^2}}{\frac{1}{\eta_0}\sqrt{\left|E_x\right|^2 + \left|E_y\right|^2}} = \eta_0$$

The ratio of the electric and magnetic field magnitudes of a **single** plane wave (but **only** for a **single** plane wave!) is wave impedance η_0.

More importantly are the **units** of this value, which confirms that it is indeed an "impedance" value.

$$\frac{\left|\mathbf{E}(\bar{r},t)\right|}{\left|\mathbf{H}(\bar{r},t)\right|} = \eta_0 \quad \Rightarrow \quad \frac{V/m}{A/m} = \frac{V}{A} = Ohms$$

Now, let's (**finally!**) get to the **point** (no pun intended) of this handout—**The Poynting Vector**.

The Poynting Vector is defined as:

$$\mathbf{W}(\bar{r}) = \frac{1}{2}\text{Re}\left\{\mathbf{E}(\bar{r},t) \times \mathbf{H}^*(\bar{r},t)\right\}$$

Note the Poynting Vector is a **real-valued** vector!

For our **plane wave** example, the Poynting Vector is:

$$\begin{aligned}
\mathbf{W}(\bar{r}) &= \frac{1}{2}\text{Re}\left\{\mathbf{E}(\bar{r},t) \times \mathbf{H}^*(\bar{r},t)\right\} \\
&= \frac{1}{2\eta_0}\text{Re}\left\{\left(E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}}\right) \times \left(-E_y^*\,\hat{\mathbf{x}} + E_x^*\,\hat{\mathbf{y}}\right)\right\} \\
&= \frac{1}{2\eta_0}\text{Re}\left\{\left(E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}}\right) \times \left(-E_y^*\,\hat{\mathbf{x}} + E_x^*\,\hat{\mathbf{y}}\right)\right\} \\
&= \frac{1}{2\eta_0}\text{Re}\left\{\left|E_x\right|^2 + \left|E_y\right|^2\right\}\hat{\mathbf{z}} \\
&= \frac{\hat{\mathbf{z}}}{2\eta_0}\left(\left|E_x\right|^2 + \left|E_y\right|^2\right) \\
&= \frac{\hat{\mathbf{z}}}{2\eta_0}\left|\mathbf{E}(\bar{r},t)\right|^2
\end{aligned}$$

Q: *Great. Do mind telling me what exactly this &%$!@ result* ***means!?!***

A: Let's again do a **dimensional** analysis and see what we find. Since the Poynting Vector is a (cross) **product** of an electric field and a magnetic field, the **units** of the Poynting Vector will be the product of V/m and A/m:

$$\frac{V}{m}\cdot\frac{A}{m}=\frac{V\cdot A}{m^2}=\frac{Watts}{m^2}$$

The Poynting Vector has units of **Watts per square meter**—power per unit area. These are the units of **power density**.

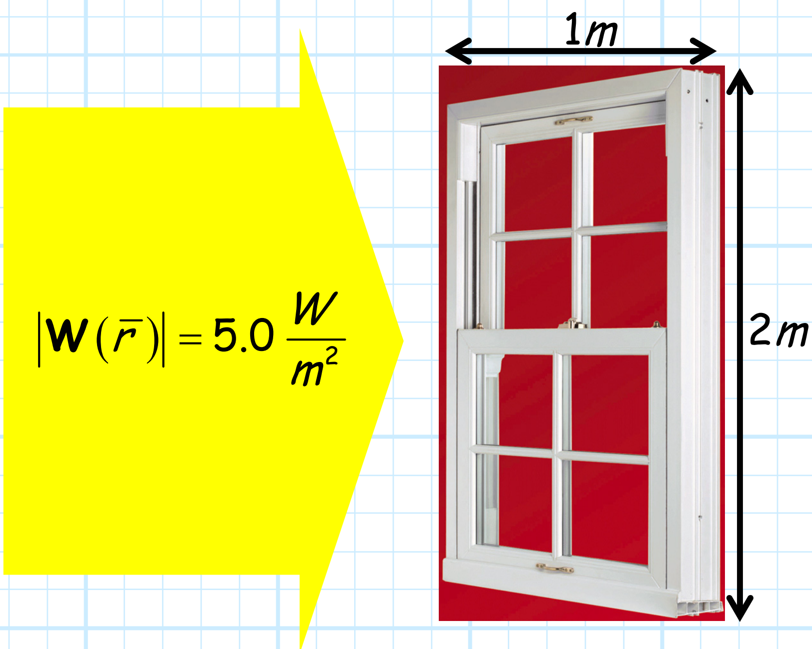
> Thus, the Poynting Vector describes the magnitude and direction of the **power flow** associated with a propagating electromagnetic wave.

This is why the Poynting Vector is a **real**-valued vector—**power** is a real-valued quantity!

* The **magnitude** of the Poynting Vector (i.e., $|\mathbf{W}(\bar{r})|$) describes the power flow in terms of its spatial density.

For **example**, say a propagating wave has a power density of **$5.0\, mW/m^2$**. Consider also a **window** whose surface area is **2 square meters**.

If this electromagnetic wave is propagating toward this window, then we will find that electromagnetic **energy** is passing **through** this window at a rate of **10.0 milli-Joules** every second!



$$P = \left(5.0\,\frac{W}{m^2}\right)\left(2.0\,m^2\right)$$
$$= 10\,\frac{mJ}{sec}$$

* The **direction** of the Poynting Vector indicates the direction of this power flow—the direction of the propagating wave.

Note that the power density of a **plane wave** is a constant:

$$\mathbf{W}(\bar{r}) = \frac{\hat{\mathbf{z}}}{2\eta_0}\left(\left|E_x\right|^2 + \left|E_y\right|^2\right)$$
$$= \frac{\hat{\mathbf{z}}}{2\eta_0}\left|\mathbf{E}(\bar{r},t)\right|^2$$

In other words, the magnitude and direction of a **plane-wave** Poynting Vector is **identical** at every point in the entire universe!

Q: *Is this likewise true for **all** propagating electromagnetic waves?*

A: Absolutely **not**! For example, the Poynting vector of a **spherical wave** is:

$$\mathbf{W}(\bar{r}) = \frac{\hat{\mathbf{r}}}{2\eta_0} \frac{|E_\theta(\theta,\phi)|^2 + |E_\phi(\theta,\phi)|^2}{r^2}$$

Here we will make the definition:

$$U(\theta,\phi) = \frac{|E_\theta(\theta,\phi)|^2 + |E_\phi(\theta,\phi)|^2}{2\eta_0} \quad \frac{Watts}{Steradian}$$

Therefore, the power density of a **spherical wave** is:

$$\boxed{\mathbf{W}(\bar{r}) = U(\theta,\phi)\frac{\hat{\mathbf{r}}}{r^2}}$$

Clearly, this power density is **not constant**, but instead diminishes (as $1/r^2$) as we move away from the origin.

Q: *So what's up with this function $U(\theta,\phi)$?*

A: The real, scalar function $U(\theta,\phi)$ is called the **intensity** of a **spherical** wave. We will find that it is a very important function in determining the performance of an **antenna**.

Q: *Antenna? What does all this have to do with antennas?*

A: A **radiating antenna** in fact launches a **spherical** wave. The expression above thus describes the **power density** produced by a radiating antenna (when located at the origin).