Phase and Frequency

Consider the trig functions **sin**x and **cos**x.

Q: What are the units of x??

A: The units of x **must** be **radians**.

In other words x is phase ϕ , i.e., **cos** ϕ and **sin** ϕ.

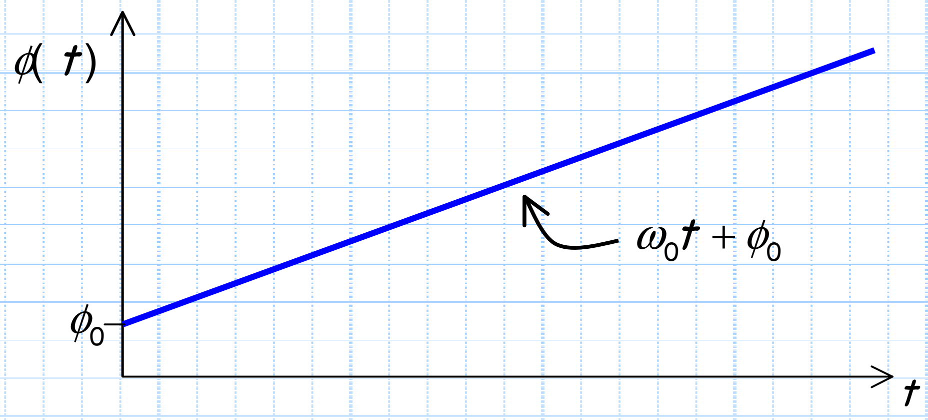
Phase can of course be a function of **time**, i.e., cos $\phi(t)$. For example:

$$\cos\left(\omega_0 t + \phi_0\right)$$

In other words, the signal **phase** $\phi(t)$ is $\phi(t) = \omega_0 t + \phi_0$!

A: Time for some **definitions**!

We call $\phi(t) = \omega_0 t + \phi_0$ the **total**, or absolute phase of the sinusoidal signal. Note the **total** phase is a **linearly increasing** function of time !



The **slope** of this line is ω_0, while the ***y*-intercept** is ϕ_0.

We can define the **relative phase** $\phi_r(t)$ as:

$$\phi_r(t) = \phi(t) - \omega_0 t$$

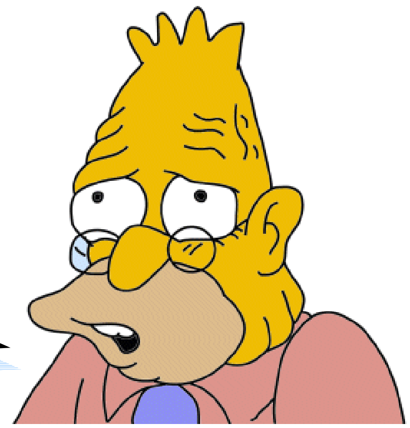
Thus, **if** $\phi(t) = \omega_0 t + \phi_0$, then $\phi_r(t) = \phi_0$.

But, the relative phase need not be a **constant**. In general, we can write:

$$\cos\left[\omega_0 t + \phi_r(t)\right]$$

Therefore, the relative phase is in general some arbitrary **function of time**.

Q: *O.K., so you have made **phase** really complicated, but at least the signal **frequency** is still ω_0, right ??*



A: **Wrong** ! Frequency too is a little more **complicated** than you might have imagined.

Angular frequency is **defined** as the rate of (**total**) phase change with respect to time. As a result, it is measured in units of **radians/second**.

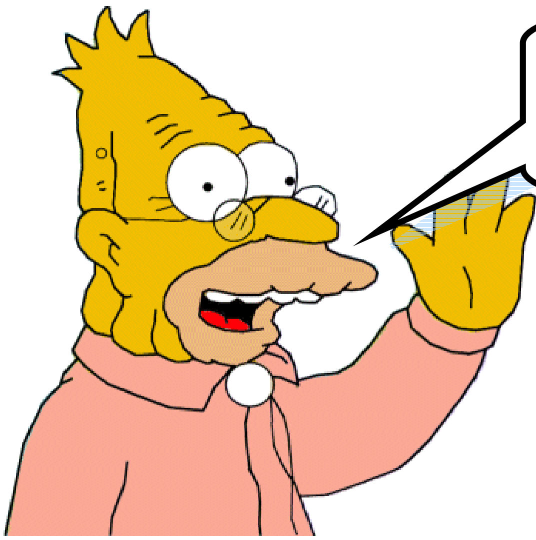
How do we **determine** the rate of phase change with respect to time?

We take the **derivative** of $\phi(t)$ with respect to t ! I.E.,

$$\omega(t) = \frac{d\,\phi(t)}{dt} \quad \text{(radians/sec)}$$

For example, if $\phi(t) = \omega_0 t + \phi_0$, then:

$$\omega(t) = \frac{d(\omega_0 t + \phi_0)}{dt} = \omega_0$$



Q: See! I **told** you! The frequency **is** ω_0 after all !

A: Not so fast! The frequency (i.e., the rate of phase change) is equal to ω_0 **only** if total phase is $\phi(t) = \omega_0 t + \phi_0$. In other words, the frequency is equal to ω_0 **if** the **relative phase** is a constant ϕ_0. Otherwise:

$$\begin{aligned}\omega(t) &= \frac{d\left[\omega_0 t + \phi_r(t)\right]}{dt} \\ &= \frac{d(\omega_0 t)}{dt} + \frac{d\phi_r(t)}{dt} \\ &= \omega_0 + \frac{d\phi_r(t)}{dt} \\ &= \omega_0 + \omega_r(t)\end{aligned}$$

In other words, the **total** frequency $\omega(t)$ is the sum of the **carrier** frequency ω_0 and the **relative** frequency $\omega_r(t)$.

→ The signal frequency can change with **time** !

Remember, we can also express frequency in **cycles/second** (i.e., Hz) if we divide by 2π.

$$f(t) = \frac{\omega(t)}{2\pi} \quad \text{(Hz)}$$

Therefore, we can write:

$$f(t) = f_0 + f_r(t)$$