The Scattering Matrix

At **"low"** frequencies, we can completely characterize a **linear** device or network using an **impedance** matrix, which relates the currents and voltages at **each** device terminal to the currents and voltages at **all** other terminals.

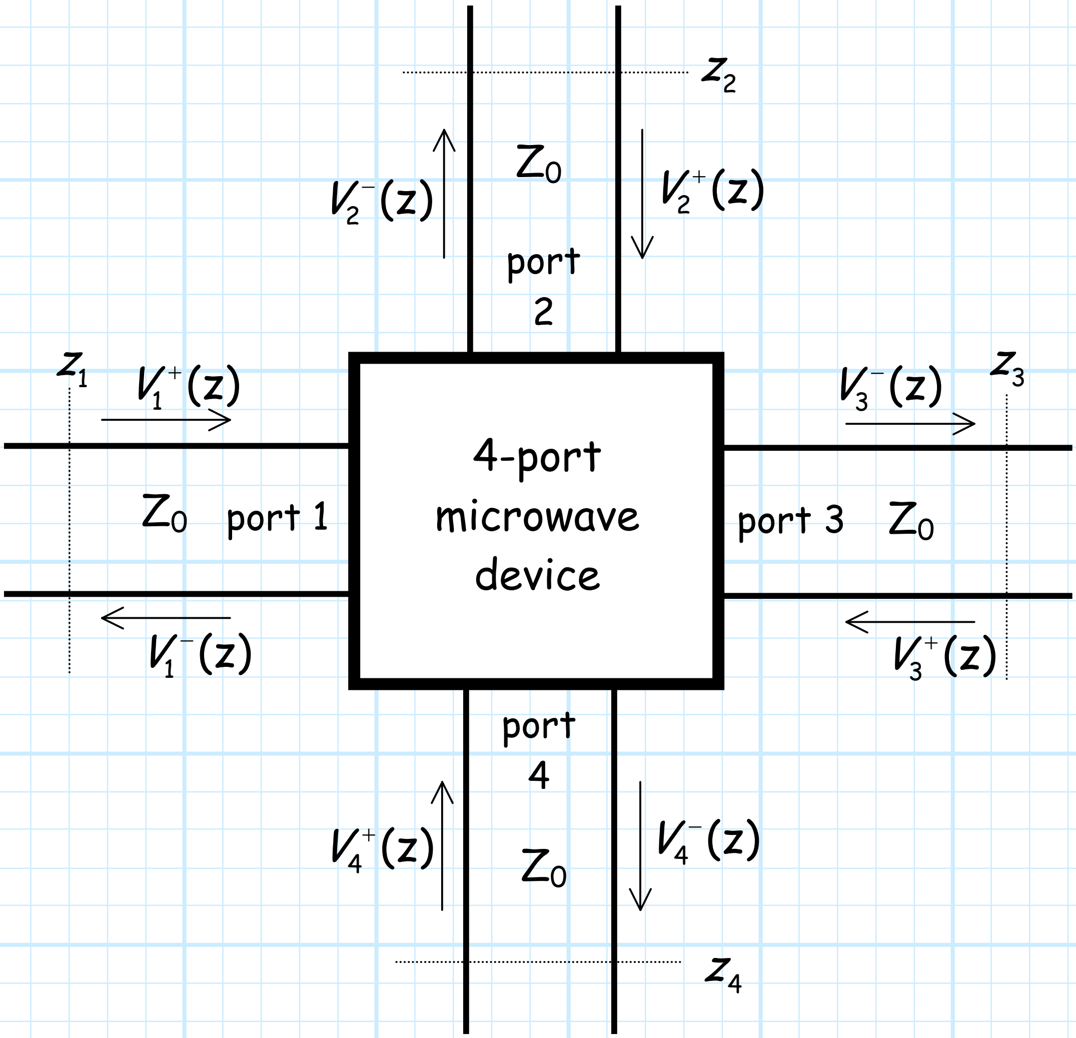
But, at microwave frequencies, it is **difficult** to measure currents and voltages!

* Instead, we can measure the **incident** and **reflected** waves $V_0^+(z)$ and $V_0^-(z)$.

* In other words, we can determine the relationship between the incident and reflected wave at **each** device terminal to the incident and reflected waves at **all** other terminals.

These relationships are completely represented by the **scattering matrix**. It **completely** describes the behavior of a linear, multi-port device at frequency ω.

Consider the **4-port** microwave device shown below:



Say we know that there exists an **incident** wave on **port 1** (i.e, $V_1^+ \neq 0$), while the incident waves on all other ports are known to be **zero** (i.e., $V_2^+ = V_3^+ = V_4^+ = 0$). Say we then measure the wave **out** of **port 2** (i.e., determine V_2^-). The complex ratio between $V_1^+(z = z_1)$ and $V_2^-(z = z_2)$ is know as the **scattering parameter** S_{21}:

$$S_{21} = \frac{V_2^-(z = z_2)}{V_1^+(z = z_1)} = \frac{V_2^- e^{+j\beta z_2}}{V_1^+ e^{-j\beta z_1}} = \frac{V_2^-}{V_1^+} e^{+j\beta(z_2+z_1)}$$

Likewise, the scattering parameters S_{31} and S_{41} are:

$$S_{31} = \frac{V_3^-(z=z_3)}{V_1^+(z=z_1)} \qquad \text{and} \qquad S_{41} = \frac{V_4^-(z=z_4)}{V_1^+(z=z_1)}$$

We of course could also define, say, scattering parameter S_{34} as the ratio between the complex values $V_4^+(z=z_4)$ (describing the wave **into** port 4) and $V_3^-(z=z_3)$ (describing the wave **out of** port 3), given that the input to all other ports (1,2, and 3) are zero.

Thus, more **generally**, the ratio of the wave incident on port n to the wave emerging from port m is:

$$\boxed{S_{mn} = \frac{V_m^-(z=z_m)}{V_n^+(z=z_n)} \qquad \text{(given that } V_k^+ = 0 \text{ for all } k \neq n)}$$

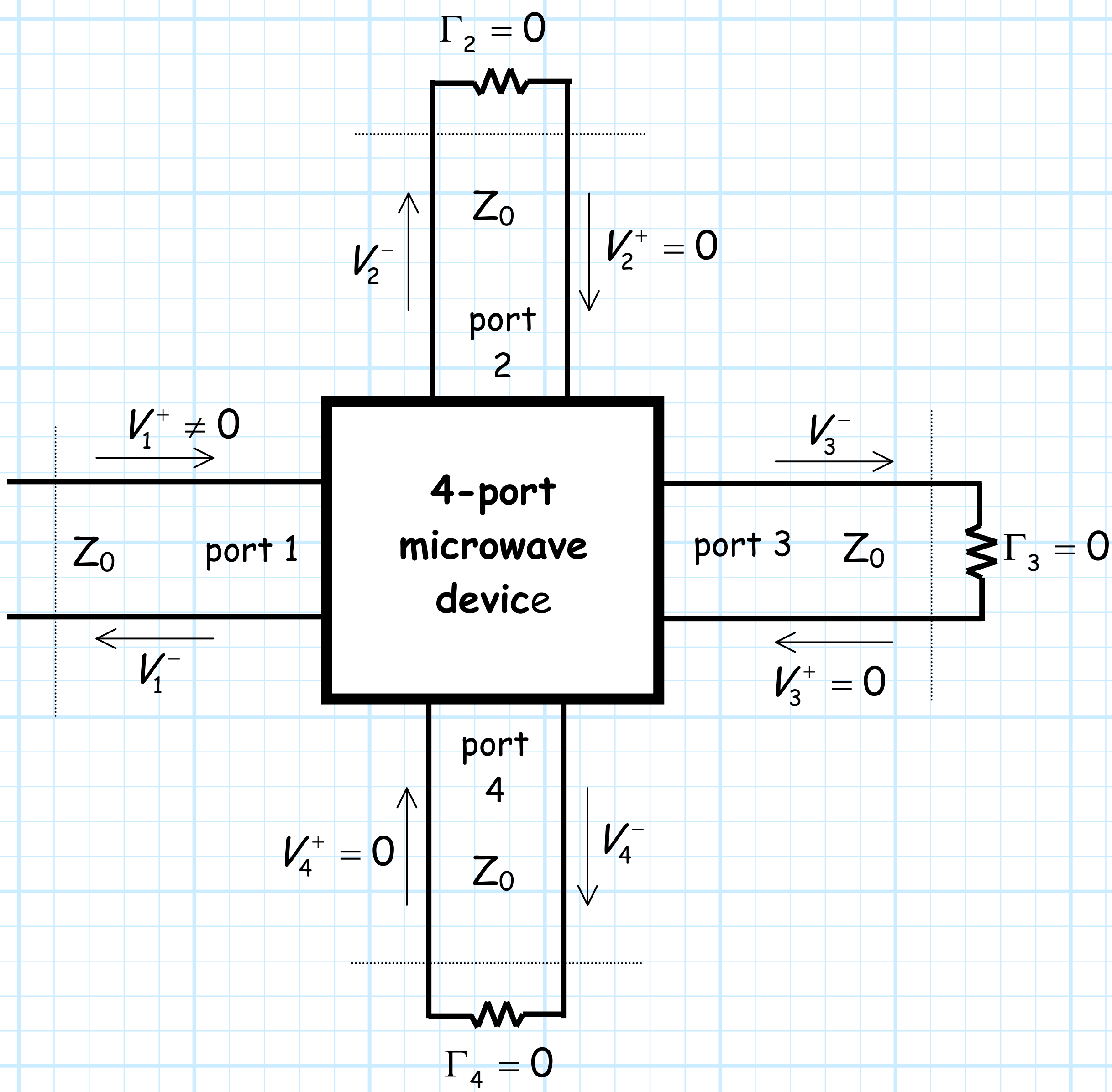
Note that frequently the port boundary locations are assigned a zero value (e.g., $z_1 = 0$, $z_2 = 0$). This of course simplifies the scattering parameter calculation:

$$S_{mn} = \frac{V_m^-(z_m=0)}{V_n^+(z_n=0)} = \frac{V_m^- e^{+j\beta 0}}{V_n^+ e^{-j\beta 0}} = \frac{V_m^-}{V_n^+}$$

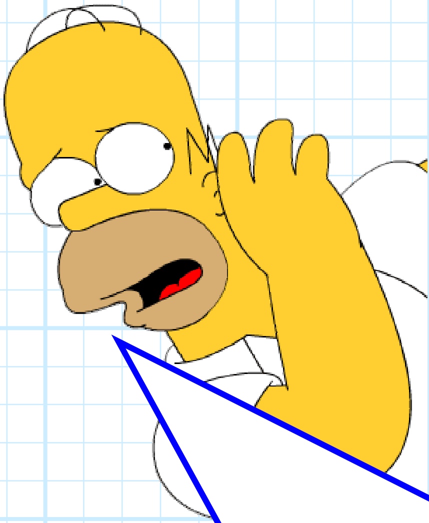
We will generally assume that the port locations are defined as $z=0$, and thus use the above notation. But remember where this expression came from!

Q: *How can we make sure that* ***only one*** *incident wave is non-zero ?*

A: **Terminate** all ports with a **matched load**!



If the ports are terminated in a **matched load** (i.e., $Z_L = Z_0$), then $\Gamma_n = 0$ and therefore:



$$V_n^+ = \Gamma_n V_n^- = 0$$

In other words, terminating a port ensures that there will be **no signal** incident on that port!

Q: *Just between you and me, I think you've messed this all up!* ***Earlier*** *you said that:*

$$V_n^- = \Gamma_n V_n^+$$

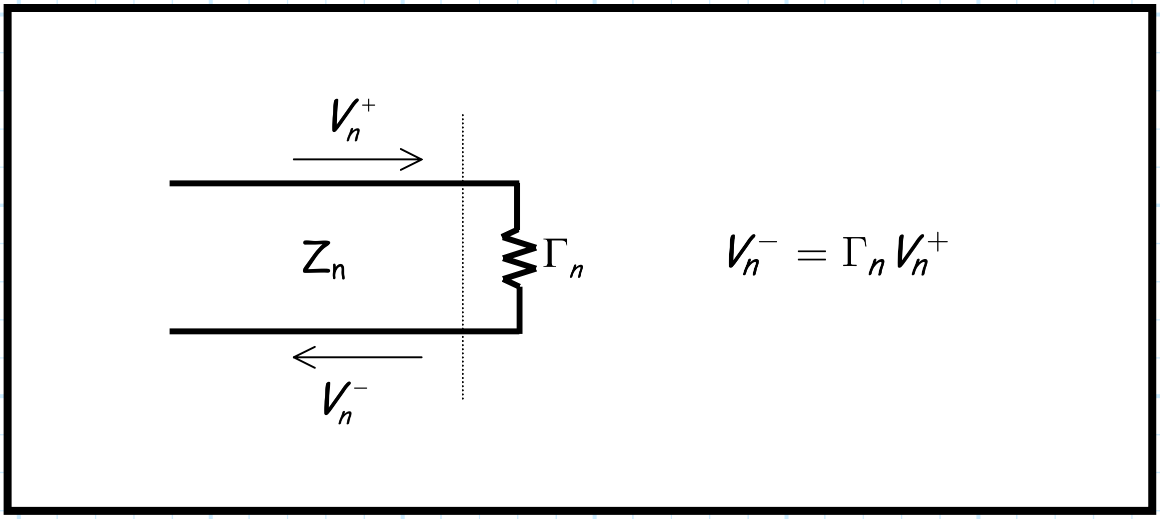
but ***now*** *you say that:*

$$V_n^+ = \Gamma_n V_n^-$$

Of course, there is no way that ***both*** *statements can be correct!*

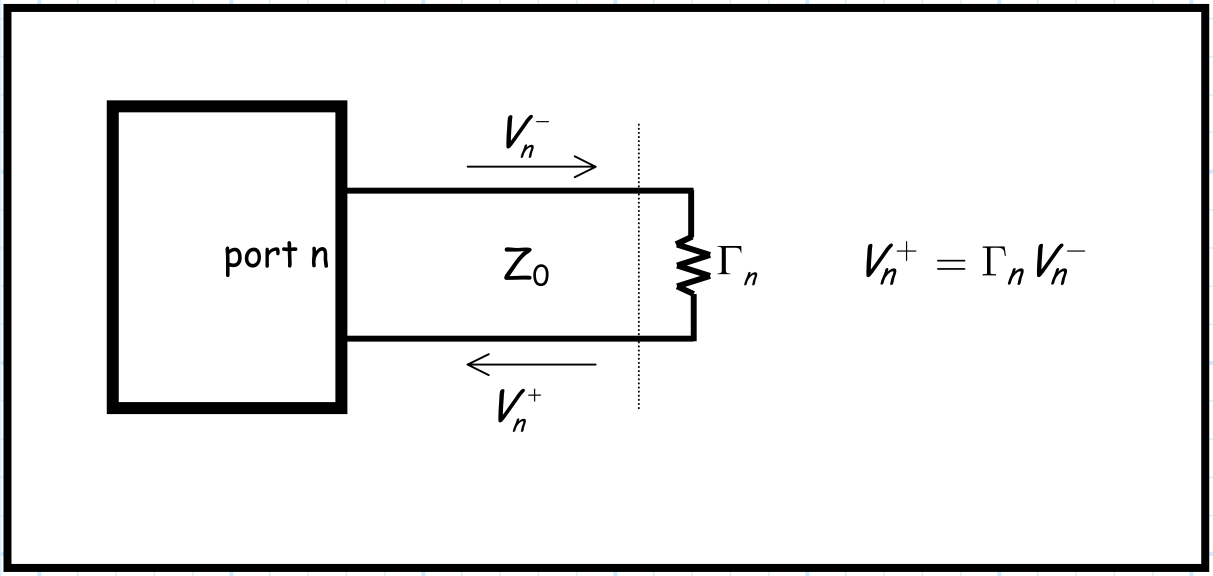
A: Actually, **both** statements are correct! You must be careful to understand the **physical definitions** of the constants V_n^+ and V_n^-.

For example, we **originally** analyzed this case:



In this case, the complex constant describing the wave **incident** on the load is V_n^+ and the constant describing the **reflected** wave is V_n^-.

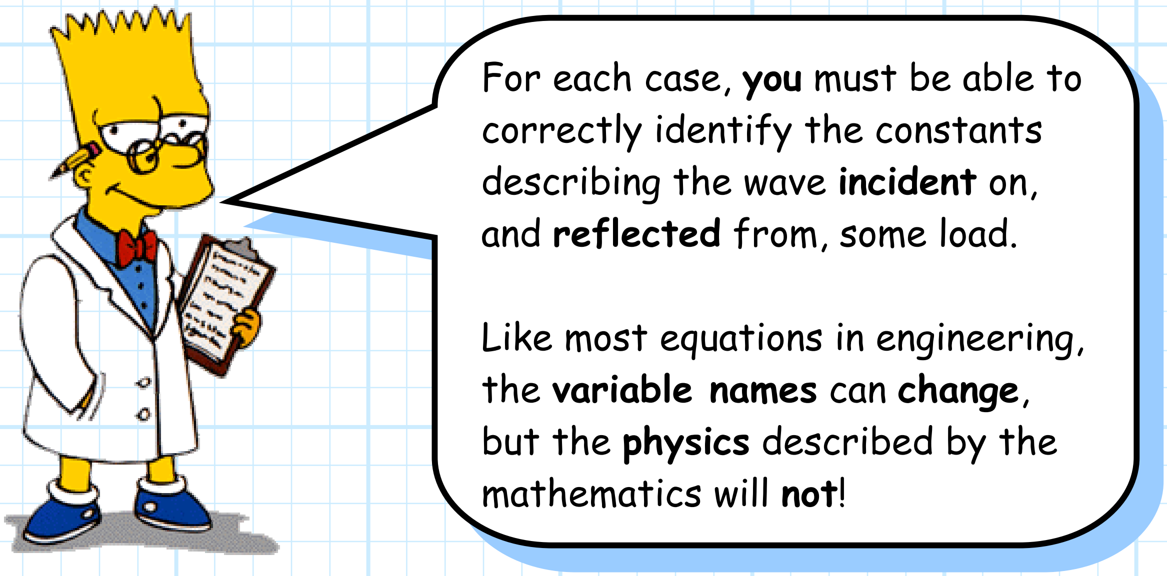
Contrast this with the case we are **now** considering:



Here, we defined the constant of the wave incident on **port** n as V_n^+. As a result, the constant V_n^+ now describes the wave **reflected** from the load. Likewise, the wave **incident** on the load is now described by constant V_n^-.

Perhaps we could more generally state that:

$$\boxed{V^{reflected} = \Gamma \; V^{incident}}$$



Now, **back** to our discussion of **S-parameters**. We found that:

$$\boxed{V_m^- = S_{mn} V_n^+ \qquad \text{(given that} \quad V_k^+ = 0 \;\; \text{for all } k \neq n)}$$

Which we can now **equivalently** state as:

$$\boxed{V_m^- = S_{mn} V_n^+ \qquad \text{(given that all ports } k \neq n \text{ are } \textbf{matched}\text{)}}$$

Say that we have waves **simultaneously** incident on each of the four ports of our device.

Q: *What then is the output at, for example, port 3??*

A: Use **superposition** !!

Since the device is **linear**, the output at port 3 due to **all** the incident waves is simply the coherent **sum** of the output at port 3 due to **each** wave:

$$V_3^- = S_{34} V_4^+ + S_{33} V_3^+ + S_{32} V_2^+ + S_{31} V_1^+$$

Or, more generally, the output at port m of an N-port device is:

$$V_m^- = \sum_{n=1}^{N} S_{mn} V_n^+$$

This expression can be written in matrix form as:

$$\bar{\mathbf{V}}^- = \bar{\bar{\mathbf{S}}}\, \bar{\mathbf{V}}^+$$

Where $\bar{\mathbf{V}}^-$ is the vector:

$$\bar{\mathbf{V}}^- = \left[V_1^-, V_2^-, V_3^-, \ldots, V_N^-\right]^T$$

and $\bar{\mathbf{V}}^+$ is the vector:

$$\bar{\mathbf{V}}^+ = \left[V_1^+, V_2^+, V_3^+, \ldots, V_N^+\right]^T$$

Therefore $\bar{\bar{\mathbf{S}}}$ is the **scattering matrix**:

$$\bar{\bar{\mathbf{S}}} = \begin{pmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{m1} & \cdots & S_{mn} \end{pmatrix}$$

The scattering matrix is a N by N matrix that **completely characterizes** a linear, N-port device (at frequency ω).