The Telegrapher Equations

Consider a section of “wire“:

\[ i(z,t) - i(z + \Delta z, t) \]

\[ v(z,t) - v(z + \Delta z, t) \]

Q: Huh ?! Current \( i \) and voltage \( v \) are a function of position \( z \) ?

Shouldn’t \( i(z,t) = i(z + \Delta z, t) \) and \( v(z,t) = v(z + \Delta z, t) \) ?

A: NO ! Because a wire is never a perfect conductor.

A “wire“ will have:

1) Inductance
2) Resistance
3) Capacitance
4) Conductance
i.e.,

\[
\partial + R \Delta z i(z, t) + L \Delta z \frac{\partial i(z, t)}{\partial t} = -\Delta \left( v(z, t) - G \Delta z \frac{\partial v(z, t)}{\partial t} \right)
\]

\[
\partial + C \Delta z v(z+\Delta z, t) = -\Delta \left( i(z+\Delta z, t) - C \Delta z \frac{\partial v(z+\Delta z, t)}{\partial t} \right)
\]

Where:

- \( R = \) resistance/unit length
- \( L = \) inductance/unit length
- \( C = \) capacitance/unit length
- \( G = \) conductance/unit length

\[ ∴ \] resistance of wire length \( \Delta z \) is \( R \Delta z \).

Using KVL, we find:

\[
\nu(z + \Delta z, t) - \nu(z, t) = -R \Delta z i(z, t) - L \Delta z \frac{\partial i(z, t)}{\partial t}
\]

and from KCL:

\[
i(z + \Delta z, t) - i(z, t) = -G \Delta z \nu(z, t) - C \Delta z \frac{\partial \nu(z, t)}{\partial t}
\]

resistance of wire length \( \Delta z \) is \( R \Delta z \).
Dividing the first equation by $\Delta z$, and then taking the limit as $\Delta z \to 0$:

$$\lim_{\Delta z \to 0} \frac{v(z + \Delta z, t) - v(z, t)}{\Delta z} = -R i(z, t) - L \frac{\partial i(z, t)}{\partial t}$$

which, by definition of the derivative, becomes:

$$\frac{\partial v(z, t)}{\partial z} = -R i(z, t) - L \frac{\partial i(z, t)}{\partial t}$$

Similarly, the KCL equation becomes:

$$\frac{\partial i(z, t)}{\partial z} = -G v(z, t) - C \frac{\partial v(z, t)}{\partial t}$$

If $v(z, t)$ and $i(z, t)$ have the form:

$$v(z, t) = \text{Re}\{V(z)e^{j\omega t}\} \quad \text{and} \quad i(z, t) = \text{Re}\{I(z)e^{j\omega t}\}$$

then these equations become:

$$\frac{\partial V(z)}{\partial z} = -(R + j\omega L) I(z)$$

$$\frac{\partial I(z)}{\partial z} = -(G + j\omega C) V(z)$$

These equations are known as the telegrapher’s equations!
* The functions $I(z)$ and $V(z)$ are complex, where the magnitude and phase of the complex functions describe the magnitude and phase of the sinusoidal time function $e^{j\omega t}$.

* Thus, $I(z)$ and $V(z)$ describe the current and voltage along the transmission line, as a function as position $z$.

* Remember, not just any function $I(z)$ and $V(z)$ can exist on a transmission line, but rather only those functions that satisfy the telegraphers equations.

Our task, therefore, is to solve the telegrapher equations and find all solutions $I(z)$ and $V(z)$!