The Terminated, Lossless Transmission Line

Consider a lossless line, length \( \ell \), terminated with a load \( Z_L \).

\[ I(z) \quad \rightarrow \quad I_L \]

\[ \mathbf{V}(z) \quad Z_0, \beta \quad \mathbf{V}_L \quad Z_L \]

\[ z = -\ell \quad \ell \quad z = 0 \]

We know from the telegrapher’s equations that:

\[ V(z = 0) = V_0^+ e^{-\beta \ell} + V_0^- e^{\beta \ell} = V_0^+ + V_0^- \]

\[ I(z = 0) = \frac{V_0^+}{Z_0} e^{-\beta \ell} - \frac{V_0^-}{Z_0} e^{\beta \ell} = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0} \]

We also know that the load voltage and current must be related by “Ohms Law”:

\[ \frac{V_L}{I_L} = Z_L \]
BUT, we notice that the transmission line current at \( z = 0 \) is the current flowing into the load, while the transmission line voltage at \( z = 0 \) is the voltage across the load:

\[
V(z = 0) = V_L
\]

\[
I(z = 0) = I_L
\]

These are the **boundary conditions** of transmission line problem, and result in yet another equation that \( I(z) \) and \( V(z) \) must satisfy:

\[
Z_L = \frac{V_L}{I_L} = \frac{V(z = 0)}{I(z = 0)} = \frac{(V_0^+ + V_0^-)}{(V_0^+ - V_0^-)} \left( \frac{1}{Z_0} - \frac{1}{Z_0} \right)
\]

Rearranging, we find that the two complex coefficients \( V_0^+ \) and \( V_0^- \) are no longer **independent**, but instead must satisfy the following:

\[
\frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \pm \Gamma
\]

The value \( \Gamma \) is a complex coefficient known as the **reflection coefficient**. It relates the magnitude and phase of the wave incident on the load (\( V_0^+ \)) to the magnitude and phase of the wave emerging (i.e., reflected from) the load (\( V_0^- \)).

\[
V_0^- = \Gamma V_0^+
\]
Some interesting things to note about the reflection coefficient:

1) Since \( \text{Re} \{Z_L\} > 0 \), \(|\Gamma| \leq 1\).

2) The current and voltage along a terminated transmission line can be written as:

\[
V(z) = V_0^+ \left[ e^{-j\beta z} + \Gamma e^{+j\beta z} \right]
\]

\[
I(z) = \frac{V_0^+}{Z_0} \left[ e^{-j\beta z} - \Gamma e^{+j\beta z} \right]
\]

\[
V_0^- = \Gamma V_0
\]

Q: How do we determine \( V_0^+ \)?

A: We require a second boundary condition to determine \( V_0^+ \). The only boundary left is at the other end of the transmission line. Typically, a source of some sort is located there. This makes physical sense, as something must generate the incident wave!