The Transmission Line Wave Equation

So, what functions $I(z)$ and $V(z)$ **do** satisfy both telegrapher's equations??

To make this easier, we will combine the telegrapher equations to form **one** differential equation for $V(z)$ and **another** for $I(z)$.

First, take the **derivative** with respect to z of the **first** telegrapher equation:

$$\frac{\partial}{\partial z}\left\{\frac{\partial V(z)}{\partial z} = -(R+j\omega L)I(z)\right\}$$
$$= \frac{\partial^2 V(z)}{\partial z^2} = -(R+j\omega L)\frac{\partial I(z)}{\partial z}$$

Note that the **second** telegrapher equation expresses the derivative of $I(z)$ in terms of $V(z)$:

$$\frac{\partial I(z)}{\partial z} = -(G+j\omega C)\,V(z)$$

Combining these two equations, we get an equation involving $V(z)$ **only**:

$$\frac{\partial^2 V(z)}{\partial z^2} = (R+j\omega L)(G+j\omega C)\,V(z)$$
$$= \gamma^2 V(z)$$

where it is apparent that $\gamma^2 = (R + j\omega L)(G + j\omega C)$.

In a **similar** manner (i.e., begin by taking the derivative of the **second** telegrapher equation), we can derive the differential equation:

$$\frac{\partial^2 I(z)}{\partial z} = \gamma^2 I(z)$$

We have **decoupled** the telegrapher's equations, such that we now have **two** equations involving **one** function only:

$$\boxed{\begin{gathered}\frac{\partial^2 V(z)}{\partial z} = \gamma^2 V(z) \\[2ex] \frac{\partial^2 I(z)}{\partial z} = \gamma^2 I(z)\end{gathered}}$$

BUT ! Again we ask, what functions satisfy **these** differential equations ??

Note only **special** functions satisfy these equations: if we take the double derivative of the function, the result is the **original function** (to within a constant)!

Such functions do exist ! For example, $e^{-\gamma z}$ and $e^{+\gamma z}$.

Therefore, the **general** solution to these differential equations (and thus the telegrapher equations) are a **linear superposition** of these two solutions:

$$V(z) = V_0^+ \, e^{-\gamma z} + V_0^- \, e^{+\gamma z}$$

$$I(z) = I_0^+ \, e^{-\gamma z} + I_0^- \, e^{+\gamma z}$$

where V_0^+, V_0^-, I_0^+, I_0^-, and γ are **complex** constants.

Q: *How do we determine* V_0^+, V_0^-, I_0^+, and I_0^- *??*

A: We apply boundary conditions !

The solutions describe **two waves** propagating in the transmission line, one propagating in a direction (***+z***) and one propagating in the other direction (***-z***).

Therefore, we call the differential equations introduced in this handout the transmission line **wave equations**.