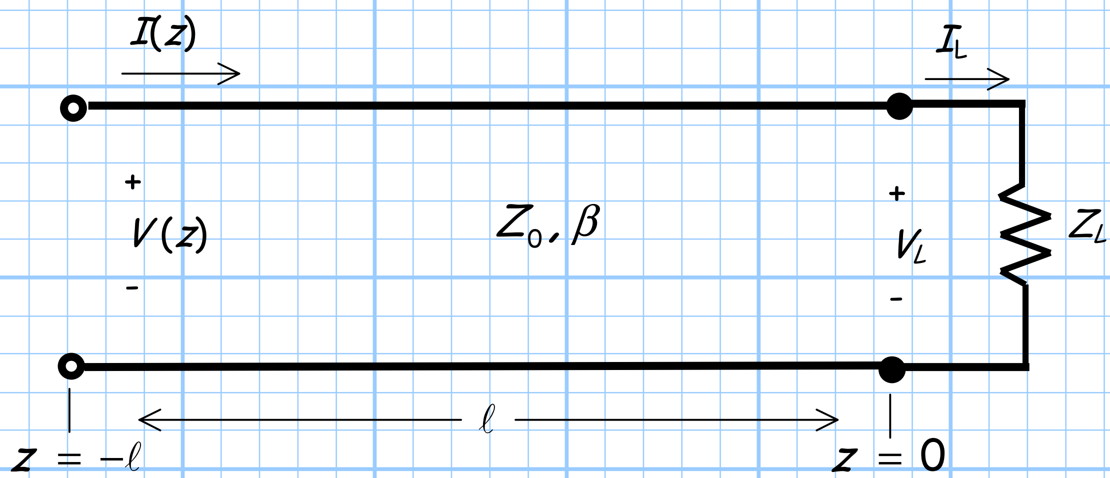
Transmission Line Input Impedance

Consider a **lossless** line, length ℓ, terminated with a **load** Z_L.



What is the **input impedance** of this line?

Q: *Just what do you mean by **input** impedance?*

A: The line impedance seen at the **beginning** $(z = -\ell)$ of the transmission line, i.e.:

$$Z_{in} = Z(z = -\ell) = \frac{V(z = -\ell)}{I(z = -\ell)}$$

Note Z_{in} equal to **neither** the load impedance Z_L **nor** the characteristic impedance Z_0!

$$Z_{in} \neq Z_L \quad \text{and} \quad Z_{in} \neq Z_0$$

To determine exactly what Z_{in} is, we first must determine the voltage and current at the **beginning** of the transmission line ($z = -\ell$).

$$V(z=-\ell) = V_0^+ \left[e^{+j\beta\ell} + \Gamma e^{-j\beta\ell} \right]$$

$$I(z=-\ell) = \frac{V_0^+}{Z_0} \left[e^{+j\beta\ell} - \Gamma e^{-j\beta\ell} \right]$$

Therefore:

$$Z_{in} = \frac{V(z=-\ell)}{I(z=-\ell)} = Z_0 \left(\frac{e^{+j\beta\ell} + \Gamma e^{-j\beta\ell}}{e^{+j\beta\ell} - \Gamma e^{-j\beta\ell}} \right)$$

We can explicitly write Z_{in} in terms of load Z_L using the relationship:

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Combining these two expressions, we get:

$$\begin{aligned} Z_{in} &= Z_0 \frac{(Z_L + Z_0) e^{+j\beta\ell} + (Z_L - Z_0) e^{-j\beta\ell}}{(Z_L + Z_0) e^{+j\beta\ell} - (Z_L - Z_0) e^{-j\beta\ell}} \\ &= Z_0 \left(\frac{Z_L \left(e^{+j\beta\ell} + e^{-j\beta\ell} \right) + Z_0 \left(e^{+j\beta\ell} - e^{-j\beta\ell} \right)}{Z_L \left(e^{+j\beta\ell} + e^{-j\beta\ell} \right) - Z_0 \left(e^{+j\beta\ell} - e^{-j\beta\ell} \right)} \right) \end{aligned}$$

Now, recall **Euler's equations**:

$$e^{+j\beta\ell} = \cos\beta\ell + j\sin\beta\ell$$
$$e^{-j\beta\ell} = \cos\beta\ell - j\sin\beta\ell$$

Using Euler's relationships, we can likewise write the input impedance without the **complex** exponentials:

$$
\begin{aligned}
Z_{in} &= Z_0 \left(\frac{Z_L \cos \beta\ell + j\, Z_0 \sin \beta\ell}{Z_0 \cos \beta\ell + j\, Z_L \sin \beta\ell} \right) \\
&= Z_0 \left(\frac{Z_L + j\, Z_0 \tan \beta\ell}{Z_0 + j\, Z_L \tan \beta\ell} \right)
\end{aligned}
$$

Note that depending on the values of β, Z_0 and ℓ, the input impedance can be **radically** different from the load impedance Z_L!

Special Cases

1. $\ell = \lambda/2$

If the length of the transmission line is exactly **one-half** wavelength ($\ell = \lambda/2$), we find that:

$$\beta\ell = \frac{2\pi}{\lambda}\frac{\lambda}{2} = \pi$$

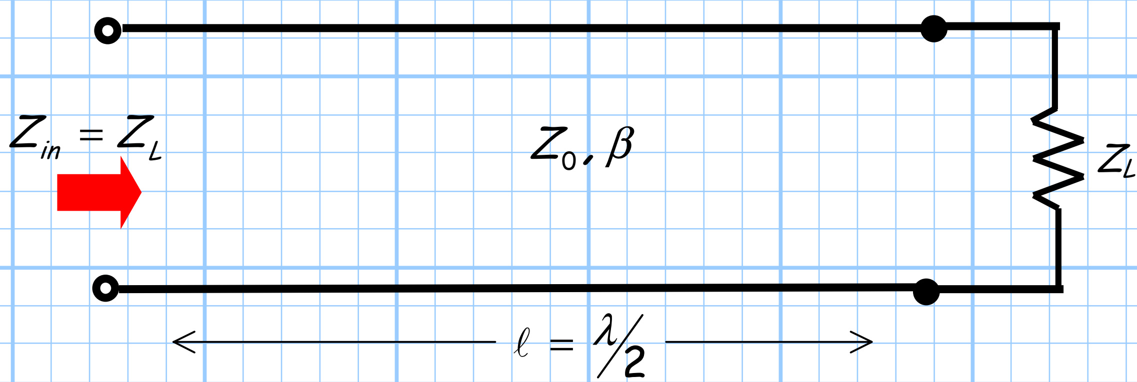
meaning that:

$$\cos \beta\ell = \cos \pi = -1 \qquad \text{and} \qquad \sin \beta\ell = \sin \pi = 0$$

and therefore:

$$
\begin{aligned}
Z_{in} &= Z_0\left(\frac{Z_L\cos\beta\ell + j\,Z_0\sin\beta\ell}{Z_0\cos\beta\ell + j\,Z_L\sin\beta\ell}\right)\\
&= Z_0\left(\frac{Z_L\,(-1)+j\,Z_L\,(0)}{Z_0\,(-1)+j\,Z_L\,(0)}\right)\\
&= Z_L
\end{aligned}
$$

In other words, if the transmission line is precisely **one-half wavelength** long, the **input** impedance is equal to the **load** impedance, **regardless** of Z_0 or β.



2. $\ell = \lambda/4$

If the length of the transmission line is exactly **one-quarter** wavelength ($\ell = \lambda/4$), we find that:

$$\beta\ell = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$$

meaning that:

$$\cos\beta\ell = \cos\ \pi/2 = 0 \qquad \text{and} \qquad \sin\beta\ell = \sin\ \pi/2 = 1$$

and therefore:

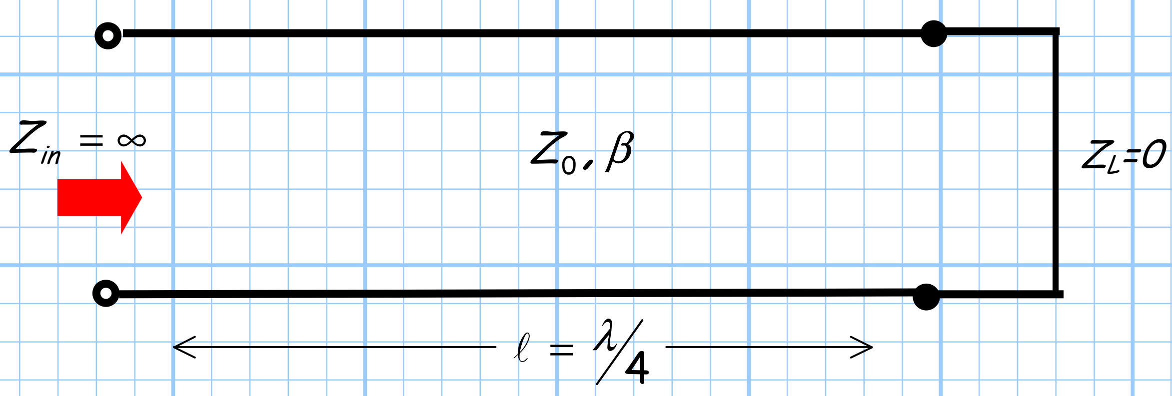
$$
\begin{aligned}
Z_{in} &= Z_0 \left(\frac{Z_L \cos\beta\ell + j\, Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j\, Z_L \sin\beta\ell} \right) \\
&= Z_0 \left(\frac{Z_L\,(0) + j\, Z_0\,(1)}{Z_0\,(0) + j\, Z_L\,(1)} \right) \\
&= \frac{(Z_0)^2}{Z_L}
\end{aligned}
$$

In other words, if the transmission line is precisely **one-quarter wavelength** long, the **input** impedance is **inversely** proportional to the **load** impedance.

Think about what this means! Say the load impedance is a **short** circuit, such that $Z_L = 0$. The **input impedance** at beginning of the $\lambda/4$ transmission line is therefore:

$$
Z_{in} = \frac{(Z_0)^2}{Z_L} = \frac{(Z_0)^2}{0} = \infty
$$

$Z_{in} = \infty$! This is an **open** circuit! The quarter-wave transmission line **transforms** a short-circuit into an open-circuit—and vice versa!

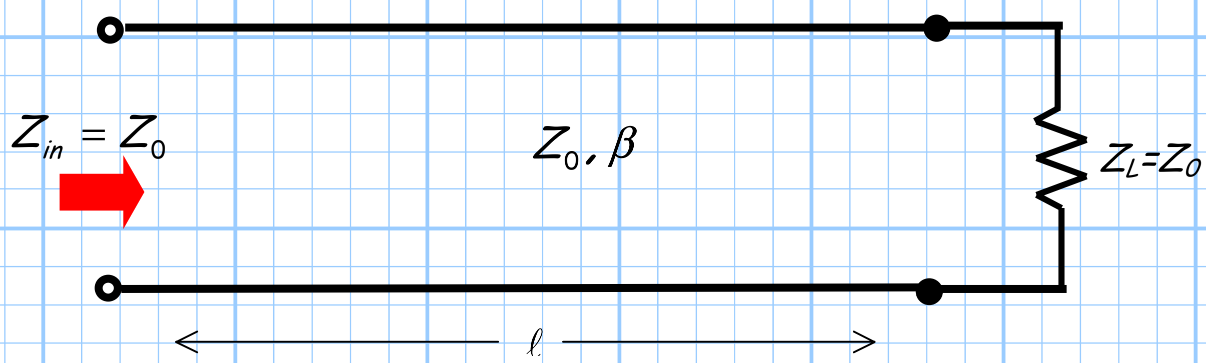


3. $Z_L = Z_0$

If the load is **numerically equal** to the characteristic impedance of the transmission line (a real value), we find that the input impedance becomes:

$$\begin{aligned} Z_{in} &= Z_0 \left(\frac{Z_L \cos \beta\ell + j\, Z_0 \sin \beta\ell}{Z_0 \cos \beta\ell + j\, Z_L \sin \beta\ell} \right) \\ &= Z_0 \left(\frac{Z_0 \cos \beta\ell + j\, Z_0 \sin \beta\ell}{Z_0 \cos \beta\ell + j\, Z_0 \sin \beta\ell} \right) \\ &= Z_0 \end{aligned}$$

In other words, if the **load** impedance is equal to the transmission line **characteristic** impedance, the **input** impedance will be likewise be equal to Z_0 regardless of the transmission line length ℓ.

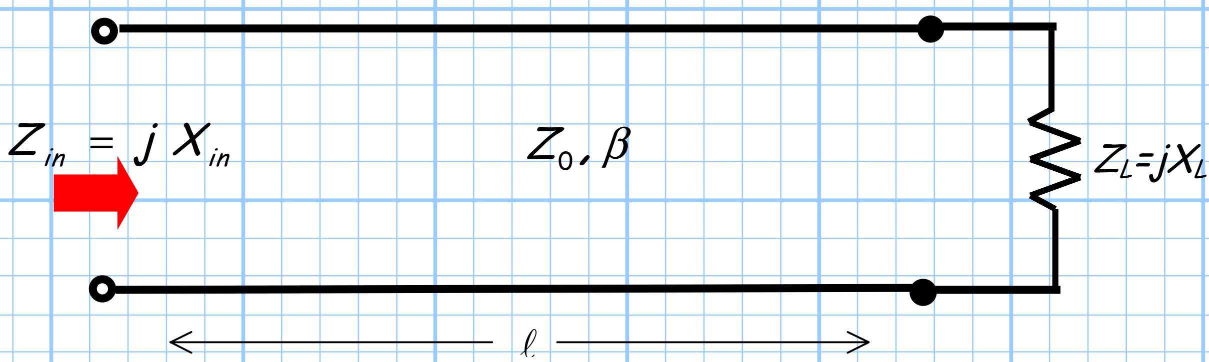


4. $Z_L = j\, X_L$

If the load is **purely reactive** (i.e., the resistive component is zero), the input impedance is:

$$\begin{aligned} Z_{in} &= Z_0 \left(\frac{Z_L \cos \beta\ell + j\, Z_0 \sin \beta\ell}{Z_0 \cos \beta\ell + j\, Z_L \sin \beta\ell} \right) \\ &= Z_0 \left(\frac{j\, X_L \cos \beta\ell + j\, Z_0 \sin \beta\ell}{Z_0 \cos \beta\ell + j^2\, X_L \sin \beta\ell} \right) \\ &= j\, Z_0 \left(\frac{X_L \cos \beta\ell + Z_0 \sin \beta\ell}{Z_0 \cos \beta\ell - X_L \sin \beta\ell} \right) \end{aligned}$$

In other words, if the load is purely reactive, then the input impedance will **likewise** be purely reactive, **regardless** of the line length ℓ.



Note that the **opposite** is **not** true: even if the load is **purely resistive** ($Z_L = R$), the input impedance will be **complex** (both resistive and reactive components).

Q: Why is this?

5. $\ell \ll \lambda$

If the transmission line is **electrically small**—its length ℓ is small with respect to signal wavelength λ--we find that:

$$\beta\ell = \frac{2\pi}{\lambda}\ell = 2\pi\frac{\ell}{\lambda} \approx 0$$

and thus:

$$\cos\beta\ell = \cos 0 = 1 \qquad \text{and} \qquad \sin\beta\ell = \sin 0 = 0$$

so that the input impedance is:

$$\begin{aligned} Z_{in} &= Z_0\left(\frac{Z_L\cos\beta\ell + j\,Z_0\sin\beta\ell}{Z_0\cos\beta\ell + j\,Z_L\sin\beta\ell}\right) \\ &= Z_0\left(\frac{Z_L\,(1) + j\,Z_L\,(0)}{Z_0\,(1) + j\,Z_L\,(0)}\right) \\ &= Z_L \end{aligned}$$

In other words, if the transmission line length is much smaller than a wavelength, the **input** impedance Z_{in} will **always** be equal to the **load** impedance Z_L.

This is the assumption we used in all previous circuits courses (e.g., EECS 211, 212, 312, 412)! In those courses, we assumed that the signal frequency ω is relatively **low**, such that the signal wavelength λ is **very large** ($\lambda \gg \ell$).

Note also for this case (the electrically short transmission line), the voltage and current at each end of the transmission line are approximately the same!

$$V(z=-\ell) \approx V(z=0) \quad \text{and} \quad I(z=-\ell) \approx I(z=0) \quad \text{if} \quad \ell \ll \lambda$$

If $\ell \ll \lambda$, our "wire" behaves **exactly** as it did in EECS 211 !