2.1 -The Lumped Element Circuit Model for Transmission Lines

Reading Assignment: *pp. 1-5, 49-52*

Q: *So just what is a **transmission line**?*

A:

→

Q: *Oh, so it's simply a conducting **wire**, right?*

A:

HO: The Telegraphers Equations

Q: *So, what complex functions I(z) and V(z) **do** satisfy both telegrapher equations?*

A:

HO: The Transmission Line Wave Equations

Q: *Are the solutions for I(z) and V(z) completely **independent**, or are they **related** in any way ?*

A:

HO: The Transmission Line Characteristic Impedance

Q: *So what is the significance of the complex constant γ? What does it tell us?*

A:

HO: The Complex Propagation Constant

Q: *Is **characteristic** impedance Z_0 the same as the concept of impedance I learned about in **circuits** class?*

A:

HO: Line Impedance

Q: *These **wave** functions $V^{+}(z)$ and $V^{-}(z)$ seem to be important. How are **they** related?*

A:

HO: The Reflection Coefficient

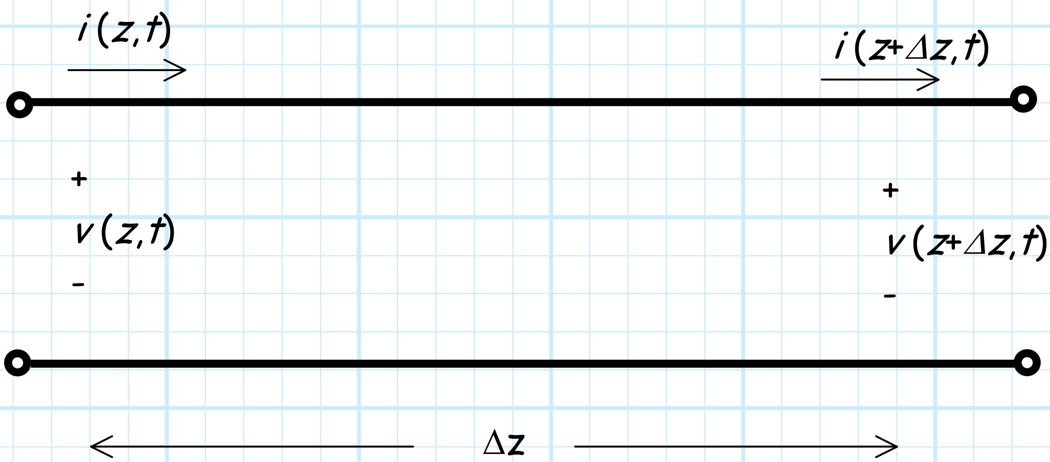
Q: *Now,* ***you*** *said earlier that* ***characteristic*** *impedance Z_0 is a* ***complex*** *value. But I recall engineers referring to a transmission line as simply a "50 Ohm line", or a "300 Ohm line". But these are* ***real*** *values; are they* ***not*** *referring to characteristic impedance Z_0 ??*

A:

HO: The Lossless Transmission Line

The Telegrapher Equations

Consider a section of "wire":



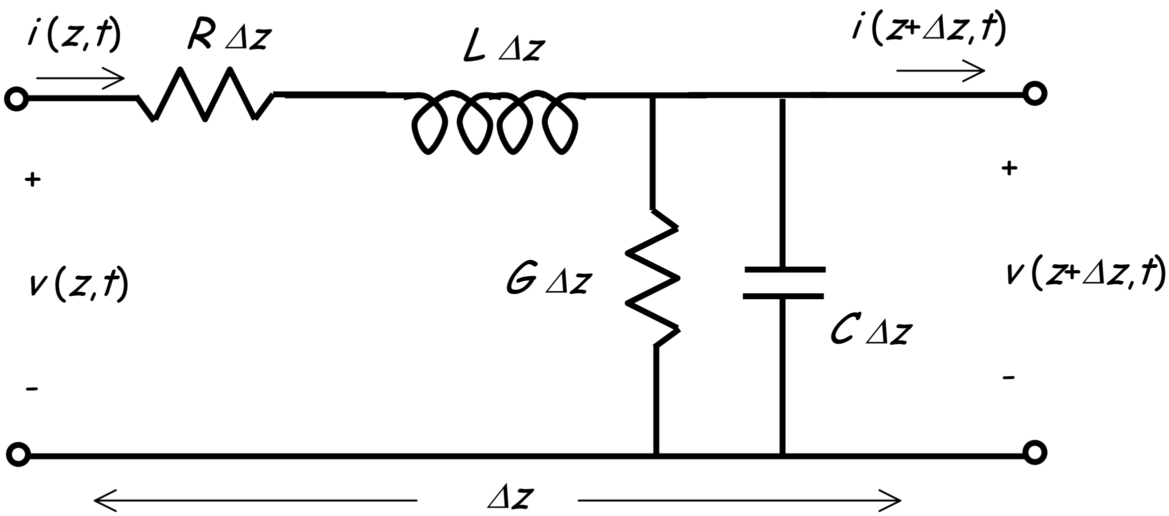
Q: *Huh ?! Current i and voltage v are a function of* ***position z*** *??* *Shouldn't* $i(z,t) = i(z+\Delta z,t)$ *and* $v(z,t) = v(z+\Delta z,t)$ *?*

A: NO ! Because a wire is never a **perfect** conductor.

A "wire" will have:

1) Inductance
2) Resistance
3) Capacitance
4) Conductance

i.e.,



Where:

R = resistance/unit length
L = inductance/unit length
C = capacitance/unit length
G = conductance/unit length

$\therefore$ resistance of wire length Δz is $R\Delta z$.

Using KVL, we find:

$$v(z+\Delta z,t)-v(z,t)=-R\Delta z\, i(z,t)-L\Delta z\frac{\partial i(z,t)}{\partial t}$$

and from KCL:

$$i(z+\Delta z,t)-i(z,t)=-G\Delta z\, v(z,t)-C\Delta z\frac{\partial v(z,t)}{\partial t}$$

Dividing the first equation by Δz, and then taking the limit as $\Delta z \rightarrow 0$:

$$\lim_{\Delta z \rightarrow 0} \frac{v(z+\Delta z,t)-v(z,t)}{\Delta z} = -R\,i(z,t) - L\frac{\partial i(z,t)}{\partial t}$$

which, by definition of the derivative, becomes:

$$\frac{\partial v(z,t)}{\partial z} = -R\,i(z,t) - L\frac{\partial i(z,t)}{\partial t}$$

Similarly, the KCL equation becomes:

$$\frac{\partial i(z,t)}{\partial z} = -G\,v(z,t) - C\frac{\partial v(z,t)}{\partial t}$$

If $v(z,t)$ and $i(z,t)$ have the form:

$$v(z,t) = \mathrm{Re}\left\{V(z)e^{j\omega t}\right\} \quad \text{and} \quad i(z,t) = \mathrm{Re}\left\{I(z)e^{j\omega t}\right\}$$

then these equations become:

$$\boxed{\begin{aligned} \frac{\partial V(z)}{\partial z} &= -(R+j\omega L)\,I(z) \\ \frac{\partial I(z)}{\partial z} &= -(G+j\omega C)\,V(z) \end{aligned}}$$

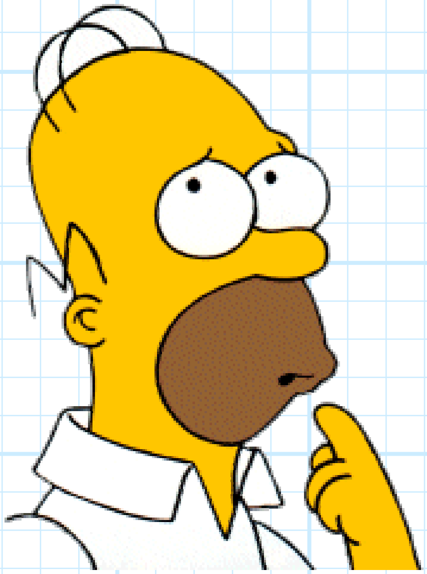
These equations are known as the **telegrapher's equations** !

* The functions $I(z)$ and $V(z)$ are **complex,** where the **magnitude** and **phase** of the complex functions describe the **magnitude** and **phase** of the sinusoidal time function $e^{j\omega t}$.

* Thus, $I(z)$ and $V(z)$ describe the current and voltage along the transmission line, as a function as position z.

* **Remember**, not just **any** function $I(z)$ and $V(z)$ can exist on a transmission line, but rather **only** those functions that satisfy the **telegraphers equations**.

Our task, therefore, is to **solve** the telegrapher equations and find **all** solutions $I(z)$ and $V(z)$!



The Transmission Line Wave Equation

Q: *So, what functions $I(z)$ and $V(z)$ **do** satisfy both telegrapher's equations??*

A: To make this easier, we will combine the telegrapher equations to form **one** differential equation for $V(z)$ and **another** for $I(z)$.

First, take the **derivative** with respect to z of the **first** telegrapher equation:

$$\frac{\partial}{\partial z}\left\{\frac{\partial V(z)}{\partial z} = -(R + j\omega L) I(z)\right\}$$

$$= \frac{\partial^2 V(z)}{\partial z^2} = -(R + j\omega L)\frac{\partial I(z)}{\partial z}$$

Note that the **second** telegrapher equation expresses the derivative of $I(z)$ in terms of $V(z)$:

$$\frac{\partial I(z)}{\partial z} = -(G + j\omega C)\, V(z)$$

Combining these two equations, we get an equation involving $V(z)$ **only**:

$$\frac{\partial^2 V(z)}{\partial z^2} = (R + j\omega L)(G + j\omega C)\, V(z)$$
$$= \gamma^2 V(z)$$

where it is apparent that:

$$\boxed{\gamma^2 \doteq (R + j\omega L)(G + j\omega C)}$$

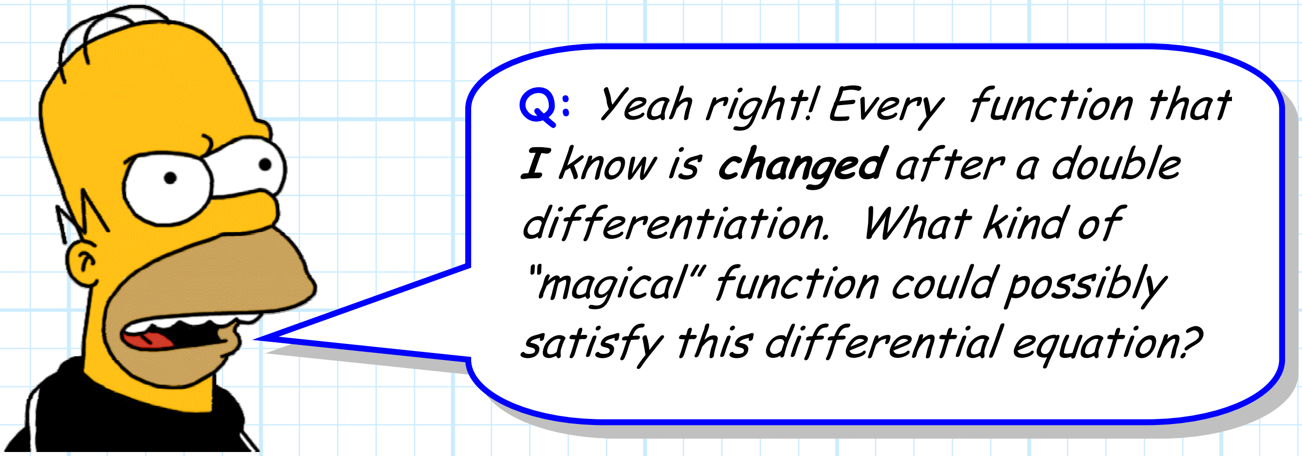
In a **similar** manner (i.e., begin by taking the derivative of the **second** telegrapher equation), we can derive the differential equation:

$$\frac{\partial^2 I(z)}{\partial z} = \gamma^2 I(z)$$

We have **decoupled** the telegrapher's equations, such that we now have **two** equations involving **one** function only:

$$\boxed{\begin{gathered} \frac{\partial^2 V(z)}{\partial z} = \gamma^2 V(z) \\ \frac{\partial^2 I(z)}{\partial z} = \gamma^2 I(z) \end{gathered}}$$

Note only **special** functions satisfy these equations: if we take the double derivative of the function, the result is the **original function** (to within a constant)!



A: Such functions **do** exist !

For example, the functions $V(z) = e^{-\gamma z}$ and $V(z) = e^{+\gamma z}$ each satisfy this transmission line wave equation (**insert** these into the differential equation and see for **yourself**!).

Likewise, since the transmission line wave equation is a **linear** differential equation, a weighted **superposition** of the two solutions is **also a solution** (again, **insert** this solution to and see for **yourself**!):

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

In fact, it turns out that **any** and **all** possible solutions to the differential equations can be expressed in **this** simple form!

Therefore, the **general** solution to these wave equations (and thus the telegrapher equations) are:

$$\boxed{\begin{aligned} V(z) &= V_0^+\, e^{-\gamma z} + V_0^-\, e^{+\gamma z} \\ I(z) &= I_0^+\, e^{-\gamma z} + I_0^-\, e^{+\gamma z} \end{aligned}}$$

where V_0^+, V_0^-, I_0^+, I_0^-, and γ are **complex constants**.

→ It is **unfathomably** important that **you** understand what this result means!

It means that the functions $V(z)$ and $I(z)$, describing the current and voltage at **all** points z along a transmission line, can **always** be **completely** specified with just **four complex constants** (V_0^+, V_0^-, I_0^+, I_0^-)!!

We can **alternatively** write these solutions as:

$$V(z) = V^+(z) + V^-(z)$$

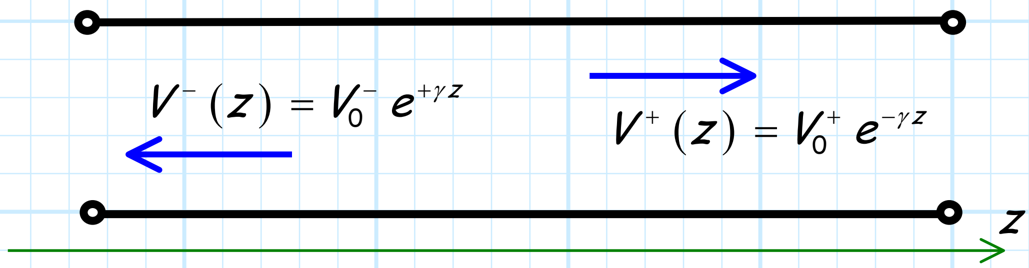
$$I(z) = I^+(z) + I^-(z)$$

where:

$$V^+(z) \doteq V_0^+\, e^{-\gamma z} \qquad V^-(z) \doteq V_0^-\, e^{+\gamma z}$$

$$I^+(z) \doteq I_0^+\, e^{-\gamma z} \qquad I^-(z) \doteq I_0^-\, e^{+\gamma z}$$

The two terms in each solution describe **two waves** propagating in the transmission line, **one** wave ($V^+(z)$ or $I^+(z)$) propagating in one direction (**+*z***) and the **other** wave ($V^-(z)$ or $I^-(z)$) propagating in the **opposite** direction (**-*z***).



Therefore, we call the differential equations introduced in this handout the **transmission line wave equations**.

Q: *So just what **are** the complex values V_0^+, V_0^-, I_0^+, I_0^- ?*

A: Consider the wave solutions at **one** specific point on the transmission line—the point $z = 0$. For example, we find that:

$$\begin{aligned} V^+(z=0) &= V_0^+ e^{-\gamma(z=0)} \\ &= V_0^+ e^{-(0)} \\ &= V_0^+ (1) \\ &= V_0^+ \end{aligned}$$

In other words, V_0^+ is simply the **complex** value of the wave function $V^+(z)$ at the point z =0 on the transmission line!

Likewise, we find:

$$V_0^- = V^-(z=0)$$

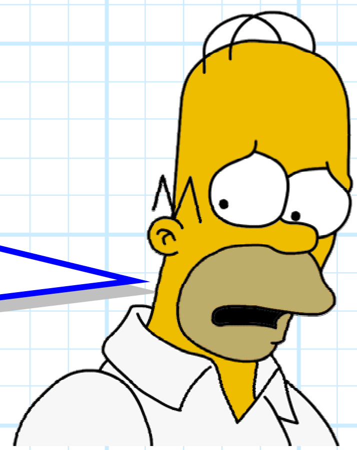
$$I_0^+ = I^+(z=0)$$

$$I_0^- = I^-(z=0)$$

Again, the four complex values V_0^+, I_0^+, V_0^-, I_0^- are **all** that is needed to determine the voltage and current at any and all points on the transmission line.

More specifically, **each** of these four complex constants completely specifies **one** of the four transmission line wave functions $V^+(z)$, $I^+(z)$, $V^-(z)$, $I^-(z)$.

Q: *But what **determines** these wave functions? How do we **find** the values of constants V_0^+, I_0^+, V_0^-, I_0^-?*



A: As you might expect, the voltage and current on a transmission line is determined by the devices **attached** to it on either end (e.g., active sources and/or passive loads)!

The precise values of V_0^+, I_0^+, V_0^-, I_0^- are therefore determined by satisfying the **boundary conditions** applied at **each end** of the transmission line—much more on this **later**!

The Characteristic Impedance of a Transmission Line

So, from the telegrapher's differential equations, we know that the complex current $I(z)$ and voltage $V(z)$ **must** have the form:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$$

Let's insert the expression for $V(z)$ into the first telegrapher's equation, and **see what happens**!

$$\frac{dV(z)}{dz} = -\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{+\gamma z} = -(R + j\omega L) I(z)$$

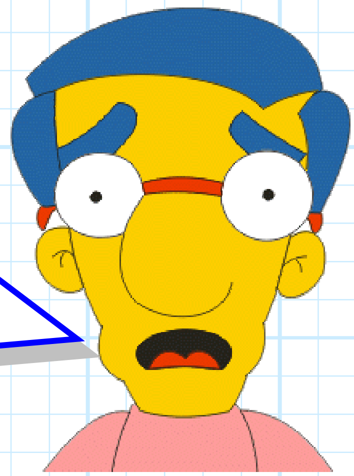
Therefore, rearranging, $I(z)$ must be:

$$I(z) = \frac{\gamma}{R + j\omega L}\left(V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}\right)$$

Q: ***But wait***! *I thought we already knew current I(z). Isn't it:*

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} \quad ??$$

How can ***both*** *expressions for I(z) be true??*



A: Easy ! Both expressions for current are **equal** to each other.

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} = \frac{\gamma}{R + j\omega L}\left(V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}\right)$$

For the above equation to be true for **all** z, I_0 and V_0 must be related as:

$$I_0^+ e^{-\gamma z} = \left(\frac{\gamma}{R + j\omega L}\right) V_0^+ e^{-\gamma z} \quad \text{and} \quad I_0^- e^{+\gamma z} = \left(\frac{-\gamma}{R + j\omega L}\right) V_0^- e^{+\gamma z}$$

Or—recalling that $V_0^+ e^{-\gamma z} = V^+(z)$ (etc.)—we can express this in terms of the **two propagating waves**:

$$I^+(z) = \left(\frac{\gamma}{R + j\omega L}\right) V^+(z) \quad \text{and} \quad I^-(z) = \left(\frac{-\gamma}{R + j\omega L}\right) V^-(z)$$

Now, we note that since:

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

We find that:

$$\frac{\gamma}{R+j\omega L}=\frac{\sqrt{(R+j\omega L)(G+j\omega C)}}{R+j\omega L}=\sqrt{\frac{G+j\omega C}{R+j\omega L}}$$

Thus, we come to the **startling** conclusion that:

$$\boxed{\frac{V^+(z)}{I^+(z)}=\sqrt{\frac{R+j\omega L}{G+j\omega C}} \quad \text{and} \quad \frac{-V^-(z)}{I^-(z)}=\sqrt{\frac{R+j\omega L}{G+j\omega C}}}$$

Q: *What's so startling about **this** conclusion?*

A: Note that although the magnitude and phase of each propagating wave is a **function** of transmission line **position** z (e.g., $V^+(z)$ and $I^+(z)$), the **ratio** of the voltage and current of **each wave** is independent of position—a **constant** with respect to position z!

Although $V_0^\pm$ and $I_0^\pm$ are determined by **boundary conditions** (i.e., what's connected to either end of the transmission line), the **ratio** $V_0^\pm / I_0^\pm$ is determined by the parameters of the transmission line **only** (R, L, G, C).

→ This ratio is an important **characteristic of a transmission line**, called its **Characteristic Impedance** Z_0.

$$Z_0 \doteq \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

We can therefore describe the current and voltage along a transmission line as:

$$V(z) = V_0^+ \, e^{-\gamma z} + V_0^- \, e^{+\gamma z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{+\gamma z}$$

or equivalently:

$$V(z) = Z_0 \, I_0^+ \, e^{-\gamma z} - Z_0 \, I_0^- \, e^{+\gamma z}$$

$$I(z) = I_0^+ \, e^{-\gamma z} + I_0^- \, e^{+\gamma z}$$

Note that instead of characterizing a transmission line with **real** parameters *R*, *G*, *L*, and *C*, we can (and typically do!) describe a transmission line using **complex** parameters Z_0 and γ.

The Complex Propagation Constant γ

Recall that the current and voltage along a transmission line have the form:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{+\gamma z}$$

where Z_0 and γ are **complex constants** that describe the properties of a transmission line. Since γ is complex, we can consider both its **real** and **imaginary** components.

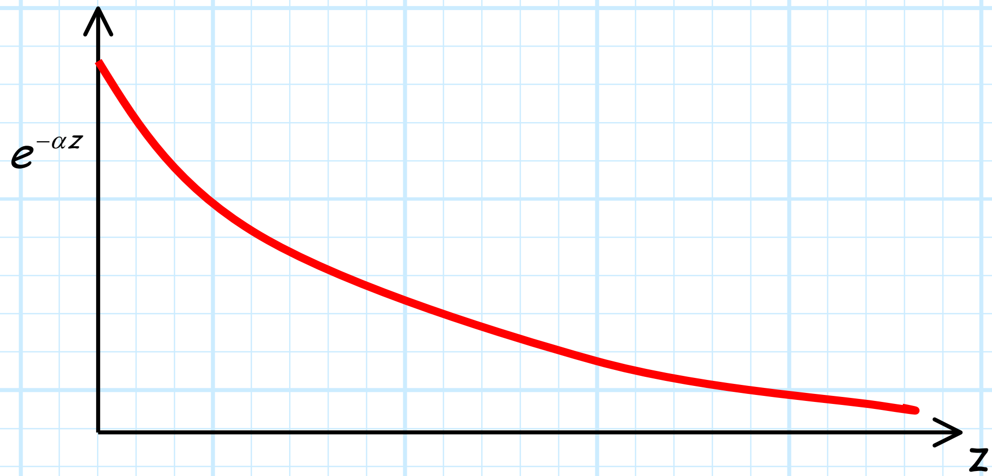
$$\begin{aligned}\gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} \\ &\doteq \alpha + j\beta\end{aligned}$$

where $\alpha = Re\{\gamma\}$ and $\beta = Im\{\gamma\}$. Therefore, we can write:

$$e^{-\gamma z} = e^{-(\alpha + j\beta) z} = e^{-\alpha z} e^{-j\beta z}$$

Since $\left|e^{-j\beta z}\right|$=1, then $e^{-\alpha z}$ alone determines the **magnitude** of $e^{-\gamma z}$.

I.E., $\left|e^{-\gamma z}\right| = e^{-\alpha z}$.



Therefore, α expresses the **attenuation** of the signal due to the loss in the transmission line.

Since $e^{-\alpha z}$ is a real function, it expresses the **magnitude** of $e^{-\gamma z}$ only. The **relative phase** $\phi(z)$ of $e^{-\gamma z}$ is therefore determined by $e^{-j\beta z} = e^{-j\phi(z)}$ only (recall $\left|e^{-j\beta z}\right| = 1$).

From Euler's equation:

$$e^{j\phi(z)} = e^{j\beta z} = \cos(\beta z) + j\sin(\beta z)$$

Therefore, βz represents the **relative phase** $\phi(z)$ of the oscillating signal, as a function of transmission line position z. Since phase $\phi(z)$ is expressed in radians, and z is distance (in meters), the value β must have units of :

$$\beta = \frac{\phi}{z} \qquad \frac{\text{radians}}{\text{meter}}$$

The **wavelength** λ of the signal is the distance $\Delta z_{2\pi}$ over which the relative phase changes by 2π radians. So:

$$2\pi = \phi(z + \Delta z_{2\pi}) - \phi(z) = \beta\,\Delta z_{2\pi} = \beta\lambda$$

or, rearranging:

$$\boxed{\beta = \frac{2\pi}{\lambda}}$$

Since the signal is oscillating in **time** at rate ω rad/sec, the **propagation velocity** of the wave is:

$$v_p = \frac{\omega}{\beta} = \frac{\omega\lambda}{2\pi} = f\lambda \qquad \left(\frac{m}{sec} = \frac{rad}{sec}\,\frac{m}{rad}\right)$$

where f is **frequency** in cycles/sec.

Recall we originally considered the transmission line current and voltage as a function of time and position (i.e., $v(z,t)$ and $i(z,t)$). We assumed the time function was sinusoidal, oscillating with frequency ω:

$$v(z,t) = \mathbf{Re}\left\{V(z)e^{j\omega t}\right\}$$

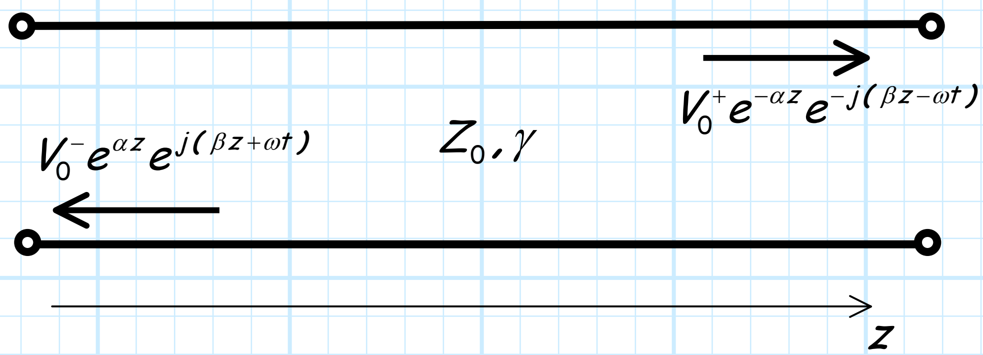
$$i(z,t) = \mathbf{Re}\left\{I(z)e^{j\omega t}\right\}$$

Now that we **know** *V(z)* and *I(z)*, we can write the original functions as:

$$v(z,t) = Re\left\{V_0^+ e^{-\alpha z} e^{-j(\beta z-\omega t)} + V_0^- e^{\alpha z} e^{j(\beta z+\omega t)}\right\}$$

$$i(z,t) = Re\left\{\frac{V_0^+}{Z_0} e^{-\alpha z} e^{-j(\beta z-\omega t)} - \frac{V_0^+}{Z_0} e^{\alpha z} e^{j(\beta z+\omega t)}\right\}$$

The first term in each equation describes a wave **propagating** in the ***+z*** direction, while the second describes a wave propagating in the **opposite** (-z) direction.



Each wave has **wavelength**:

$$\lambda = \frac{2\pi}{\beta}$$

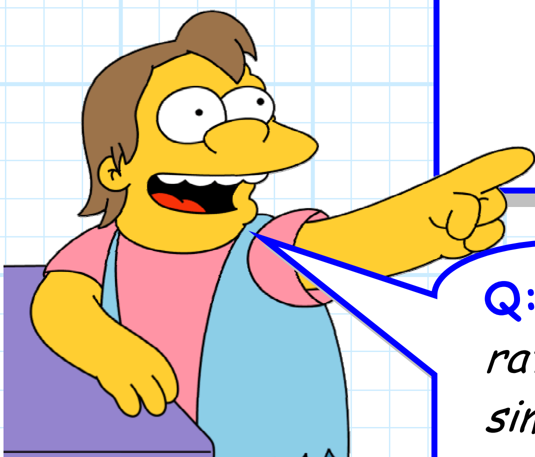
And **velocity**:

$$v_p = \frac{\omega}{\beta}$$

Line Impedance

Now let's define **line impedance** $Z(z)$, which is simply the ratio of the complex line voltage and complex line current:

$$Z(z) = \frac{V(z)}{I(z)}$$



Q: *Hey! I know what this is! The ratio of the voltage to current is simply the **characteristic impedance** Z_0, right ???*

A: NO! The line impedance $Z(z)$ is (generally speaking) **NOT** the transmission line **characteristic** impedance Z_0 !!!

→ It is **unfathomably important** that you understand this!!!!

To see why, recall that:

$$V(z) = V^{+}(z) + V^{-}(z)$$

And that:

$$I(z) = \frac{V^+(z) - V^-(z)}{Z_0}$$

Therefore:

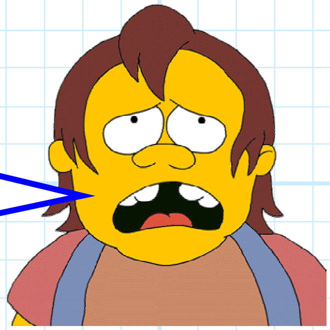
$$\boxed{Z(z) = \frac{V(z)}{I(z)} = Z_0\left(\frac{V^+(z) + V^-(z)}{V^+(z) - V^-(z)}\right) \neq Z_0}$$

Or, more specifically, we can write:

$$Z(z) = Z_0\left(\frac{V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}}{V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}}\right)$$

Q: *I'm confused! Isn't:*

$$V^+(z)/I^+(z) = Z_0\ ???$$



A: Yes! That **is** true! The ratio of the voltage to current for **each** of the two propagating waves is $\pm Z_0$. However, the ratio of the **sum** of the two voltages to the **sum** of the two currents is **not** equal to Z_0 (generally speaking)!

This is actually confirmed by the equation above. Say that $V^-(z) = 0$, so that only **one** wave ($V^+(z)$) is propagating on the line.

In this case, the ratio of the **total** voltage to the total current is simply the ratio of the voltage and current of the **one** remaining wave—the **characteristic impedance** Z_0!

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \left(\frac{V^+(z)}{V^+(z)} \right) = \frac{V^+(z)}{I^+(z)} = Z_0 \quad \text{(when } V^+(z)\text{)}$$

Q: *So, it appears to me that characteristic impedance Z_0 is a **transmission line parameter**, depending **only** on the transmission line values R, G, L, and C.*

*Whereas **line impedance** is $Z(z)$ depends the magnitude and phase of the two propagating waves $V^+(z)$ and $V^-(z)$--values that depend **not only** on the transmission line, but also on the two things **attached** to either **end** of the transmission line!*

Right !?

A: **Exactly!** Moreover, note that characteristic impedance Z_0 is simply a **number**, whereas line impedance $Z(z)$ is a **function** of position (z) on the transmission line.

The Reflection Coefficient

So, we know that the transmission line **voltage** $V(z)$ and the transmission line **current** $I(z)$ can be related by the **line impedance** $Z(z)$:

$$V(z) = Z(z)\, I(z)$$

or equivalently:

$$I(z) = \frac{V(z)}{Z(z)}$$

Expressing the "activity" on a transmission line in terms of voltage, current and impedance is of course **perfectly** valid. However, let us look **closer** at the expression for each of these quantities:

$$V(z) = V^+(z) + V^-(z)$$

$$I(z) = \frac{V^+(z) - V^-(z)}{Z_0}$$

$$Z(z) = Z_0 \left(\frac{V^+(z) + V^-(z)}{V^+(z) - V^-(z)} \right)$$

It is evident that we can **alternatively** express all "activity" on the transmission line in terms of the two transmission line **waves** $V^+(z)$ and $V^-(z)$.

In other words, we can describe transmission line activity in terms of:

$$V^+(z) \text{ and } V^-(z)$$

instead of:

$$V(z) \text{ and } I(z)$$

Q: *But $V(z)$ and $I(z)$ are related by line impedance $Z(z)$:*

$$Z(z) = \frac{V(z)}{I(z)}$$

How are $V^+(z)$ and $V^-(z)$ related?

A: Similar to line impedance, we can define a new parameter—the **reflection coefficient** $\Gamma(z)$--as the **ratio** of the two quantities:

$$\boxed{\Gamma(z) \doteq \frac{V^-(z)}{V^+(z)}}$$

More specifically, we can express $\Gamma(z)$ as:

$$\Gamma(z) = \frac{V_0^- \, e^{+\gamma z}}{V_0^+ \, e^{-\gamma z}} = \frac{V_0^-}{V_0^+} \, e^{+2\gamma z}$$

Note then, the value of the reflection coefficient at z=0 is:

$$\Gamma(z=0) = \frac{V_0^-}{V_0^+} \, e^{+2\gamma(0)}$$
$$= \frac{V_0^-}{V_0^+}$$

We define this value as Γ_0, where:

$$\boxed{\Gamma_0 \doteq \Gamma(z=0) = \frac{V_0^-}{V_0^+}}$$

Note then that we can alternatively write $\Gamma(z)$ as:

$$\boxed{\Gamma(z) = \Gamma_0 \, e^{+2\gamma z}}$$

Thus, we now know:

$$V^{-}(z) = \Gamma(z)\, V^{+}(z)$$

and therefore we can express line current and voltage as:

$$\boxed{\begin{aligned} V(z) &= V^{+}(z)\big(1+\Gamma(z)\big) \\ I(z) &= \frac{V^{+}(z)}{Z_0}\big(1-\Gamma(z)\big) \end{aligned}}$$

Or, more explicitly, since $V_0^{-} = \Gamma_0\, V_0^{+}$:

$$\boxed{\begin{aligned} V(z) &= V_0^{+}\left(e^{-\gamma z} + \Gamma_0 e^{+\gamma z}\right) \\ I(z) &= \frac{V_0^{+}}{Z_0}\left(e^{-\gamma z} - \Gamma_0 e^{+\gamma z}\right) \end{aligned}}$$

More importantly, we find that **line impedance** $Z(z) = V(z)/I(z)$ is:

$$\boxed{Z(z) = Z_0\left(\frac{1+\Gamma(z)}{1-\Gamma(z)}\right)}$$

Look what happened—the line impedance can be **completely** and explicitly expressed in terms of **reflection coefficient** $\Gamma(z)$!

Or, rearranging, we find that the reflection coefficient $\Gamma(z)$ can **likewise** be written in terms of line impedance:

$$\boxed{\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0}}$$

Thus, the values $\Gamma(z)$ and $Z(z)$ are **equivalent** parameters—if we know **one**, then we can determine the **other**!

Likewise, the relationships:

$$V(z) = Z(z)\, I(z)$$

and:

$$V^-(z) = \Gamma(z)\, V^+(z)$$

are **equivalent** relationships—we can use **either** when describing an transmission line.

The Lossless Transmission Line

Say a transmission line is **lossless** (i.e., $R=G=0$); the transmission line equations are then **significantly** simplified!

Characteristic Impedance

$$\begin{aligned} Z_0 &= \sqrt{\frac{R + j\omega L}{G + j\omega C}} \\ &= \sqrt{\frac{j\omega L}{j\omega C}} \\ &= \sqrt{\frac{L}{C}} \end{aligned}$$

Note the characteristic impedance of a **lossless** transmission line is purely **real** (i.e., Im{Z_0} =0)!

Propagation Constant

$$\begin{aligned} \gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} \\ &= \sqrt{(j\omega L)(j\omega C)} \\ &= \sqrt{-\omega^2 LC} \\ &= j\omega\sqrt{LC} \end{aligned}$$

The wave propagation constant is purely **imaginary**!

In other words, for a **lossless** transmission line:

$$\alpha = 0 \quad \text{and} \quad \beta = \omega\sqrt{LC}$$

Voltage and Current

The **complex functions** describing the magnitude and phase of the voltage/current at every location *z* along a transmission line are for a **lossless** line are:

$$V(z) = V_0^+ \, e^{-j\beta z} + V_0^- \, e^{+j\beta z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z}$$

Line Impedance

The **complex function** describing the impedance at every point along a **lossless** transmission line is:

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \, \frac{V_0^+ \, e^{-j\beta z} + V_0^- \, e^{+j\beta z}}{V_0^+ e^{-j\beta z} - V_0^- \, e^{+j\beta z}}$$

Reflection Coefficient

The **complex function** describing the reflection at every point along a **lossless** transmission line is:

$$\Gamma(z) = \frac{V_0^- \, e^{+j\beta z}}{V_0^+ \, e^{-j\beta z}} = \frac{V_0^-}{V_0^+} \, e^{+j2\beta z}$$

Wavelength and Phase Velocity

We can now **explicitly** write the wavelength and propagation velocity of the two transmission line waves in terms of transmission line parameters L and C:

$$\lambda = \frac{2\pi}{\beta} = \frac{1}{f\sqrt{LC}}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

Q: *Oh **please**, continue wasting my valuable time. We both know that a **perfectly** lossless transmission line is a physical **impossibility**.*

A: True! However, a **low-loss** line **is** possible—in fact, it is **typical**! If $R \ll \omega L$ and $G \ll \omega C$, we find that the lossless transmission line equations are excellent **approximations**!

Unless otherwise indicated, we will use the lossless equations to **approximate** the behavior of a **low-loss** transmission line.

The lone **exception** is when determining the attenuation of a **long** transmission line. For that case we will use the approximation:

$$\alpha \approx \frac{1}{2}\left(\frac{R}{Z_0} + GZ_0\right)$$

where $Z_0 = \sqrt{L/C}$.

A summary of lossless transmission line equations

$$Z_0 = \sqrt{\frac{L}{C}} \qquad \gamma = j\omega\sqrt{LC}$$

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z} \qquad I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z}$$

$$Z(z) = Z_0 \frac{V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}}{V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}}$$

$$V^+(z) = V_0^+ e^{-j\beta z} \qquad V^-(z) = V_0^- e^{+j\beta z}$$

$$\Gamma(z) = \frac{V_0^-}{V_0^+} e^{+j2\beta z}$$

$$\beta = \omega\sqrt{LC} \qquad \lambda = \frac{1}{f\sqrt{LC}} \qquad v_p = \frac{1}{\sqrt{LC}}$$