2.1 - The Lumped Element Circuit Model for Transmission Lines

Reading Assignment: pp. 1-5, 49-52

Q: So just what is a transmission line?

A:

→

Q: Oh, so it's simply a conducting wire, right?

A:

HO: The Telegraphers Equations

Q: So, what complex functions \( I(z) \) and \( V(z) \) do satisfy both telegrapher equations?

A:

HO: The Transmission Line Wave Equations

Q: Are the solutions for \( I(z) \) and \( V(z) \) completely independent, or are they related in any way?
A:

**HO: The Transmission Line Characteristic Impedance**

**Q:** So what is the significance of the complex constant $\gamma$? What does it tell us?

A:

**HO: The Complex Propagation Constant**

**Q:** Is characteristic impedance $Z_0$ the same as the concept of impedance I learned about in circuits class?

A:

**HO: Line Impedance**

**Q:** These wave functions $V^+(z)$ and $V^-(z)$ seem to be important. How are they related?

A:

**HO: The Reflection Coefficient**
Q: Now, you said earlier that characteristic impedance $Z_0$ is a complex value. But I recall engineers referring to a transmission line as simply a “50 Ohm line”, or a “300 Ohm line”. But these are real values; are they not referring to characteristic impedance $Z_0$??

A:

HO: The Lossless Transmission Line
The Telegrapher Equations

Consider a section of “wire”:

\[ i(z,t) \xrightarrow{+} i(z+\Delta z,t) \]

\[ v(z,t) \xrightarrow{-} v(z+\Delta z,t) \]

Q: Huh ?! Current \( i \) and voltage \( v \) are a function of position \( z \) ?? Shouldn’t \( i(z,t) = i(z+\Delta z,t) \) and \( v(z,t) = v(z+\Delta z,t) \) ?

A: NO ! Because a wire is never a perfect conductor.

A “wire” will have:

1) Inductance
2) Resistance
3) Capacitance
4) Conductance
i.e.,

Where:

- $R = \text{resistance/unit length}$
- $L = \text{inductance/unit length}$
- $C = \text{capacitance/unit length}$
- $G = \text{conductance/unit length}$

∴ resistance of wire length $\Delta z$ is $R \Delta z$.

Using KVL, we find:

$$v(z + \Delta z, t) - v(z, t) = -R \Delta z \cdot i(z, t) - L \Delta z \cdot \frac{\partial i(z, t)}{\partial t}$$

and from KCL:

$$i(z + \Delta z, t) - i(z, t) = -G \Delta z \cdot v(z, t) - C \Delta z \cdot \frac{\partial v(z, t)}{\partial t}$$
Dividing the first equation by $\Delta z$, and then taking the limit as $\Delta z \to 0$:

$$
\lim_{\Delta z \to 0} \frac{v(z + \Delta z, t) - v(z, t)}{\Delta z} = -R i(z, t) - L \frac{\partial i(z, t)}{\partial t}
$$

which, by definition of the derivative, becomes:

$$
\frac{\partial v(z, t)}{\partial z} = -R i(z, t) - L \frac{\partial i(z, t)}{\partial t}
$$

Similarly, the KCL equation becomes:

$$
\frac{\partial i(z, t)}{\partial z} = -G v(z, t) - C \frac{\partial v(z, t)}{\partial t}
$$

If $v(z, t)$ and $i(z, t)$ have the form:

$$
v(z, t) = \text{Re}\{V(z)e^{j\omega t}\} \quad \text{and} \quad i(z, t) = \text{Re}\{I(z)e^{j\omega t}\}
$$

then these equations become:

$$
\frac{\partial V(z)}{\partial z} = -(R + j\omega L) I(z)
$$

$$
\frac{\partial I(z)}{\partial z} = -(G + j\omega C) V(z)
$$

These equations are known as the telegrapher’s equations!
* The functions $I(z)$ and $V(z)$ are complex, where the magnitude and phase of the complex functions describe the magnitude and phase of the sinusoidal time function $e^{j\omega t}$.

* Thus, $I(z)$ and $V(z)$ describe the current and voltage along the transmission line, as a function as position $z$.

* Remember, not just any function $I(z)$ and $V(z)$ can exist on a transmission line, but rather only those functions that satisfy the telegraphers equations.

Our task, therefore, is to solve the telegrapher equations and find all solutions $I(z)$ and $V(z)$!
The Transmission Line Wave Equation

**Q:** So, what functions \( I(z) \) and \( V(z) \) do satisfy both telegrapher's equations??

**A:** To make this easier, we will combine the telegrapher equations to form one differential equation for \( V(z) \) and another for \( I(z) \).

First, take the derivative with respect to \( z \) of the first telegrapher equation:

\[
\frac{\partial}{\partial z} \left( \frac{\partial V(z)}{\partial z} \right) = -(R + j\omega L) I(z)
\]

\[
= \frac{\partial^2 V(z)}{\partial z^2} = -(R + j\omega L) \frac{\partial I(z)}{\partial z}
\]

Note that the second telegrapher equation expresses the derivative of \( I(z) \) in terms of \( V(z) \):

\[
\frac{\partial I(z)}{\partial z} = -(G + j\omega C) V(z)
\]

Combining these two equations, we get an equation involving \( V(z) \) only:
\[
\frac{\partial^2 V(z)}{\partial z^2} = (R + j\omega L)(G + j\omega C) V(z) \\
= \gamma^2 V(z)
\]

where it is apparent that:

\[
\gamma^2 = (R + j\omega L)(G + j\omega C)
\]

In a similar manner (i.e., begin by taking the derivative of the second telegrapher equation), we can derive the differential equation:

\[
\frac{\partial^2 I(z)}{\partial z^2} = \gamma^2 I(z)
\]

We have decoupled the telegrapher's equations, such that we now have two equations involving one function only:

\[
\frac{\partial^2 V(z)}{\partial z^2} = \gamma^2 V(z) \\
\frac{\partial^2 I(z)}{\partial z^2} = \gamma^2 I(z)
\]

Note only special functions satisfy these equations: if we take the double derivative of the function, the result is the original function (to within a constant)!
A: Such functions do exist!

For example, the functions $V(z) = e^{-rz}$ and $V(z) = e^{+r z}$ each satisfy this transmission line wave equation (insert these into the differential equation and see for yourself!).

Likewise, since the transmission line wave equation is a linear differential equation, a weighted superposition of the two solutions is also a solution (again, insert this solution to and see for yourself!):

$$V(z) = V_0^+ e^{-rz} + V_0^- e^{+rz}$$

In fact, it turns out that any and all possible solutions to the differential equations can be expressed in this simple form!
Therefore, the general solution to these wave equations (and thus the telegrapher equations) are:

\[
V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}
\]
\[
I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}
\]

where \(V_0^+, V_0^-, I_0^+, I_0^-\), and \(\gamma\) are complex constants.

\(\Rightarrow\) It is unfathomably important that you understand what this result means!

It means that the functions \(V(z)\) and \(I(z)\), describing the current and voltage at all points \(z\) along a transmission line, can always be completely specified with just four complex constants \((V_0^+, V_0^-, I_0^+, I_0^-)\)!!

We can alternatively write these solutions as:

\[
V(z) = V^+(z) + V^-(z)
\]
\[
I(z) = I^+(z) + I^-(z)
\]

where:

\[
V^+(z) \doteq V_0^+ e^{-\gamma z} \quad \quad V^-(z) \doteq V_0^- e^{+\gamma z}
\]
\[
I^+(z) \doteq I_0^+ e^{-\gamma z} \quad \quad I^-(z) \doteq I_0^- e^{+\gamma z}
\]
The two terms in each solution describe two waves propagating in the transmission line, one wave \((V^+(z) \text{ or } I^+(z))\) propagating in one direction \((+z)\) and the other wave \((V^-(z) \text{ or } I^-(z))\) propagating in the opposite direction \((-z)\).

Therefore, we call the differential equations introduced in this handout the transmission line wave equations.

**Q:** So just what are the complex values \(V_0^+, V_0^-, I_0^+, I_0^-\) ?

**A:** Consider the wave solutions at one specific point on the transmission line—the point \(z = 0\). For example, we find that:

\[
V^+(z = 0) = V_0^+ e^{-\gamma z}
\]

\[
= V_0^+ e^{-(0)}
\]

\[
= V_0^+ (1)
\]

\[
= V_0^+
\]

In other words, \(V_0^+\) is simply the complex value of the wave function \(V^+(z)\) at the point \(z = 0\) on the transmission line!
Likewise, we find:

\[ V_0^- = V^-(z = 0) \]

\[ I_0^+ = I^+(z = 0) \]

\[ I_0^- = I^-(z = 0) \]

Again, the four complex values \( V_0^+, I_0^+, V_0^-, I_0^- \) are all that is needed to determine the voltage and current at any and all points on the transmission line.

More specifically, each of these four complex constants completely specifies one of the four transmission line wave functions \( V^+(z), I^+(z), V^-(z), I^-(z) \).

Q: But what determines these wave functions? How do we find the values of constants \( V_0^+, I_0^+, V_0^-, I_0^- \)?

A: As you might expect, the voltage and current on a transmission line is determined by the devices attached to it on either end (e.g., active sources and/or passive loads)!

The precise values of \( V_0^+, I_0^+, V_0^-, I_0^- \) are therefore determined by satisfying the boundary conditions applied at each end of the transmission line—much more on this later!
The Characteristic Impedance of a Transmission Line

So, from the telegrapher's differential equations, we know that the complex current $I(z)$ and voltage $V(z)$ must have the form:

$$V(z) = V_0^+ e^{-r_1 z} + V_0^- e^{r_2 z}$$

$$I(z) = I_0^+ e^{-r_1 z} + I_0^- e^{r_2 z}$$

Let's insert the expression for $V(z)$ into the first telegrapher's equation, and see what happens!

$$\frac{dV(z)}{dz} = -\gamma V_0^+ e^{-r_1 z} + \gamma V_0^- e^{r_2 z} = -(R + j\omega L)I(z)$$

Therefore, rearranging, $I(z)$ must be:

$$I(z) = \frac{\gamma}{R + j\omega L} (V_0^+ e^{-r_1 z} - V_0^- e^{r_2 z})$$
**Q:** But wait! I thought we already knew current \( I(z) \). Isn’t it:

\[
I(z) = I_0^+ e^{-r_0 z} + I_0^- e^{r_0 z}
\]

How can both expressions for \( I(z) \) be true??

**A:** Easy! Both expressions for current are equal to each other.

\[
I(z) = I_0^+ e^{-r_0 z} + I_0^- e^{r_0 z} = \frac{\gamma}{R + j\omega L} (V_0^+ e^{-r_0 z} - V_0^- e^{r_0 z})
\]

For the above equation to be true for all \( z \), \( I_0 \) and \( V_0 \) must be related as:

\[
I_0^+ e^{-r_0 z} = \left( \frac{\gamma}{R + j\omega L} \right) V_0^+ e^{-r_0 z} \quad \text{and} \quad I_0^- e^{r_0 z} = \left( \frac{-\gamma}{R + j\omega L} \right) V_0^- e^{r_0 z}
\]

Or—recalling that \( V_0^+ e^{-r_0 z} = V^+(z) \) (etc.)—we can express this in terms of the two propagating waves:

\[
I^+(z) = \left( \frac{\gamma}{R + j\omega L} \right) V^+(z) \quad \text{and} \quad I^-(z) = \left( \frac{-\gamma}{R + j\omega L} \right) V^-(z)
\]

Now, we note that since:

\[
\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}
\]
We find that:

\[
\gamma = \frac{\sqrt{(R + j\omega L)(G + j\omega C)}}{R + j\omega L} = \frac{G + j\omega C}{\sqrt{R + j\omega L}}
\]

Thus, we come to the **startling** conclusion that:

\[
\frac{V^+(z)}{I^+(z)} = \frac{R + j\omega L}{\sqrt{G + j\omega C}} \quad \text{and} \quad \frac{-V^-(z)}{I^-(z)} = \frac{R + j\omega L}{\sqrt{G + j\omega C}}
\]

**Q:** What’s so startling about this conclusion?

**A:** Note that although the magnitude and phase of each propagating wave is a **function** of transmission line position \(z\) (e.g., \(V^+(z)\) and \(I^+(z)\)), the **ratio** of the voltage and current of each wave is independent of position—a **constant** with respect to position \(z\)!

Although \(V_0^\pm\) and \(I_0^\pm\) are determined by **boundary conditions** (i.e., what’s connected to either end of the transmission line), the **ratio** \(V_0^\pm/I_0^\pm\) is determined by the parameters of the transmission line **only** \((R, L, G, C)\).

\(\Rightarrow\) This ratio is an important **characteristic** of a transmission line, called its **Characteristic Impedance** \(Z_0\).
We can therefore describe the current and voltage along a transmission line as:

\[ V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \]

\[ I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{+\gamma z} \]

or equivalently:

\[ V(z) = Z_0 I_0^+ e^{-\gamma z} - Z_0 I_0^- e^{+\gamma z} \]

\[ I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} \]

Note that instead of characterizing a transmission line with real parameters \( R, G, L, \) and \( C \), we can (and typically do!) describe a transmission line using complex parameters \( Z_0 \) and \( \gamma \).
The Complex Propagation Constant $\gamma$

Recall that the current and voltage along a transmission line have the form:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{+\gamma z}$$

where $Z_0$ and $\gamma$ are complex constants that describe the properties of a transmission line. Since $\gamma$ is complex, we can consider both its real and imaginary components.

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta$$

where $\alpha = Re\{\gamma\}$ and $\beta = Im\{\gamma\}$. Therefore, we can write:

$$e^{-\gamma z} = e^{-(\alpha + j\beta)z} = e^{-\alpha z}e^{-j\beta z}$$

Since $|e^{j\beta z}|=1$, then $e^{-\alpha z}$ alone determines the magnitude of $e^{-\gamma z}$. 

I.E., \( |e^{-\gamma z}| = e^{-\alpha z} \).

Therefore, \( \alpha \) expresses the **attenuation** of the signal due to the loss in the transmission line.

Since \( e^{-\alpha z} \) is a real function, it expresses the **magnitude** of \( e^{-\gamma z} \) only. The **relative phase** \( \phi(z) \) of \( e^{-\gamma z} \) is therefore determined by \( e^{-j\beta z} = e^{-j\phi(z)} \) only (recall \( |e^{-j\beta z}| = 1 \)).

From Euler's equation:

\[
e^{j\phi(z)} = e^{j\beta z} = \cos(\beta z) + j \sin(\beta z)
\]

Therefore, \( \beta z \) represents the **relative phase** \( \phi(z) \) of the oscillating signal, as a function of transmission line position \( z \). Since phase \( \phi(z) \) is expressed in radians, and \( z \) is distance (in meters), the value \( \beta \) must have units of:

\[
\beta = \frac{\phi}{z} \quad \text{radians/meter}
\]
The wavelength $\lambda$ of the signal is the distance $\Delta z_{2\pi}$ over which the relative phase changes by $2\pi$ radians. So:

$$2\pi = \phi(z + \Delta z_{2\pi}) - \phi(z) = \beta \Delta z_{2\pi} = \beta \lambda$$

or, rearranging:

$$\beta = \frac{2\pi}{\lambda}$$

Since the signal is oscillating in time at rate $\omega$ rad/sec, the propagation velocity of the wave is:

$$v_p = \frac{\omega}{\beta} = \frac{\omega \lambda}{2\pi} = f \lambda \left( \frac{m}{\text{sec}} = \frac{\text{rad}}{\text{sec} \text{ rad}} \right)$$

where $f$ is frequency in cycles/sec.

Recall we originally considered the transmission line current and voltage as a function of time and position (i.e., $v(z,t)$ and $i(z,t)$). We assumed the time function was sinusoidal, oscillating with frequency $\omega$:

$$v(z,t) = \text{Re}\{V(z)e^{j\omega t}\}$$

$$i(z,t) = \text{Re}\{I(z)e^{j\omega t}\}$$
Now that we know \( V(z) \) and \( I(z) \), we can write the original functions as:

\[
\begin{align*}
\nu(z, t) &= \text{Re} \left\{ V_0^+ e^{-\alpha z} e^{-j(\beta z - \omega t)} + V_0^- e^{\alpha z} e^{j(\beta z + \omega t)} \right\} \\
i(z, t) &= \text{Re} \left\{ \frac{V_0^+}{Z_0} e^{-\alpha z} e^{-j(\beta z - \omega t)} - \frac{V_0^+}{Z_0} e^{\alpha z} e^{j(\beta z + \omega t)} \right\}
\end{align*}
\]

The first term in each equation describes a wave propagating in the \(+z\) direction, while the second describes a wave propagating in the opposite \((-z)\) direction.

Each wave has **wavelength**:

\[
\lambda = \frac{2\pi}{\beta}
\]

And **velocity**:

\[
v_p = \frac{\omega}{\beta}
\]
Line Impedance

Now let’s define line impedance \( Z(z) \), which is simply the ratio of the complex line voltage and complex line current:

\[
Z(z) = \frac{V(z)}{I(z)}
\]

Q: Hey! I know what this is! The ratio of the voltage to current is simply the characteristic impedance \( Z_0 \), right???

A: NO! The line impedance \( Z(z) \) is (generally speaking) NOT the transmission line characteristic impedance \( Z_0 \)!!!

→ It is unfathomably important that you understand this!!!!

To see why, recall that:

\[
V(z) = V^+(z) + V^-(z)
\]
And that:

\[ I(z) = \frac{V^+(z) - V^-(z)}{Z_0} \]

Therefore:

\[ Z(z) = \frac{V(z)}{I(z)} = Z_0 \left( \frac{V^+(z) + V^-(z)}{V^+(z) - V^-(z)} \right) \neq Z_0 \]

Or, more specifically, we can write:

\[ Z(z) = Z_0 \left( \frac{V^+_0 e^{-rz} + V^-_0 e^{+rz}}{V^+_0 e^{-rz} - V^-_0 e^{+rz}} \right) \]

Q: I'm confused! Isn’t:

\[ \frac{V^+(z)}{I^+(z)} = Z_0 \]

A: Yes! That is true! The ratio of the voltage to current for each of the two propagating waves is \( \pm Z_0 \). However, the ratio of the sum of the two voltages to the sum of the two currents is not equal to \( Z_0 \) (generally speaking)!

This is actually confirmed by the equation above. Say that \( V^-(z) = 0 \), so that only one wave \( (V^+(z)) \) is propagating on the line.
In this case, the ratio of the total voltage to the total current is simply the ratio of the voltage and current of the one remaining wave—the characteristic impedance $Z_0$!

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \left( \frac{V^+(z)}{V^+(z)} \right) = \frac{V^+(z)}{I^+(z)} = Z_0 \quad \text{(when } V^+(z))$$

**Q:** So, it appears to me that characteristic impedance $Z_0$ is a transmission line parameter, depending only on the transmission line values $R$, $G$, $L$, and $C$.

Whereas line impedance is $Z(z)$ depends the magnitude and phase of the two propagating waves $V^+(z)$ and $V^-(z)$—values that depend not only on the transmission line, but also on the two things attached to either end of the transmission line!

Right !?

**A:** Exactly! Moreover, note that characteristic impedance $Z_0$ is simply a number, whereas line impedance $Z(z)$ is a function of position ($z$) on the transmission line.
The Reflection Coefficient

So, we know that the transmission line voltage $V(z)$ and the transmission line current $I(z)$ can be related by the line impedance $Z(z)$:

$$V(z) = Z(z) I(z)$$

or equivalently:

$$I(z) = \frac{V(z)}{Z(z)}$$

Expressing the “activity” on a transmission line in terms of voltage, current and impedance is of course perfectly valid. However, let us look closer at the expression for each of these quantities:

$$V(z) = V^+(z) + V^-(z)$$

$$I(z) = \frac{V^+(z) - V^-(z)}{Z_0}$$

$$Z(z) = Z_0 \left( \frac{V^+(z) + V^-(z)}{V^+(z) - V^-(z)} \right)$$
It is evident that we can **alternatively** express all “activity” on the transmission line in terms of the two transmission line waves $V^+(z)$ and $V^-(z)$.

In other words, we can describe transmission line activity in terms of:

$$V^+(z) \text{ and } V^-(z)$$

instead of:

$$V(z) \text{ and } I(z)$$

**Q:** But $V(z)$ and $I(z)$ are related by line impedance $Z(z)$:

$$Z(z) = \frac{V(z)}{I(z)}$$

*How are $V^+(z)$ and $V^-(z)$ related?*

**A:** Similar to line impedance, we can define a new parameter—the **reflection coefficient** $\Gamma(z)$—as the ratio of the two quantities:

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)}$$
More specifically, we can express $\Gamma(z)$ as:

$$\Gamma(z) = \frac{V_0^-}{V_0^+} e^{\gamma z} = \frac{V_0^-}{V_0^+} e^{2\gamma z}$$

Note then, the value of the reflection coefficient at $z = 0$ is:

$$\Gamma(z = 0) = \frac{V_0^-}{V_0^+} e^{2\gamma(0)}$$

$$= \frac{V_0^-}{V_0^+}$$

We define this value as $\Gamma_0$, where:

$$\Gamma_0 \doteq \Gamma(z = 0) = \frac{V_0^-}{V_0^+}$$

Note then that we can alternatively write $\Gamma(z)$ as:

$$\Gamma(z) = \Gamma_0 e^{2\gamma z}$$
Thus, we now know:

\[ V^-(z) = \Gamma(z) V^+(z) \]

and therefore we can express line current and voltage as:

\[
\begin{align*}
V(z) &= V^+(z) (1 + \Gamma(z)) \\
I(z) &= \frac{V^+(z)}{Z_0} (1 - \Gamma(z))
\end{align*}
\]

Or, more explicitly, since \( V_0^- = \Gamma_0 V_0^+ \):

\[
\begin{align*}
V(z) &= V_0^+ (e^{-\gamma z} + \Gamma_0 e^{+\gamma z}) \\
I(z) &= \frac{V_0^+}{Z_0} (e^{-\gamma z} - \Gamma_0 e^{+\gamma z})
\end{align*}
\]

More importantly, we find that line impedance \( Z(z) = V(z)/I(z) \) is:

\[
Z(z) = Z_0 \left( \frac{1 + \Gamma(z)}{1 - \Gamma(z)} \right)
\]
Look what happened—the line impedance can be completely and explicitly expressed in terms of reflection coefficient $\Gamma(z)$!

Or, rearranging, we find that the reflection coefficient $\Gamma(z)$ can likewise be written in terms of line impedance:

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0}$$

Thus, the values $\Gamma(z)$ and $Z(z)$ are equivalent parameters—if we know one, then we can determine the other!

Likewise, the relationships:

$$V(z) = Z(z) I(z)$$

and:

$$V^-(z) = \Gamma(z) V^+(z)$$

are equivalent relationships—we can use either when describing an transmission line.

Based on circuits experience, you might be tempted to always use the first relationship. However, we will find that it is also very useful (as well as simple) to describe activity on a transmission line in terms of the second relationship—in terms of the two propagating transmission line waves!
The Lossless Transmission Line

Say a transmission line is lossless (i.e., $R=G=0$); the transmission line equations are then significantly simplified!

**Characteristic Impedance**

$$Z_0 = \frac{R + j\omega L}{\sqrt{G + j\omega C}}$$

$$= \sqrt{\frac{j\omega L}{j\omega C}}$$

$$= \sqrt{\frac{L}{C}}$$

Note the characteristic impedance of a lossless transmission line is purely real (i.e., $\text{Im}(Z_0) = 0$)

**Propagation Constant**

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$= \sqrt{(j\omega L)(j\omega C)}$$

$$= \sqrt{-\omega^2 LC}$$

$$= j\omega \sqrt{LC}$$

The wave propagation constant is purely imaginary!
In other words, for a lossless transmission line:

\[ \alpha = 0 \quad \text{and} \quad \beta = \omega \sqrt{LC} \]

**Voltage and Current**

The complex functions describing the magnitude and phase of the voltage/current at every location \( z \) along a transmission line are for a lossless line are:

\[ V(z) = V_0^+ e^{-j \beta z} + V_0^- e^{+j \beta z} \]

\[ I(z) = \frac{V_0^+}{Z_0} e^{-j \beta z} - \frac{V_0^-}{Z_0} e^{+j \beta z} \]

**Line Impedance**

The complex function describing the impedance at every point along a lossless transmission line is:

\[ Z(z) = \frac{V(z)}{I(z)} = \frac{V_0^+ e^{-j \beta z} + V_0^- e^{+j \beta z}}{V_0^+ e^{-j \beta z} - V_0^- e^{+j \beta z}} \]

**Reflection Coefficient**

The complex function describing the reflection at every point along a lossless transmission line is:

\[ \Gamma(z) = \frac{V_0^- e^{+j \beta z}}{V_0^+ e^{-j \beta z}} = \frac{V_0^-}{V_0^+} e^{+j 2 \beta z} \]
**Wavelength and Phase Velocity**

We can now explicitly write the wavelength and propagation velocity of the two transmission line waves in terms of transmission line parameters $L$ and $C$:

$$\lambda = \frac{2\pi}{\beta} = \frac{1}{f\sqrt{LC}}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

Q: *Oh please, continue wasting my valuable time. We both know that a perfectly lossless transmission line is a physical impossibility.*

A: True! However, a **low-loss** line is possible—in fact, it is typical! If $R \ll \omega L$ and $G \ll \omega C$, we find that the lossless transmission line equations are excellent approximations!

Unless otherwise indicated, we will use the lossless equations to **approximate** the behavior of a **low-loss** transmission line.
The lone exception is when determining the attenuation of a long transmission line. For that case we will use the approximation:

\[ \alpha \approx \frac{1}{2} \left( \frac{R}{Z_0} + GZ_0 \right) \]

where \( Z_0 = \sqrt{L/C} \).

**A summary of lossless transmission line equations**

\[ Z_0 = \sqrt{\frac{L}{C}} \quad \quad \gamma = j \omega \sqrt{LC} \]

\[ V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z} \quad I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z} \]

\[ Z(z) = Z_0 \left( \frac{V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}}{V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}} \right) \]

\[ V^+(z) = V_0^+ e^{-j\beta z} \quad V^-(z) = V_0^- e^{+j\beta z} \]

\[ \Gamma(z) = \frac{V_0^-}{V_0^+} e^{+j2\beta z} \]

\[ \beta = \omega \sqrt{LC} \quad \lambda = \frac{1}{f\sqrt{LC}} \quad \nu_p = \frac{1}{\sqrt{LC}} \]