2.3 - The Terminated, Lossless Transmission Line

Reading Assignment: pp. 57-64

We now know that a lossless transmission line is completely characterized by real constants $Z_0$ and $\beta$.

Likewise, the 2 waves propagating on a transmission line are completely characterized by complex constants $V_0^+$ and $V_0^-$.  

Q: $Z_0$ and $\beta$ are determined from $L$, $C$, and $\omega$. How do we find $V_0^+$ and $V_0^-$?

A:

Every transmission line has 2 “boundaries”

1)
2)

Typically, there is a source at one end of the line, and a load at the other.

→

Let’s apply the load boundary condition!
HO: The Terminated, Lossless Transmission Line

HO: Special Values of Load Impedance

Q: So the line impedance at the end of a line must be load impedance $Z_L$ (i.e., $Z(z = z_L) = Z_L$): what is the line impedance at the beginning of the line (i.e., $Z(z = z_L - l) = ?$)?

A:

HO: Transmission Line Input Impedance

Example: Input Impedance

Q: For a given $Z_L$ we can determine an equivalent $\Gamma_L$. Is there an equivalent $\Gamma_{in}$ for each $Z_{in}$?

A: HO: The Reflection Coefficient Transformation

Q: So, the purpose of the transmission line is to transfer E.M. energy from the source to the load. Exactly how much power is flowing in the transmission line, and how much is delivered to the load?

A: HO: Power Flow and Return Loss

Note that we can specify a load with:
1) 
2) 
3) 

**HO: VSWR**  

**Q:** What happens if our transmission line is terminated by something other than a load? Is our transmission line theory still valid?  

**A:** As long as a transmission line is connected to linear devices our theory is valid. However, we must be careful to properly apply the **boundary conditions** associated with each linear device!  

**Example:** The Transmission Coefficient  

**Example:** Applying Boundary Conditions
The Terminated, Lossless Transmission Line

Now let's attach something to our transmission line. Consider a lossless line, length \( \ell \), terminated with a load \( Z_L \).

\[ I(z) \]

\[ V(z) \]

\[ Z_0, \beta \]

\[ L \]

\[ Z = Z_L - \ell \]

\[ Z = Z_L \]

**Q:** What is the current and voltage at each and every point on the transmission line (i.e., what is \( I(z) \) and \( V(z) \) for all points \( z \) where \( z_L - \ell \leq z \leq z_L \)?)?

**A:** To find out, we must apply boundary conditions!

In other words, at the end of the transmission line (\( z = z_L \))—where the load is attached—we have many requirements that all must be satisfied!
1. To begin with, the voltage and current \( I(z = z_L) \) and \( V(z = z_L) \) must be consistent with a valid transmission line solution:

\[
V(z = z_L) = V^+(z = z_L) + V^-(z = z_L) \\
= V_0^+ e^{-j\beta z_L} + V_0^- e^{+j\beta z_L}
\]

\[
I(z = z_L) = \frac{V_0^+(z = z_L)}{Z_0} - \frac{V_0^-(z = z_L)}{Z_0} \\
= \frac{V_0^+}{Z_0} e^{-j\beta z_L} - \frac{V_0^-}{Z_0} e^{+j\beta z_L}
\]

2. Likewise, the load voltage and current must be related by Ohm's law:

\[
V_L = Z_L I_L
\]

3. Most importantly, we recognize that the values \( I(z = z_L) \), \( V(z = z_L) \) and \( I_L, V_L \) are not independent, but in fact are strictly related by Kirchoff's Laws!
From KVL and KCL we find these requirements:

\[ V(z = z_L) = V_L \]
\[ I(z = z_L) = I_L \]

These are the **boundary conditions** for this particular problem.

⇒ **Careful!** Different transmission line problems lead to **different** boundary conditions—you must access each problem **individually** and **independently**!

**Combining** these equations and boundary conditions, we find that:

\[ V_L = Z_L \cdot I_L \]
\[ V(z = z_L) = Z_L \cdot I (z = z_L) \]
\[ V^+(z = z_L) + V^-(z = z_L) = \frac{Z_L}{Z_0} \left( V^+(z = z_L) - V^-(z = z_L) \right) \]

Rearranging, we can conclude:

\[ \frac{V^-(z = z_L)}{V^+(z = z_L)} = \frac{Z_L - Z_0}{Z_L + Z_0} \]
Q: Hey wait as second! We earlier defined $V^{-}(z)/V^{+}(z)$ as reflection coefficient $\Gamma(z)$. How does this relate to the expression above?

A: Recall that $\Gamma(z)$ is a function of transmission line position $z$. The value $V^{-}(z = z_L)/V^{+}(z = z_L)$ is simply the value of function $\Gamma(z)$ evaluated at $z = z_L$ (i.e., evaluated at the end of the line):

$$
\frac{V^{-}(z = z_L)}{V^{+}(z = z_L)} = \Gamma(z = z_L) = \frac{Z_L - Z_0}{Z_L + Z_0}
$$

This value is of fundamental importance for the terminated transmission line problem, so we provide it with its own special symbol ($\Gamma_L$):

$$
\Gamma_L \doteq \Gamma(z = z_L) = \frac{Z_L - Z_0}{Z_L + Z_0}
$$

Q: Wait! We earlier determined that:

$$
\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0}
$$

so it would seem that:

$$
\Gamma_L = \Gamma(z = z_L) = \frac{Z(z = z_L) - Z_0}{Z(z = z_L) + Z_0}
$$

Which expression is correct??
A: They both are! It is evident that the two expressions:

\[ \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \quad \text{and} \quad \Gamma_L = \frac{Z(z = z_L) - Z_0}{Z(z = z_L) + Z_0} \]

are equal if:

\[ Z(z = z_L) = Z_L \]

And since we know that from Ohm's Law:

\[ Z_L = \frac{V_L}{I_L} \]

and from Kirchhoff's Laws:

\[ \frac{V_L}{I_L} = \frac{V(z = z_L)}{I(z = z_L)} \]

and that line impedance is:

\[ \frac{V(z = z_L)}{I(z = z_L)} = Z(z = z_L) \]

we find it apparent that the line impedance at the end of the transmission line is equal to the load impedance:

\[ Z(z = z_L) = Z_L \]

The above expression is essentially another expression of the boundary condition applied at the end of the transmission line.
Q: I'm confused! Just what are we trying to accomplish in this handout?

A: We are trying to find $V(z)$ and $I(z)$ when a lossless transmission line is terminated by a load $Z_L$!

We can now determine the value of $V_0^-$ in terms of $V_0^+$. Since:

$$\Gamma_L = \frac{V^-(z = Z_L)}{V^+(z = Z_L)} = \frac{V_0^- e^{j\beta z_L}}{V_0^+ e^{-j\beta z_L}}$$

We find:

$$V_0^- = e^{-2j\beta z_L} \Gamma_L V_0^+$$

And therefore we find:

$$V^-(z) = (e^{-2j\beta z_L} \Gamma_L V_0^+ ) e^{j\beta z}$$

$$V(z) = V_0^+ \left[ e^{-j\beta z} + (e^{-2j\beta z_L} \Gamma_L ) e^{j\beta z} \right]$$

$$I(z) = \frac{V_0^+}{Z_0} \left[ e^{-j\beta z} - (e^{-2j\beta z_L} \Gamma_L ) e^{j\beta z} \right]$$

where:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$
Now, we can further simplify our analysis by arbitrarily assigning the end point \( z_L \) a zero value (i.e., \( z_L = 0 \)):

\[
\begin{align*}
V(z) &= V^+(z=0) + V^-(z=0) \\
&= V_0^+ e^{-j\beta(0)} + V_0^- e^{+j\beta(0)} \\
&= V_0^+ + V_0^- \\
I(z) &= \frac{V_0^+(z=0)}{Z_0} - \frac{V_0^-(z=0)}{Z_0} \\
&= \frac{V_0^+}{Z_0} e^{-j\beta(0)} - \frac{V_0^-}{Z_0} e^{+j\beta(0)} \\
&= \frac{V_0^+ - V_0^-}{Z_0} \\
Z(z=0) &= Z_0 \left( \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} \right)
\end{align*}
\]

If the load is located at \( z=0 \) (i.e., if \( z_L = 0 \)), we find that:

\[
\begin{align*}
V(z=0) &= V^+(z=0) + V^-(z=0) \\
&= \frac{V_0^+}{Z_0} e^{-j\beta(0)} + \frac{V_0^-}{Z_0} e^{+j\beta(0)} \\
&= \frac{V_0^+ + V_0^-}{Z_0}
\end{align*}
\]
Likewise, it is apparent that if \( z_L = 0 \), \( \Gamma_L \) and \( \Gamma_0 \) are the same:

\[
\Gamma_L = \Gamma_L(z = z_L) = \frac{V^-(z = 0)}{V^+(z = 0)} = \frac{V_0^-}{V_0^+} = \Gamma_0
\]

Therefore:

\[
\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_0
\]

Thus, we can write the line current and voltage simply as:

\[
V(z) = V_0^+ \left[ e^{-j\beta z} + \Gamma_L e^{+j\beta z} \right]
\]

\[\text{[for } z_L = 0]\]

\[
I(z) = \frac{V_0^+}{Z_0} \left[ e^{-j\beta z} - \Gamma_L e^{+j\beta z} \right]
\]

**Q:** But, how do we determine \( V_0^+ \)?

**A:** We require a second boundary condition to determine \( V_0^+ \). The only boundary left is at the other end of the transmission line. Typically, a source of some sort is located there. This makes physical sense, as something must generate the incident wave!
Special Values of Load Impedance

Let's look at some specific values of load impedance $Z_L = R_L + jX_L$ and see what happens on our transmission line!

1. $Z_L = Z_0$

In this case, the load impedance is numerically equal to the characteristic impedance of the transmission line. Assuming the line is lossless, then $Z_0$ is real, and thus:

$$R_L = Z_0 \quad \text{and} \quad X_L = 0$$

It is evident that the resulting load reflection coefficient is zero:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_0 - Z_0}{Z_0 + Z_0} = 0$$

This result is very interesting, as it means that there is no reflected wave $V^-(z)$!

$$V^-(z) = (e^{-2j\beta z_L} \Gamma_L V_0^*) e^{+j\beta z}$$
$$= (e^{-2j\beta z_L} (0) V_0^*) e^{+j\beta z}$$
$$= 0$$
Thus, the total voltage and current along the transmission line is simply voltage and current of the incident wave:

\[
V(z) = V^+(z) = V_0^+ e^{-j\beta z}
\]

\[
I(z) = I^+(z) = \frac{V_0^+}{Z_0} e^{-j\beta z}
\]

Meaning that the line impedance is likewise numerically equal to the characteristic impedance of the transmission line for all line position \( z \):

\[
Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{V_0^+ e^{-j\beta z}}{V_0^+ e^{-j\beta z}} = Z_0
\]

And likewise, the reflection coefficient is zero at all points along the line:

\[
\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{0}{V^+(z)} = 0
\]

We call this condition (when \( Z_L = Z_0 \)) the matched condition, and the load \( Z_L = Z_0 \) a matched load.

2. \( Z_L = 0 \)

A device with no impedance is called a short circuit! I.E.:  

\[
R_L = 0 \quad \text{and} \quad X_L = 0
\]
In this case, the **voltage** across the load—and thus the voltage at the end of the transmission line—is **zero**:

\[ V_L = Z_L I_L = 0 \quad \text{and} \quad V(z = z_L) = 0 \]

Note that this does **not** mean that the **current** is zero!

\[ I_L = I(z = z_L) \neq 0 \]

For a **short**, the resulting load reflection coefficient is therefore:

\[ \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{0 - Z_0}{0 + Z_0} = -1 \]

Meaning (assuming \( z_L = 0 \)):

\[ V_0^- = -V_0^+ \]

As a result, the total **voltage** and **current** along the transmission line is simply:

\[ V(z) = V_0^+ \left( e^{-j \beta z} - e^{+j \beta z} \right) = -j 2V_0^+ \sin(\beta z) \]

\[ I(z) = \frac{V_0^+}{Z_0} \left( e^{-j \beta z} + e^{+j \beta z} \right) = \frac{2V_0^+}{Z_0} \cos(\beta z) \]

Meaning that the line **impedance** can likewise be written in terms of a **trigonometric** function:
\[ Z(z) = \frac{V(z)}{I(z)} = -jZ_0 \tan(\beta z) \]

Note that this impedance is purely reactive. This means that the current and voltage on the transmission line will be everywhere 90° out of phase.

Hopefully, this was likewise apparent to you when you observed the expressions for \( V(z) \) and \( I(z) \)!

Note at the end of the line (i.e., \( z = z_L = 0 \)), we find that

\[ V(z = 0) = -j2V_0^+ \sin(0) = 0 \]

\[ I(z = 0) = \frac{2V_0^+}{Z_0} \cos(0) = \frac{2V_0^+}{Z_0} \]

As expected, the voltage is zero at the end of the transmission line (i.e. the voltage across the short). Likewise, the current at the end of the line (i.e., the current through the short) is at a maximum!

Finally, we note that the line impedance at the end of the transmission line is:

\[ Z(z = 0) = -jZ_0 \tan(0) = 0 \]

Just as we expected—a short circuit!
Finally, the reflection coefficient function is (assuming $z_L = 0$):

$$\Gamma(z) = \frac{V_-(z)}{V_+(z)} = \frac{-V_0^+ e^{j\beta z}}{V_0^+ e^{-j\beta z}} = -e^{j\beta z}$$

Note that for this case $|\Gamma(z)| = 1$, meaning that:

$$|V_-(z)| = |V_+(z)|$$

In other words, the magnitude of each wave on the transmission line is the same—the reflected wave is just as big as the incident wave!

3. $Z_L = \infty$

A device with infinite impedance is called an open circuit!

I.E.:

$$R_L = \infty \quad \text{and/or} \quad X_L = \pm \infty$$

In this case, the current through the load—and thus the current at the end of the transmission line—is zero:

$$I_L = \frac{V_L}{Z_L} = 0 \quad \text{and} \quad I(z = z_L) = 0$$

Note that this does not mean that the voltage is zero!

$$V_L = V(z = z_L) \neq 0$$
For an open, the resulting load reflection coefficient is:

\[ \Gamma_L = \lim_{Z_L \to \infty} \frac{Z_L - Z_0}{Z_L + Z_0} = \lim_{Z_L \to \infty} \frac{Z_L}{Z_L} = 1 \]

Meaning (assuming \( z_L = 0 \)):

\[ V_0^- = V_0^+ \]

As a result, the total voltage and current along the transmission line is simply (assuming \( z_L = 0 \)):

\[ V(z) = V_0^+ \left( e^{-j\beta z} + e^{+j\beta z} \right) = 2V_0^+ \cos(\beta z) \]

\[ I(z) = \frac{V_0^+}{Z_0} \left( e^{-j\beta z} - e^{+j\beta z} \right) = -j \frac{2V_0^+}{Z_0} \sin(\beta z) \]

Meaning that the line impedance can likewise be written in terms of trigonometric function:

\[ Z(z) = \frac{V(z)}{I(z)} = jZ_0 \cot(\beta z) \]

Again note that this impedance is purely reactive—\( V(z) \) and \( I(z) \) are again 90° out of phase!

Note at the end of the line (i.e., \( z = z_L = 0 \)), we find that
\[ V(z = 0) = 2V_0^+ \cos(0) = \frac{2V_0^+}{Z_0} \]

\[ I(z = 0) = -j \frac{2V_0^+}{Z_0} \sin(0) = 0 \]

As expected, the current is zero at the end of the transmission line (i.e. the current through the open). Likewise, the voltage at the end of the line (i.e., the voltage across the open) is at a maximum!

Finally, we note that the line impedance at the end of the transmission line is:

\[ Z(z = 0) = jZ_0 \cot(0) = \infty \]

Just as we expected—an open circuit!

Finally, the reflection coefficient is (assuming \( z_L = 0 \)):

\[ \Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{V_0^+ e^{j\beta z}}{V_0^+ e^{-j\beta z}} = e^{j2\beta z} \]

Note that likewise for this case \( |\Gamma(z)| = 1 \), meaning again that:

\[ |V^-(z)| = |V^+(z)| \]

In other words, the magnitude of each wave on the transmission line is the same—the reflected wave is just as big as the incident wave!
4. $R_L = 0$

For this case, the load impedance is purely reactive (e.g. a capacitor of inductor):

$$Z_L = jX_L$$

Thus, both the current through the load, and voltage across the load, are non-zero:

$$I_L = I(z = z_L) \neq 0 \quad V_L = V(z = z_L) \neq 0$$

The resulting load reflection coefficient is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{jX_L - Z_0}{jX_L + Z_0}$$

Given that $Z_0$ is real (i.e., the line is lossless), we find that this load reflection coefficient is generally some complex number.

We can rewrite this value explicitly in terms of its real and imaginary part as:

$$\Gamma_L = \frac{jX_L - Z_0}{jX_L + Z_0} = \left(\frac{X_L^2 - Z_0^2}{X_L^2 + Z_0^2}\right) + j\left(\frac{2Z_0X_L}{X_L^2 + Z_0^2}\right)$$

_Yuck! This isn't much help!_
Let's instead write this complex value $\Gamma_L$ in terms of its **magnitude** and **phase**. For **magnitude** we find a much more straightforward result!

\[
|\Gamma_L|^2 = \left| \frac{jX_L - Z_0}{jX_L + Z_0} \right|^2 = \frac{X_L^2 + Z_0^2}{X_L^2 + Z_0^2} = 1
\]

Its magnitude is **one**! Thus, we find that for reactive loads, the reflection coefficient can be simply expressed as:

\[
\Gamma_L = e^{j\theta_r}
\]

where

\[
\theta_r = \tan^{-1} \left[ \frac{2Z_0X_L}{X_L^2 - Z_0^2} \right]
\]

We can therefore conclude that for a **reactive** load:

\[
V_0^- = e^{j\theta_r} V_0^+
\]

As a result, the total voltage and current along the transmission line is simply (assuming $Z_L = 0$):

\[
V(z) = V_0^+ \left( e^{-j\beta z} + e^{+j\theta_L} e^{+j\beta z} \right) = V_0^+ e^{+j\theta_L/2} \left( e^{-j(\beta z + \theta_L/2)} + e^{+j(\beta z + \theta_L/2)} \right) = 2V_0^+ e^{+j\theta_L/2} \cos(\beta z + \theta_L/2)
\]
\[ I(z) = \frac{V_0^+}{Z_0} (e^{-j\beta z} - e^{+j\beta z}) \]
\[ = \frac{V_0^+}{Z_0} e^{+j\theta_L/2} \left( e^{-j(\beta z + \theta_L/2)} - e^{+j(\beta z + \theta_L/2)} \right) \]
\[ = -j \frac{2V_0^+}{Z_0} e^{+j\theta_L/2} \sin(\beta z + \theta_L/2) \]

Meaning that the line impedance can again be written in terms of trigonometric function:

\[ Z(z) = \frac{V(z)}{I(z)} = jZ_0 \cot(\beta z + \theta_T/2) \]

Again note that this impedance is purely reactive—\( V(z) \) and \( I(z) \) are once again 90° out of phase!

Note at the end of the line (i.e., \( z = z_L = 0 \)), we find that

\[ V(z = 0) = 2V_0^+ \cos(\theta_T/2) \]
\[ I(z = 0) = -j \frac{2V_0^+}{Z_0} \sin(\theta_T/2) \]

As expected, neither the current nor voltage at the end of the line are zero.

We also note that the line impedance at the end of the transmission line is:
\[ Z(z=0) = jZ_0 \cot(\theta_\Gamma/2) \]

With a little trigonometry, we can show (trust me!) that:

\[ \cot(\theta_\Gamma/2) = \frac{X_L}{Z_0} \]

and therefore:

\[ Z(z=0) = jZ_0 \cot(\theta_\Gamma/2) = jX_L = Z_L \]

Just as we expected!

Finally, the reflection coefficient function is (assuming \( z_L = 0 \)):

\[ \Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{V_0^+ e^{j\theta_\Gamma} e^{+j\beta z}}{V_0^- e^{-j\beta z}} = e^{+j2(\beta z + \theta_\Gamma/2)} \]

Note that likewise for this case \( |\Gamma(z)| = 1 \), meaning once again:

\[ |V^-(z)| = |V^+(z)| \]

In other words, the magnitude of each wave on the transmission line is the same—the reflected wave is just as big as the incident wave!

**Q:** Gee, a reactive load leads to results very similar to that of an open or short circuit. Is this just coincidence?
**A:** Hardly! An open and short are in fact reactive loads—they cannot absorb power (think about this!).

Specifically, for an open, we find $\theta_{\Gamma} = 0$, so that:

$$\Gamma_L = e^{j\theta_{\Gamma}} = 1$$

Likewise, for a short, we find that $\theta_{\Gamma} = \pi$, so that:

$$\Gamma_L = e^{j\theta_{\Gamma}} = -1$$

5. $X_L = 0$

For this case, the load impedance is purely real (e.g. a resistor):

$$Z_L = R_L$$

Thus, both the current through the load, and voltage across the load, are non-zero:

$$I_L = I(z = z_L) \neq 0 \quad V_L = V(z = z_L) \neq 0$$

The resulting load reflection coefficient is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{R - Z_0}{R + Z_0}$$

Given that $Z_0$ is real (i.e., the line is lossless), we find that this load reflection coefficient must be a purely real value!
In other words:

\[ \text{Re}\{\Gamma_L\} = \frac{R - Z_0}{R + Z_0} \quad \text{Im}\{\Gamma_L\} = 0 \]

The magnitude is thus:

\[ |\Gamma_L| = \frac{|R - Z_0|}{|R + Z_0|} \]

whereas the phase \( \theta_\Gamma \) can take on one of two values:

\[ \theta_\Gamma = \begin{cases} 0 & \text{if } \text{Re}\{\Gamma_L\} > 0 \quad (\text{i.e., if } R_L > Z_0) \\ \pi & \text{if } \text{Re}\{\Gamma_L\} < 0 \quad (\text{i.e., if } R_L < Z_0) \end{cases} \]

For this case, the impedance at the end of the line must be real \((Z(z = Z_L) = R_L)\). Thus, the current and the voltage at this point are precisely in phase.

However, even though the load impedance is real, the line impedance at all other points on the line is generally complex!

Moreover, the general current and voltage expressions, as well as reflection coefficient function, cannot be further simplified for the case where \(Z_L = R_L\).
Q: Why is that? When the load was purely imaginary (reactive), we were able to simply our general expressions, and likewise deduce all sorts of interesting results!

A: True! And here’s why. Remember, a lossless transmission line has series inductance and shunt capacitance only. In other words, a length of lossless transmission line is a purely reactive device (it absorbs no energy!).

* If we attach a purely reactive load at the end of the transmission line, we still have a completely reactive system (load and transmission line). Because this system has no resistive (i.e., real) component, the general expressions for line impedance, line voltage, etc. can be significantly simplified.

* However, if we attach a purely real load to our reactive transmission line, we now have a complex system, with both real and imaginary (i.e., resistive and reactive) components. This complex case is exactly what our general expressions already describes—no further simplification is possible!

5. \[ Z_L = R_L + jX_L \]

Now, let’s look at the general case, where the load has both a real (resitive) and imaginary (reactive) component.

Q: Haven’t we already determined all the general expressions (e.g., \( \Gamma_L, V(z), I(z), Z(z), \Gamma(z) \)) for this general case? Is there anything else left to be determined?
**A:** There is one last thing we need to discuss. It seems trivial, but its ramifications are very important!

For you see, the “general” case is not, in reality, quite so general. Although the reactive component of the load can be either positive or negative \((-\infty < X_L < \infty)\), the resistive component of a passive load must be positive \((R_L > 0)\)—there’s no such thing as negative resistor!

This leads to one very important and useful result. Consider the load reflection coefficient:

\[
\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \\
= \frac{(R_L + jX_L) - Z_0}{(R_L + jX_L) + Z_0} \\
= \frac{(R_L - Z_0) + jX_L}{(R_L + Z_0) + jX_L}
\]

Now let’s look at the magnitude of this value:
\[ |\Gamma_L|^2 = \frac{(R_L - Z_0) + jX_L}{(R_L + Z_0) + jX_L}^2 \]
\[ = \frac{(R_L - Z_0)^2 + X_L^2}{(R_L + Z_0)^2 + X_L^2} \]
\[ = \frac{(R_L^2 - 2R_L Z_0 + Z_0^2) + X_L^2}{(R_L^2 + 2R_L Z_0 + Z_0^2) + X_L^2} \]
\[ = \frac{(R_L^2 + Z_0^2 + X_L^2) - 2R_L Z_0}{(R_L^2 + Z_0^2 + X_L^2) + 2R_L Z_0} \]

It is apparent that since both \(R_L\) and \(Z_0\) are positive, the numerator of the above expression must be less than (or equal to) the denominator of the above expression.

\[ \Rightarrow \quad \text{In other words, the magnitude of the load reflection coefficient is always less than or equal to one!} \]

\[ |\Gamma_L| \leq 1 \quad (\text{for } R_L \geq 0) \]

Moreover, we find that this means the reflection coefficient function likewise always has a magnitude less than or equal to one, for all values of position \(z\).

\[ |\Gamma(z)| \leq 1 \quad (\text{for all } z) \]
Which means, of course, that the reflected wave will always have a magnitude less than that of the incident wave magnitude:

\[ |V^{-}(z)| \leq |V^{+}(z)| \quad \text{(for all } z) \]

We will find out later that this result is consistent with conservation of energy—the reflected wave from a passive load cannot be larger than the wave incident on it.
Consider a lossless line, length \( \ell \), terminated with a load \( Z_L \).

Let's determine the input impedance of this line!

**Q:** Just what do you mean by input impedance?

**A:** The input impedance is simply the line impedance seen at the beginning \( z = -\ell \) of the transmission line, i.e.:

\[
Z_{in} = Z(z = -\ell) = \frac{V(z = -\ell)}{I(z = -\ell)}
\]

Note \( Z_{in} \) equal to neither the load impedance \( Z_L \) nor the characteristic impedance \( Z_0 \)!

\[
Z_m \neq Z_L \quad \text{and} \quad Z_m \neq Z_0
\]
To determine exactly what $Z_{in}$ is, we first must determine the voltage and current at the beginning of the transmission line ($z = -\ell$).

\[
V(z = -\ell) = V_0^+ [e^{+j\beta \ell} + \Gamma_L e^{-j\beta \ell}]
\]

\[
I(z = -\ell) = \frac{V_0^+}{Z_0} [e^{+j\beta \ell} - \Gamma_L e^{-j\beta \ell}]
\]

Therefore:

\[
Z_{in} = \frac{V(z = -\ell)}{I(z = -\ell)} = Z_0 \left( \frac{e^{+j\beta \ell} + \Gamma_L e^{-j\beta \ell}}{e^{+j\beta \ell} - \Gamma_L e^{-j\beta \ell}} \right)
\]

We can explicitly write $Z_{in}$ in terms of load $Z_L$ using the previously determined relationship:

\[
\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}
\]

Combining these two expressions, we get:

\[
Z_{in} = Z_0 \left( \frac{Z_L + Z_0}{Z_L + Z_0} + \frac{Z_L - Z_0}{Z_L - Z_0} \right)
\]

\[
= Z_0 \left( \frac{Z_L (e^{+j\beta \ell} + e^{-j\beta \ell}) + Z_0 (e^{+j\beta \ell} - e^{-j\beta \ell})}{Z_L (e^{+j\beta \ell} + e^{-j\beta \ell}) - Z_0 (e^{+j\beta \ell} - e^{-j\beta \ell})} \right)
\]

Now, recall Euler's equations:

\[
e^{+j\beta \ell} = \cos \beta \ell + j \sin \beta \ell
\]

\[
e^{-j\beta \ell} = \cos \beta \ell - j \sin \beta \ell
\]
Using Euler's relationships, we can likewise write the input impedance without the complex exponentials:

\[
Z_{in} = Z_0 \left( \frac{Z_L \cos \beta \ell + j Z_0 \sin \beta \ell}{Z_0 \cos \beta \ell + j Z_L \sin \beta \ell} \right) \\
= Z_0 \left( \frac{Z_L + j Z_0 \tan \beta \ell}{Z_0 + j Z_L \tan \beta \ell} \right)
\]

Note that depending on the values of \( \beta \), \( Z_0 \) and \( \ell \), the input impedance can be radically different from the load impedance \( Z_L \)!

**Special Cases**

Now let's look at the \( Z_{in} \) for some important load impedances and line lengths.

→ You should commit these results to memory!

1. \( \ell = \frac{\lambda}{2} \)

If the length of the transmission line is exactly one-half wavelength (\( \ell = \lambda/2 \)), we find that:

\[
\beta \ell = \frac{2\pi \lambda}{\lambda} = \pi
\]

meaning that:

\[
\cos \beta \ell = \cos \pi = -1 \quad \text{and} \quad \sin \beta \ell = \sin \pi = 0
\]
and therefore:

\[
Z_{in} = Z_0 \left( \frac{Z_L \cos \beta \ell + j \ Z_0 \sin \beta \ell}{Z_0 \cos \beta \ell + j \ Z_L \sin \beta \ell} \right) \\
= Z_0 \left( \frac{Z_L (-1) + j \ Z_L (0)}{Z_0 (-1) + j \ Z_L (0)} \right) \\
= Z_L
\]

In other words, if the transmission line is precisely one-half wavelength long, the input impedance is equal to the load impedance, regardless of \( Z_0 \) or \( \beta \).

2. \( \ell = \frac{\lambda}{4} \)

If the length of the transmission line is exactly one-quarter wavelength (\( \ell = \frac{\lambda}{4} \)), we find that:

\[
\beta \ell = \frac{2\pi \ \lambda}{4 \ \lambda} = \frac{\pi}{2}
\]

meaning that:

\[
\cos \beta \ell = \cos \frac{\pi}{2} = 0 \quad \text{and} \quad \sin \beta \ell = \sin \frac{\pi}{2} = 1
\]
and therefore:

\[
Z_{\text{in}} = Z_0 \left( \frac{Z_L \cos \beta \ell + j Z_0 \sin \beta \ell}{Z_0 \cos \beta \ell + j Z_L \sin \beta \ell} \right)
\]

\[
= Z_0 \left( \frac{Z_L (0) + j Z_0 (1)}{Z_0 (0) + j Z_L (1)} \right)
\]

\[
= \left( \frac{Z_0}{Z_L} \right)^2
\]

In other words, if the transmission line is precisely one-quarter wavelength long, the input impedance is inversely proportional to the load impedance.

Think about what this means! Say the load impedance is a short circuit, such that \( Z_L = 0 \). The input impedance at beginning of the \( \lambda/4 \) transmission line is therefore:

\[
Z_{\text{in}} = \left( \frac{Z_0}{Z_L} \right)^2 = \left( \frac{Z_0}{0} \right)^2 = \infty
\]

\( Z_{\text{in}} = \infty \)! This is an open circuit! The quarter-wave transmission line transforms a short-circuit into an open-circuit—and vice versa!
3. \( Z_L = Z_0 \)

If the load is numerically equal to the characteristic impedance of the transmission line (a real value), we find that the input impedance becomes:

\[
Z_{in} = Z_0 \left( \frac{Z_L \cos \beta \ell + j Z_0 \sin \beta \ell}{Z_0 \cos \beta \ell + j Z_L \sin \beta \ell} \right)
\]

\[
= Z_0 \left( \frac{Z_0 \cos \beta \ell + j Z_0 \sin \beta \ell}{Z_0 \cos \beta \ell + j Z_0 \sin \beta \ell} \right)
\]

\[
= Z_0
\]

In other words, if the load impedance is equal to the transmission line characteristic impedance, the input impedance will be likewise be equal to \( Z_0 \) regardless of the transmission line length \( \ell \).

4. \( Z_L = j X_L \)

If the load is purely reactive (i.e., the resistive component is zero), the input impedance is:
In other words, if the load is purely reactive, then the input impedance will likewise be purely reactive, regardless of the line length $\ell$.

Note that the opposite is not true: even if the load is purely resistive ($Z_L = R$), the input impedance will be complex (both resistive and reactive components).

**Q:** *Why is this?*

**A:**
5. $\ell \ll \lambda$

If the transmission line is electrically small—its length $\ell$ is small with respect to signal wavelength $\lambda$--we find that:

$$\beta \ell = \frac{2\pi}{\lambda} \ell = 2\pi \frac{\ell}{\lambda} \approx 0$$

and thus:

$$\cos \beta \ell = \cos 0 = 1 \quad \text{and} \quad \sin \beta \ell = \sin 0 = 0$$

so that the input impedance is:

$$Z_{in} = Z_0 \left( \frac{Z_L \cos \beta \ell + j Z_0 \sin \beta \ell}{Z_0 \cos \beta \ell + j Z_L \sin \beta \ell} \right)
= Z_0 \left( \frac{Z_L (1) + j Z_L (0)}{Z_0 (1) + j Z_L (0)} \right)
= Z_L$$

In other words, if the transmission line length is much smaller than a wavelength, the input impedance $Z_{in}$ will always be equal to the load impedance $Z_L$.

This is the assumption we used in all previous circuits courses (e.g., EECS 211, 212, 312, 412)! In those courses, we assumed that the signal frequency $\omega$ is relatively low, such that the signal wavelength $\lambda$ is very large ($\lambda \gg \ell$).
Note also for this case (the electrically short transmission line), the voltage and current at each end of the transmission line are approximately the same!

\[ V(z = -l) \approx V(z = 0) \quad \text{and} \quad I(z = -l) \approx I(z = 0) \quad \text{if} \quad l \ll \lambda \]

If \( l \ll \lambda \), our “wire” behaves exactly as it did in EECS 211!
Example: Input Impedance

Consider the following circuit:

If we ignored our new μ-wave knowledge, we might erroneously conclude that the input impedance of this circuit is:

Therefore:

Of course, this is not the correct answer!

We must use our transmission line theory to determine an accurate value. Define $Z_j$ as the input impedance of the last section:
we find that $Z_f$ is:

$$Z_1 = Z_0 \left( \frac{Z_L \cos \beta \ell + j Z_0 \sin \beta \ell}{Z_0 \cos \beta \ell + j Z_L \sin \beta \ell} \right)$$

$$= 2 \left( \frac{(1 + j2) \cos \left( \frac{\pi}{4} \right) + j 2 \sin \left( \frac{\pi}{4} \right)}{2 \cos \left( \frac{\pi}{4} \right) + j(1 + j2) \sin \left( \frac{\pi}{4} \right)} \right)$$

$$= 2 \left( \frac{1 + j4}{j} \right)$$

$$= 8 - j2$$

Therefore, our circuit now becomes:

Note the resistor is in **series** with impedance $Z_f$. We can combine these two into one impedance defined as $Z_2$:

$$Z_2 = 2 + Z_1 = 2 + (8 - j2) = 10 - j2$$
Now let's define the input impedance of the middle transmission line section as $Z_3$:

\[ Z_3 = \frac{Z_0^2}{Z_L} = \frac{Z_0^2}{Z_2} = \frac{1.5^2}{10 - j2} = 0.21 + j0.043 \]

Thus, we can further simplify the original circuit as:

Now we find that the impedance $Z_3$ is parallel to the capacitor. We can combine the two impedances and define the result as impedance $Z_4$: 
\[ Z_4 = -j3(0.21 + j0.043) \]
\[ = -j3 + 0.21 + j0.043 \]
\[ = 0.22 + j0.028 \]

Now we are left with this equivalent circuit:

Note that the remaining transmission line section is a half wavelength! This is one of the special situations we discussed in a previous handout. Recall that the input impedance in this case is simply equal to the load impedance:

\[ Z_{in} = Z_L = Z_4 = 0.22 + j0.028 \]

Whew! We are finally done. The input impedance of the original circuit is:

\[ Z_{in} = 0.22 + j0.028 \]
Note this means that this circuit:

\[ Z_m = 0.22 + j0.028 \]

and this circuit:

\[ Z_m = 0.22 + j0.028 \]

are precisely the same! They have exactly the same impedance, and thus they “behave” precisely the same way in any circuit (but only at frequency \( \omega_0 \)!).
The Reflection Coefficient Transformation

The load at the end of some length of a transmission line (with characteristic impedance $Z_0$) can be specified in terms of its impedance $Z_L$ or its reflection coefficient $\Gamma_L$.

Note both values are complex, and either one completely specifies the load—if you know one, you know the other!

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \quad \text{and} \quad Z_L = Z_0 \left( \frac{1 + \Gamma_L}{1 - \Gamma_L} \right)$$

Recall that we determined how a length of transmission line transformed the load impedance into an input impedance of a (generally) different value:
where:

\[
Z_{in} = Z_0 \left( \frac{Z_L \cos \beta \ell + j Z_0 \sin \beta \ell}{Z_0 \cos \beta \ell + j Z_L \sin \beta \ell} \right)
\]

\[
= Z_0 \left( \frac{Z_L + j Z_0 \tan \beta \ell}{Z_0 + j Z_L \tan \beta \ell} \right)
\]

Q: Say we know the load in terms of its reflection coefficient. How can we express the input impedance in terms its reflection coefficient (call this \( \Gamma_{in} \))?

A: Well, we could execute these three steps:

1. Convert \( \Gamma_L \) to \( Z_L \):

\[
Z_L = Z_0 \left( \frac{1+\Gamma_L}{1-\Gamma_L} \right)
\]

2. Transform \( Z_L \) down the line to \( Z_{in} \):

\[
Z_{in} = Z_0 \left( \frac{Z_L \cos \beta \ell + j Z_0 \sin \beta \ell}{Z_0 \cos \beta \ell + j Z_L \sin \beta \ell} \right)
\]
3. Convert $Z_{in}$ to $\Gamma_{in}$:

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

**Q:** Yikes! This is a ton of complex arithmetic—isn’t there an easier way?

**A:** Actually, there is!

Recall in an earlier handout that the input impedance of a transmission line length $\ell$, terminated with a load $\Gamma_L$, is:

$$Z_{in} = \frac{V(z = -\ell)}{I(z = -\ell)} = Z_0 \left( \frac{e^{+j\beta\ell} + \Gamma_L e^{-j\beta\ell}}{e^{+j\beta\ell} - \Gamma_L e^{-j\beta\ell}} \right)$$

Note this directly relates $\Gamma_L$ to $Z_{in}$ (steps 1 and 2 combined!).

If we directly insert this equation into:

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

we get an equation directly relating $\Gamma_L$ to $\Gamma_{in}$:
\[
\Gamma_m = \frac{Z_0 \left( e^{+j\beta L} + \Gamma_L e^{-j\beta L} \right) - \left( e^{+j\beta L} - \Gamma_L e^{-j\beta L} \right)}{Z_0 \left( e^{+j\beta L} + \Gamma_L e^{-j\beta L} \right) + \left( e^{+j\beta L} - \Gamma_L e^{-j\beta L} \right)}
\]

\[
= \frac{2\Gamma_L e^{-j\beta L}}{2e^{+j\beta L}}
\]

\[
= \Gamma_L e^{-j\beta L} e^{-j\beta L}
\]

\[
= \Gamma_L e^{-j2\beta L}
\]

**Q:** Hey! This result looks familiar. Haven't we seen something like this before?

**A:** Absolutely! Recall that we found that the reflection coefficient function \( \Gamma(z) \) can be expressed as:

\[
\Gamma(z) = \Gamma_0 e^{2\gamma z}
\]

Now, for a lossless line, we know that \( \gamma = j\beta \), so that:

\[
\Gamma(z) = \Gamma_0 e^{j2\beta z}
\]

Evaluating this function at the beginning of the line (i.e., at \( z = z_L - \ell \)):

\[
\Gamma \left( z = z_L - \ell \right) = \Gamma_0 e^{j2\beta(z_L - \ell)}
\]

\[
= \Gamma_0 e^{j2\beta z_L} e^{-j2\beta \ell}
\]

But, we recognize that:

\[
\Gamma_0 e^{j2\beta z_L} = \Gamma(z = z_L) = \Gamma_L
\]

\[
= \Gamma_0 e^{j2\beta z_L} e^{-j2\beta \ell}
\]

\[
= \Gamma_0 e^{j2\beta z_L} e^{-j2\beta \ell}
\]

\[
= \Gamma_0 e^{j2\beta z_L} e^{-j2\beta \ell}
\]
And so:
\[
\Gamma(z = z_L - \ell) = \Gamma_0 e^{j2\beta z_L} e^{-j2\beta \ell} = \Gamma_L e^{-j2\beta \ell}
\]

Thus, we find that \( \Gamma_{in} \) is simply the value of function \( \Gamma(z) \) evaluated at the line input of \( z = z_L - \ell \):

\[
\Gamma_{in} = \Gamma(z = z_L - \ell) = \Gamma_L e^{-j2\beta \ell}
\]

Makes sense! After all, the input impedance is likewise simply the line impedance evaluated at the line input of \( z = z_L - \ell \):

\[
Z_{in} = Z(z = z_L - \ell)
\]

It is apparent that from the above expression that the reflection coefficient at the input is simply related to \( \Gamma_L \) by a phase shift of \( 2\beta \ell \).

In other words, the magnitude of \( \Gamma_{in} \) is the same as the magnitude of \( \Gamma_L \!:
\[
|\Gamma_{in}| = |\Gamma_L| e^{j(\theta_\ell - 2\beta \ell)}
\]
\[
= |\Gamma_L| (1)
\]
\[
= |\Gamma_L|
\]

If we think about this, it makes perfect sense!
Recall that the power **absorbed** by the load $\Gamma_{in}$ would be:

$$P_{abs}^{in} = \frac{|V_0^+|^2}{2Z_0} \left( 1 - |\Gamma_{in}|^2 \right)$$

while that absorbed by the load $\Gamma_L$ is:

$$P_{abs}^{L} = \frac{|V_0^+|^2}{2Z_0} \left( 1 - |\Gamma_L|^2 \right)$$

Recall, however, that a lossless transmission line can absorb **no** power! By adding a length of transmission line to load $\Gamma_L$, we have added only **reactance**. Therefore, the power absorbed by load $\Gamma_{in}$ is **equal** to the power absorbed by $\Gamma_L$:

$$P_{abs}^{in} = P_{abs}^{L}$$

$$\frac{|V_0^+|^2}{2Z_0} \left( 1 - |\Gamma_{in}|^2 \right) = \frac{|V_0^+|^2}{2Z_0} \left( 1 - |\Gamma_L|^2 \right)$$

$$1 - |\Gamma_{in}|^2 = 1 - |\Gamma_L|^2$$
Thus, we can conclude from conservation of energy that:

\[ |\Gamma_m| = |\Gamma_L| \]

Which of course is exactly the result we just found!

Finally, the phase shift associated with transforming the load \( \Gamma_L \) down a transmission line can be attributed to the phase shift associated with the wave propagating a length \( \ell \) down the line, reflecting from load \( \Gamma_L \), and then propagating a length \( \ell \) back up the line:

To emphasize this wave interpretation, we recall that by definition, we can write \( \Gamma_{in} \) as:

\[ \Gamma_{in} = \Gamma(z = z_L - \ell) = \frac{V^{-}(z = z_L - \ell)}{V^{+}(z = z_L - \ell)} \]

Therefore:

\[ V^{-}(z = z_L - \ell) = \Gamma_{in} V^{+}(z = z_L - \ell) = e^{-j\beta\ell} \Gamma_L e^{-j\beta\ell} V^{+}(z = z_L - \ell) \]
Power Flow and Return Loss

We have discovered that two waves propagate along a transmission line, one in each direction \( (V^+(z) \text{ and } V^-(z)) \).

\[
I(z) = \frac{V_0^+}{Z_0} \left[ e^{-j\beta z} - \Gamma_L e^{+j\beta z} \right]
\]

\[
V(z) = V_0^+ \left[ e^{-j\beta z} + \Gamma_L e^{+j\beta z} \right]
\]

The result is that electromagnetic energy flows along the transmission line at a given rate (i.e., power).

**Q:** How much power flows along a transmission line, and where does that power go?

**A:** We can answer that question by determining the power absorbed by the load!
The time **average** power absorbed by an impedance $Z_L$ is:

$$
\rho_{abs} = \frac{1}{2} \text{Re}\{V_L I_L^*\}
$$

$$
= \frac{1}{2} \text{Re}\{V(z = 0) I(z = 0)^*\}
$$

$$
= \frac{1}{2} \text{Re}\left((V_0^* \left[ e^{-j\beta_0} + \Gamma_L e^{+j\beta_0} \right] \left[ e^{-j\beta_0} - \Gamma_L e^{+j\beta_0} \right]^*)\right)
$$

$$
= \frac{|V_0^*|^2}{2 Z_0} \text{Re}\left\{1 - (\Gamma_L^* - \Gamma_L) - |\Gamma_L|^2\right\}
$$

$$
= \frac{|V_0^*|^2}{2 Z_0}\left(1 - |\Gamma_L|^2\right)
$$

The significance of this result can be seen by rewriting the expression as:

$$
\rho_{abs} = \frac{|V_0^*|^2}{2 Z_0}\left(1 - |\Gamma_L|^2\right)
$$

$$
= \frac{|V_0^*|^2}{2 Z_0} - \frac{|V_0^* \Gamma_L|^2}{2 Z_0}
$$

$$
= \frac{|V_0^*|^2}{2 Z_0} - \frac{|V_0^*|^2}{2 Z_0}
$$

The two terms in above expression have a very definite **physical meaning**. The first term is the time-averaged **power of the wave** propagating along the transmission line **toward the load**.
We say that this wave is **incident** on the load:

\[
P_{\text{inc}} = P_+ = \frac{|V_0^+|^2}{2Z_0}
\]

Likewise, the second term of the \( P_{\text{abs}} \) equation describes the power of the wave moving in the other direction (away from the load). We refer to this as the wave **reflected** from the load:

\[
P_{\text{ref}} = P_- = \frac{|V_0^-|^2}{2Z_0} = \frac{|\Gamma_L|^2 |V_0^+|^2}{2Z_0} = |\Gamma_L|^2 P_{\text{inc}}
\]

Thus, the power **absorbed** by the load is simply:

\[
P_{\text{abs}} = P_{\text{inc}} - P_{\text{ref}}
\]

or, rearranging, we find:

\[
P_{\text{inc}} = P_{\text{abs}} + P_{\text{ref}}
\]

This equation is simply an expression of the **conservation of energy**!

It says that power flowing **toward** the load \( P_{\text{inc}} \) is either absorbed by the load \( P_{\text{abs}} \) or reflected back from the load \( P_{\text{ref}} \).
Note that if \( |\Gamma_L|^2 = 1 \), then \( P_{inc} = P_{ref} \), and therefore no power is absorbed by the load.

This of course makes sense!

The magnitude of the reflection coefficient (\( |\Gamma_L| \)) is equal to one only when the load impedance is purely reactive (i.e., purely imaginary).

Of course, a purely reactive element (e.g., capacitor or inductor) cannot absorb any power—all the power must be reflected!

**Return Loss**

The ratio of the reflected power to the incident power is known as return loss. Typically, return loss is expressed in dB:

\[
R.L. = -10 \log_{10} \left[ \frac{P_{ref}}{P_{inc}} \right] = -10 \log_{10} |\Gamma_L|^2
\]
For example, if the return loss is 10dB, then 10% of the incident power is reflected at the load, with the remaining 90% being absorbed by the load—we “lose” 10% of the incident power.

Likewise, if the return loss is 30dB, then 0.1 % of the incident power is reflected at the load, with the remaining 99.9% being absorbed by the load—we “lose” 0.1% of the incident power.

Thus, a larger numeric value for return loss actually indicates less lost power! An ideal return loss would be $\infty$ dB, whereas a return loss of 0 dB indicates that $|\Gamma_L|=1$--the load is reactive!
Consider again the voltage along a terminated transmission line, as a function of position $z$:

$$V(z) = V_0^+ \left[ e^{-j \beta z} + \Gamma_L e^{+j \beta z} \right]$$

Recall this is a complex function, the magnitude of which expresses the magnitude of the sinusoidal signal at position $z$, while the phase of the complex value represents the relative phase of the sinusoidal signal.

Let's look at the magnitude only:

$$|V(z)| = |V_0^+| |e^{-j \beta z} + \Gamma_L e^{+j \beta z}|$$

$$= |V_0^+||e^{-j \beta z}||1 + \Gamma_L e^{+j2 \beta z}|$$

$$= |V_0^+||1 + \Gamma_L e^{+j2 \beta z}|$$

**ICBST** the largest value of $|V(z)|$ occurs at the location $z$ where:

$$\Gamma_L e^{+j2 \beta z} = |\Gamma_L| + j0$$

while the **smallest** value of $|V(z)|$ occurs at the location $z$ where:

$$\Gamma_L e^{+j2 \beta z} = -|\Gamma_L| + j0$$
As a result we can conclude that:

\[ |V(z)|_{\text{max}} = |V_0^+| \left( 1 + |\Gamma_L| \right) \]

\[ |V(z)|_{\text{min}} = |V_0^+| \left( 1 - |\Gamma_L| \right) \]

The ratio of \( |V(z)|_{\text{max}} \) to \( |V(z)|_{\text{min}} \) is known as the Voltage Standing Wave Ratio (VSWR):

\[ \text{VSWR} = \frac{|V(z)|_{\text{max}}}{|V(z)|_{\text{min}}} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \quad : \quad 1 \leq \text{VSWR} \leq \infty \]

Note if \(|\Gamma_L| = 0\) (i.e., \(Z_L = Z_0\)), then VSWR = 1. We find for this case:

\[ |V(z)|_{\text{max}} = |V(z)|_{\text{min}} = |V_0^+| \]

In other words, the voltage magnitude is a constant with respect to position \(z\).

Conversely, if \(|\Gamma_L| = 1\) (i.e., \(Z_L = jX\)), then VSWR = \(\infty\). We find for this case:

\[ |V(z)|_{\text{min}} = 0 \quad \text{and} \quad |V(z)|_{\text{max}} = 2|V_0^+| \]

In other words, the voltage magnitude varies greatly with respect to position \(z\).
As with return loss, VSWR is dependent on the magnitude of $\Gamma_L$ (i.e., $|\Gamma_L|$) only!
The Transmission Coefficient $T$

Consider this circuit:

I.E., a transmission line with characteristic impedance $Z_1$ transitions to a different transmission line at location $z = 0$. This second transmission line has different characteristic impedance $Z_2$ ($Z_1 \neq Z_2$). This second line is terminated with a load $Z_L = Z_2$ (i.e., the second line is matched).

**Q:** What is the voltage and current along each of these two transmission lines? More specifically, what are $V_{01}^+$, $V_{01}^-$, $V_{02}^+$, and $V_{02}^-$?

**A:** Since a source has not been specified, we can only determine $V_{01}^+$, $V_{02}^+$ and $V_{02}^-$ in terms of complex constant $V_{01}^+$. To accomplish this, we must apply a boundary condition at $z = 0$!
We know that the voltage along the first transmission line is:

\[ V_1(z) = V_{01}^+ e^{-j\beta_1 z} + V_{01}^- e^{+j\beta_1 z} \quad \text{[for } z < 0 \text{]} \]

while the current along that same line is described as:

\[ I_1(z) = \frac{V_{01}^+}{Z_1} e^{-j\beta_1 z} - \frac{V_{01}^-}{Z_1} e^{+j\beta_1 z} \quad \text{[for } z < 0 \text{]} \]

We likewise know that the voltage along the second transmission line is:

\[ V_2(z) = V_{02}^+ e^{-j\beta_2 z} + V_{02}^- e^{+j\beta_2 z} \quad \text{[for } z > 0 \text{]} \]

while the current along that same line is described as:

\[ I_2(z) = \frac{V_{02}^+}{Z_2} e^{-j\beta_2 z} - \frac{V_{02}^-}{Z_2} e^{+j\beta_2 z} \quad \text{[for } z > 0 \text{]} \]

Moreover, since the second line is terminated in a matched load, we know that the reflected wave from this load must be zero:

\[ V_2^-(z) = V_{02}^- e^{-j\beta_2 z} = 0 \]
The voltage and current along the second transmission line is thus simply:

\[
V_2(z) = V_2^+(z) = V_{02}^+ e^{-j\beta_2 z} \quad [\text{for } z > 0]
\]

\[
I_2(z) = I_2^+(z) = \frac{V_{02}^+}{Z_2} e^{-j\beta_2 z} \quad [\text{for } z > 0]
\]

At the location where these two transmission lines meet, the current and voltage expressions each must satisfy some specific boundary conditions:

\[
\begin{align*}
Z_1, \beta_1 \\
V_1(0) & \quad V_2(0) \\
I_1(0) & \quad I_2(0) \\
Z_L=Z_2
\end{align*}
\]

The first boundary condition comes from KVL, and states that:

\[
\begin{align*}
V_1(z = 0) &= V_2(z = 0) \\
V_{01}^+ e^{-j\beta_1(0)} + V_{01}^- e^{+j\beta_1(0)} &= V_{02}^+ e^{-j\beta_2(0)} \\
V_{01}^- + V_{01}^+ &= V_{02}^+
\end{align*}
\]
while the second boundary condition comes from KCL, and states that:

\[ I_1(z = 0) = I_2(z = 0) \]

\[ \frac{V_{01}^+}{Z_1} e^{-j\beta_1(0)} - \frac{V_{01}^-}{Z_1} e^{+j\beta_1(0)} = \frac{V_{02}^+}{Z_2} e^{-j\beta_2(0)} \]

\[ \frac{V_{01}^+}{Z_1} - \frac{V_{01}^-}{Z_1} = \frac{V_{02}^+}{Z_2} \]

We now have two equations and two unknowns (\( V_{01}^- \) and \( V_{02}^+ \))! We can solve for each in terms of \( V_{01}^+ \) (i.e., the incident wave).

From the first boundary condition we can state:

\[ V_{01}^- = V_{02}^+ - V_{01}^+ \]

Inserting this into the second boundary condition, we find an expression involving only \( V_{02}^+ \) and \( V_{01}^+ \):

\[ \frac{V_{01}^+}{Z_1} - \frac{V_{01}^-}{Z_1} = \frac{V_{02}^+}{Z_2} \]

\[ \frac{V_{01}^+}{Z_1} - \frac{V_{02}^+ - V_{01}^+}{Z_1} = \frac{V_{02}^+}{Z_2} \]

\[ \frac{2V_{01}^+}{Z_1} = \frac{V_{02}^+}{Z_2} + \frac{V_{02}^+}{Z_1} \]

Solving this expression, we find:

\[ V_{02}^+ = \left( \frac{2Z_2}{Z_1 + Z_2} \right) V_{01}^+ \]
We can therefore define a transmission coefficient, which relates $V_{02}^+$ to $V_{01}^+$:

\[ T_0 = \frac{V_{02}^+}{V_{01}^+} = \frac{2Z_2}{Z_1 + Z_2} \]

Meaning that $V_{02}^+ = T V_{01}^+$, and thus:

\[ V_2(z) = V_2^+(z) = T V_{01}^+ e^{-j\beta z} \quad \text{[for } z > 0\text{]} \]

We can likewise determine the constant $V_{01}^-$ in terms of $V_{01}^+$. We again start with the first boundary condition, from which we concluded:

\[ V_{02}^+ = V_{01}^+ + V_{01}^- \]

We can insert this into the second boundary condition, and determine an expression involving $V_{01}^-$ and $V_{01}^+$ only:

\[ \frac{V_{01}^+}{Z_1} - \frac{V_{01}^-}{Z_1} = \frac{V_{02}^+}{Z_2} \]

\[ \frac{V_{01}^+}{Z_1} - \frac{V_{01}^-}{Z_1} = \frac{V_{01}^+ + V_{01}^-}{Z_2} \]

\[ \left( \frac{1}{Z_1} - \frac{1}{Z_2} \right) V_{01}^+ = \left( \frac{1}{Z_1} + \frac{1}{Z_2} \right) V_{01}^- \]
Solving this expression, we find:

\[ V_{01}^- = \left( \frac{Z_2 - Z_1}{Z_2 + Z_1} \right) V_{01}^+ \]

We can therefore define a reflection coefficient, which relates \( V_{01}^- \) to \( V_{01}^+ \):

\[ \Gamma_0 = \frac{V_{01}^-}{V_{01}^+} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \]

This result should not surprise us!

Note that because the second transmission line is matched, its input impedance is equal to \( Z_1 \):

\[ Z_{in} = Z_2 \]
and thus we can equivalently write the entire circuit as:

\[ I(z) = V_1(z) Z_1, \beta_1 \]

\[ \Gamma_0 = \frac{Z_2 - Z_1}{Z_2 + Z_1} \]

Which is exactly the same result as we determined earlier!

The values of the reflection coefficient \( \Gamma_0 \) and the transmission coefficient \( T_0 \) are not independent, but in fact are directly related. Recall the first boundary expressed was:

\[ V_{01}^+ + V_{01}^- = V_{02}^+ \]

Dividing this by \( V_{01}^+ \):

\[ 1 + \frac{V_{01}^-}{V_{01}^+} = \frac{V_{02}^+}{V_{01}^+} \]
Since $\Gamma_0 = \nu_{01}/\nu_{01}^*$ and $T_0 = \nu_{02}^*/\nu_{01}^*$:

\[1 + \Gamma_0 = T_0\]

Note the result $T_0 = 1 + \Gamma_0$ is true for this particular circuit, and therefore is not a universally valid expression for two-port networks!
Example: Applying Boundary Conditions

Consider this circuit:

I.E., Two transmissions of identical characteristic impedance are connected by a series impedance $Z_L$. This second line is eventually terminated with a load $Z_L = Z_0$ (i.e., the second line is matched).

Q: What is the voltage and current along each of these two transmission lines? More specifically, what are $V_{01}^+, V_{01}^-, V_{02}^+$ and $V_{02}^-$?

A: Since a source has not been specified, we can only determine $V_{01}^-, V_{02}^+$ and $V_{02}^-$ in terms of complex constant $V_{01}^+$. To accomplish this, we must apply a boundary conditions at the end of each line!
\( z_1 < 0 \)

We know that the voltage along the **first** transmission line is:

\[
V_1(z_1) = V_{01}^+ e^{-j \beta z_1} + V_{01}^- e^{+j \beta z_1} \quad [\text{for } z_1 < 0]
\]

while the **current** along that same line is described as:

\[
I_1(z_1) = \frac{V_{01}^+}{Z_0} e^{-j \beta z_1} - \frac{V_{01}^-}{Z_0} e^{+j \beta z_1} \quad [\text{for } z_1 < 0]
\]

\( z_2 > 0 \)

We likewise know that the voltage along the **second** transmission line is:

\[
V_2(z_2) = V_{02}^+ e^{-j \beta z_2} + V_{02}^- e^{+j \beta z_2} \quad [\text{for } z_2 > 0]
\]

while the **current** along that same line is described as:

\[
I_2(z_2) = \frac{V_{02}^+}{Z_0} e^{-j \beta z_2} - \frac{V_{02}^-}{Z_0} e^{+j \beta z_2} \quad [\text{for } z_2 > 0]
\]

Moreover, since the second line is terminated in a **matched** load, we know that the **reflected** wave from this load must be zero:

\[
V_2^-(z_2) = V_{02}^- e^{-j \beta z_2} = 0
\]
The voltage and current along the **second** transmission line is thus simply:

\[ V_2(z_2) = V_2^+(z_2) = V_{02}^+ e^{-j\beta z_2} \quad \text{[for } z_2 > 0\text{]} \]

\[ I_2(z_2) = I_2^+(z_2) = \frac{V_{02}^+}{Z_2} e^{-j\beta z_2} \quad \text{[for } z_2 > 0\text{]} \]

At the location where these two transmission lines meet, the current and voltage expressions each **must** satisfy some specific boundary conditions:

![Diagram](image)

The **first** boundary condition comes from **KVL**, and states that:

\[ V_1(z = 0) - I_L Z_L = V_2(z = 0) \]

\[ V_{01}^+ e^{-j\beta(0)} + V_{01}^- e^{+j\beta(0)} - I_L Z_L = V_{02}^+ e^{-j\beta(0)} \]

\[ V_{01}^+ + V_{01}^- - I_L Z_L = V_{02}^- \]
the second boundary condition comes from **KCL**, and states that:

\[
I_1(z = 0) = I_L
\]

\[
\frac{V_{01}^+}{Z_0} e^{-j\beta (0)} - \frac{V_{01}^-}{Z_0} e^{+j\beta (0)} = I_L
\]

\[
V_{01}^+ - V_{01}^- = Z_0 I_L
\]

while the third boundary condition likewise comes from **KCL**, and states that:

\[
I_L = I_2(z = 0)
\]

\[
I_L = \frac{V_{02}^+}{Z_0} e^{-j\beta (0)}
\]

\[
Z_0 I_L = V_{02}^+
\]

Finally, we have Ohm's Law:

\[
V_L = Z_L I_L
\]

Note that we now have four equations and four unknowns \((V_{01}^+, V_{02}^+, V_L, I_L)! We can solve for each in terms of \(V_{01}^+\) (i.e., the incident wave).

For example, let's determine \(V_{02}^+\) (in terms of \(V_{01}^+\)). We combine the first and second boundary conditions to determine:

\[
V_{01}^+ + V_{01}^- - I_L Z_L = V_{02}^+
\]

\[
V_{01}^+ + (V_{01}^+ - Z_0 I_L) - I_L Z_L = V_{02}^+
\]

\[
2V_{01}^+ - I_L (Z_0 + Z_L) = V_{02}^+
\]
And then adding in the **third** boundary condition:

\[
2V_{01}^+ - I_L (Z_0 + Z_L) = V_{02}^+
\]

\[
2V_{01}^+ - \frac{V_{02}^+}{Z_0} (Z_0 + Z_L) = V_{02}^+
\]

\[
2V_{01}^+ = V_{02}^+ \left( \frac{2Z_0 + Z_L}{Z_0} \right)
\]

Thus, we find that \( V_{02}^+ = T_0 V_{01}^+ \):

\[
T_0 = \frac{V_{02}^+}{V_{01}^+} = \frac{2Z_0}{2Z_0 + Z_L}
\]

Now let’s determine \( V_{01}^- \) (in terms of \( V_{01}^+ \)).

**Q:** Why are you wasting our time? Don’t we already know that \( V_{01}^- = \Gamma_0 V_{01}^+ \), where:

\[
\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0}
\]

**A:** Perhaps. Humor me while I continue with our boundary condition analysis.

We combine the **first** and **third** boundary conditions to determine:
\[ V_{01}^+ + V_{01}^- - I_L Z_L = V_{02}^+ \]
\[ V_{01}^+ + V_{01}^- - I_L Z_L = Z_0 I_L \]
\[ V_{01}^+ + V_{01}^- = I_L (Z_0 + Z_L) \]

And then adding the second boundary condition:

\[ V_{01}^+ + V_{01}^- = I_L (Z_0 + Z_L) \]
\[ V_{01}^+ + V_{01}^- = \left( \frac{V_{01}^+ - V_{01}^-}{Z_0} \right) (Z_0 + Z_L) \]
\[ V_{01}^+ \left( \frac{Z_L}{Z_0} \right) = V_{01}^- \left( \frac{2Z_0 + Z_L}{Z_0} \right) \]

Thus, we find that \( V_{01}^- = \Gamma_0 V_{01}^+ \), where:

\[ \Gamma_0 \doteq \frac{V_{01}^-}{V_{01}^+} = \frac{Z_L}{Z_L + 2Z_0} \]

Note this is not the expression:

\[ \Gamma_0 \neq \frac{Z_L - Z_0}{Z_L + Z_0} \]

This is a completely different problem than the transmission line simply terminated by load \( Z_L \). Thus, the results are likewise different. This shows that you must always carefully consider the problem you are attempting to solve, and guard against using “shortcuts” with previously derived expressions that may be inapplicable.
This is why you must know why a correct answer is correct!

Q: But, isn't there some way to solve this using our previous work?

A: Actually, there is!

An alternative way for finding $\Gamma_0 = \frac{V_0^-}{V_0^+}$ is to determine the input impedance at the end of the first transmission line:

Note that since the second line is (eventually) terminated in a matched load, the input impedance at the beginning of the second line is simply equal to $Z_0$. 

$$Z_{in} = Z_0$$
Thus, the equivalent circuit becomes:

\[ Z_{in} = Z_L + Z_0 \]

And it is apparent that:

As far as the first section of transmission line is concerned, it is terminated in a load with impedance \( Z_L + Z_0 \). The current and voltage along this first transmission line is precisely the same as if it actually were!
Thus, we find that $\Gamma_0 = \nu_{01}^-/\nu_{01}^+$, where:

$$
\Gamma_0 = \frac{Z(\tau_1 = 0) - Z_0}{Z(\tau_1 = 0) + Z_0} = \frac{(Z_L + Z_0) - Z_0}{(Z_L + Z_0) + Z_0} = \frac{Z_L}{Z_L + 2Z_0}
$$

Precisely the same result as before!

Now, one more point. Recall we found in an earlier handout that $T_0 = 1 + \Gamma_0$. But for this example we find that this statement is not valid:

$$
1 + \Gamma_0 = \frac{2(Z_L + Z_0)}{Z_L + 2Z_0} \neq T_0
$$

Again, be careful when analyzing microwave circuits!

**Q:** But this seems so difficult. How will I know if I have made a mistake?

**A:** An important engineering tool that you must master is commonly referred to as the “sanity check”. 
Simply put, a sanity check is simply thinking about your result, and determining whether or not it makes sense. A great strategy is to set one of the variables to a value so that the physical problem becomes trivial—so trivial that the correct answer is obvious to you. Then make sure your results likewise provide this obvious answer!

For example, consider the problem we just finished analyzing. Say that the impedance $Z_L$ is actually a short circuit ($Z_L=0$). We find that:

$$\Gamma_0 = \left. \frac{Z_L}{Z_L + 2Z_0} \right|_{Z_L=0} = 0$$
$$T_0 = \left. \frac{2Z_0}{2Z_0 + Z_L} \right|_{Z_L=0} = 1$$

Likewise, consider the case where $Z_L$ is actually an open circuit ($Z_L=\infty$). We find that:

$$\Gamma_0 = \left. \frac{Z_L}{Z_L + 2Z_0} \right|_{Z_L=\infty} = 1$$
$$T_0 = \left. \frac{2Z_0}{2Z_0 + Z_L} \right|_{Z_L=\infty} = 0$$

Think about what these results mean in terms of the physical problem:
Q: Do these results make sense? Have we passed the sanity check?

A: I'll let you decide! What do you think?