2.5 - The Quarter-Wave Transformer

Reading Assignment: pp. 73-76

By now you’ve noticed that a quarter-wave length of transmission line \((\ell = \lambda/4, 2\beta\ell = \pi)\) appears often in microwave engineering problems.

\[\text{HO: The Quarter-Wave Transformer}\]

Q: Why does the quarter-wave matching network work—after all, the quarter-wave line is mismatched at both ends?

A: \textit{HO: Multiple Reflection Viewpoint}
The Quarter-Wave Transformer

Say the end of a transmission line with characteristic impedance $Z_0$ is terminated with a resistive (i.e., real) load.

We typically would like all power traveling down the line to be absorbed by the load $R_L$.

But if $R_L \neq Z_0$, the line is unmatched and some of the incident power will be reflected.

**Q:** Can all incident power be delivered to a resistive load if $R_L \neq Z_0$??

**A:** Yes! We can insert a matching network between the transmission line and the load.
A matching network is a **lossless**, **2-port** device. Its job is to **transform** the load \( R_L \) (or even \( Z_L \)) to a value \( Z_0 \).

In other words, we want the **input** impedance of the matching network to be \( Z_{in} = Z_0 \), so that \( \Gamma_{in} = 0 -- \text{no reflection!} \)

Since **none** of the incident power is **reflected**, and none is **absorbed** by the lossless matching network, it **all** must be absorbed by the **load** \( R_L \)!

**Q:** These matching networks sound too good to be true. Exactly how do we **build** them?

**A:** There are **many** methods and ways, but perhaps the easiest is the **quarter-wave transformer**.

First, insert a transmission line with characteristic impedance \( Z_1 \) and length \( \ell = \lambda/4 \) (i.e., a quarter-wave line) **between** the load and the \( Z_0 \) transmission line.

![Diagram of quarter wave transformer](image)

The \( \lambda/4 \) line is the **matching network**!
**Q:** But what about the characteristic impedance $Z_1$; what **should** its value be??

**A:** Remember, the quarter wavelength case is one of the **special** cases that we studied. We know that the **input** impedance of the quarter wavelength line is:

$$Z_{in} = \frac{(Z_1)^2}{Z_L} = \frac{(Z_1)^2}{R_L}$$

Thus, if we wish for $Z_{in}$ to be numerically equal to $Z_0$, we find:

$$Z_{in} = \frac{(Z_1)^2}{R_L} = Z_0$$

Solving for $Z_L$, we find its **required** value to be:

$$(Z_1)^2 / R_L = Z_0$$

$$(Z_1)^2 = Z_0 R_L$$

$$Z_1 = \sqrt{Z_0 R_L}$$

In other words, the characteristic impedance of the quarter wave line is the **geometric average** $Z_0$ and $R_L$!
Therefore, a $\lambda/4$ line with characteristic impedance $Z_1 = \sqrt{Z_0 R_L}$ will match a transmission line with characteristic impedance $Z_0$ to a resistive load $R_L$.

Thus, all power is delivered to load $R_L$!

**Important Note:** We find that $Z_{in} = Z_0$ only if the matching if the quarter-wave transmission line is exactly one-quarter wavelength in length $\ell = \lambda/4$.

The problem with this, of course, is that a physical length $\ell$ of transmission line is exactly one-quarter wavelength at only one frequency $f$!

Remember, wavelength is related to frequency as:

$$\lambda = \frac{\nu_p}{f} = \frac{1}{f \sqrt{LC}}$$

where $\nu_p$ is the propagation velocity of the wave.
For example, assuming that $v_p = c$ ($c$ = the speed of light in a vacuum), one wavelength at 1 GHz is 30 cm ($\lambda = 0.3 \text{ m}$), while one wavelength at 3 GHz is 10 cm ($\lambda = 0.1 \text{ m}$). As a result, a transmission line length $\ell = 7.5 \text{ cm}$ is a quarter wavelength for a signal at 1 GHz only.

Thus, a quarter-wave transformer provides a perfect match ($\Gamma_{in} = 0$) at one and only one signal frequency!
Multiple Reflection Viewpoint

The quarter-wave transformer brings up an interesting question in μ-wave engineering.

Q: Why is there no reflection at \( z = -\ell \)? It appears that the line is mismatched at both \( z = 0 \) and \( z = -\ell \).

A: In fact there are reflections at these mismatched interfaces—an infinite number of them!

First, let's define a few terms:
\( \Gamma_1 = \text{partial reflection coefficient of a wave incident on the } z = -\ell \text{ interface from the } Z_0 \text{ line:} \)

\[
\Gamma_1 = \frac{Z_1 - Z_0}{Z_1 + Z_0}
\]

\( \Gamma_2 = \text{partial reflection coefficient of a wave incident on the } z = -\ell \text{ interface from the } Z_1 \text{ line:} \)

\[
\Gamma_2 = \frac{Z_0 - Z_1}{Z_0 + Z_1} = -\Gamma_1
\]

\( \Gamma_3 = \text{partial reflection coefficient of a wave incident on the } z = 0 \text{ interface from the } Z_1 \text{ line:} \)

\[
\Gamma_3 = \frac{R_L - Z_1}{R_L + Z_1}
\]
$T_1 = \text{partial transmission coefficient of a wave incident on the } z = -\ell \text{ interface from the } Z_0 \text{ line:}$

$$T_1 = \frac{2Z_1}{Z_1 + Z_0}$$

$T_2 = \text{partial transmission coefficient of a wave incident on the } z = -\ell \text{ interface from the } Z_1 \text{ line:}$

$$T_2 = \frac{2Z_0}{Z_0 + Z_1}$$

Now let's try to intemperate what \textbf{physically} happens when the \textbf{incident} voltage wave:

$$V^i e^{-j\beta z}$$

reaches the interface at $z = -\ell$. 

STOP
1. At $z = -\ell$, the characteristic impedance of the transmission line changes from $Z_0$ to $Z_1$. This mismatch creates a reflected wave:

\[ V^i e^{-j\beta z} \]

\[ V_1^r e^{+j\beta z} \]

where $V_1^r = \Gamma_1 V^i$.

2. However, a portion of the incident wave is transmitted ($T_1$) across the interface at $z = -\ell$, this wave travels a distance of $\beta \ell = 90^\circ$ to the load at $z = 0$, where a portion of it is reflected ($\Gamma_3$). This wave travels back $\beta \ell = 90^\circ$ to the interface at $z = -\ell$, where a portion is again transmitted ($T_2$) across into the $Z_0$ transmission line—another reflected wave ($V_2^r$)!

\[ V^i e^{-j\beta z} \]

\[ V_2^r e^{+j\beta z} \]

where we have found that traveling $2\beta \ell = 180^\circ$ has produced a minus sign in our result:

\[ V_2^r = T_2 e^{-j90^\circ} \Gamma_3 e^{-j90^\circ} T_1 V^i \]

\[ = -T_1 T_2 \Gamma_3 V^i \]
3. However, a portion of this second wave is also reflected ($\Gamma_2$) back into the $Z_1$ transmission line at $z = -\ell$, where it again travels to $\beta\ell = 90^\circ$ the load, is partially reflected ($\Gamma_3$), travels $\beta\ell = 90^\circ$ back to $z = -\ell$, and is partially transmitted into $Z_0$ ($T_2$)—our third reflected wave!

![Diagram]

where:

$$V_3^r = T_2 e^{-j90^\circ} \Gamma_3 e^{-j90^\circ} \Gamma_2 e^{-j90^\circ} \Gamma_3 e^{-j90^\circ} T_1 V' = T_1 T_2 (\Gamma_3)^2 \Gamma_2 V'$$

$n.$ We can see that this “bouncing” back and forth can go on forever, with each trip launching a new reflected wave into the $Z_0$ transmission line.

Note however, that the power associated with each successive reflected wave is smaller than the previous, and so eventually, the power associated with the reflected waves will diminish to insignificance!

**Q:** *But, why then is $\Gamma = 0$?*
A: Each reflected wave $V_n^r$ is a coherent wave. That is, they all oscillate at same frequency $\omega$; the reflected waves differ only in terms of their magnitude and phase.

Therefore, to determine the total reflected wave, we must perform a coherent summation of each reflected wave—a operation easily performed since we have expressed our waves with complex notation:

$$V^r e^{j \beta z} = \sum_{n=1}^{\infty} V_n^r e^{j \beta z}$$

It can be shown that this infinite series converges, with the result:

$$V^r = \left( \frac{\Gamma_1 + \Gamma_1 \Gamma_2 \Gamma_3 - T_1 T_2 \Gamma_3}{1 + \Gamma_2 \Gamma_3} \right) V^i$$

Thus, the total reflection coefficient is:

$$\Gamma = \frac{V^r}{V^i} = \frac{\Gamma_1 + \Gamma_1 \Gamma_2 \Gamma_3 - T_1 T_2 \Gamma_3}{1 + \Gamma_2 \Gamma_3}$$

Using our definitions, it can likewise be shown that the numerator of the above expression is:

$$\Gamma_1 + \Gamma_1 \Gamma_2 \Gamma_3 - T_1 T_2 \Gamma_3 = \frac{2(Z_1^2 - Z_0 R_L)}{(Z_1 + Z_0)(R_L + Z_1)}$$
It is evident that the numerator (and therefore $\Gamma$) will be zero if:

$$Z_1^2 - Z_0 R_L \Rightarrow Z_1 = \sqrt{Z_0 R_L}$$

Just as we expected!

Physically, this results insure that all the reflected waves add coherently together to produce a zero value!

A simple example of this phenomenon is the addition of two waves with equal magnitude and opposite phase (i.e., their phase difference is $180^\circ$).

$$\cos(\omega t) + \cos(\omega t + 180^\circ) = \cos(\omega t) - \cos(\omega t) = 0$$

Note all of our transmission line analysis has been steady-state analysis. We assume our signals are sinusoidal, of the form $\exp(j \omega t)$. Note this signal exists for all time $t$—the signal is assumed to have been “on” forever, and assumed to continue on forever.

In other words, in steady-state analysis, all the multiple reflections have long since occurred, and thus have reached a steady state—the reflected wave is zero!