4.2 – The Scattering Matrix

Reading Assignment: ***pp. 174-183***

Admittance and Impedance matricies use the quantities $I(z)$, $V(z)$, and $Z(z)$ (or $Y(z)$).

Q: *Is there an **equivalent** matrix for transmission line activity expressed in terms of $V^+(z)$, $V^-(z)$, and $\Gamma(z)$?*

A:

HO: The Scattering Matrix

Q: *Can we likewise determine something **physical** about our device or network by simply **looking** at its scattering matrix?*

A: HO: Matched, Reciprocal, Lossless

Example: A Lossless, Reciprocal Device

Q:

A:

Example: Determining the Scattering Matrix

Example: The Scattering Matrix

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4.3 – The Scattering Matrix

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Example: A Lossless, Reciprocal Device

Q: *OK, but how can we determine the scattering matrix of a device? And how are scattering parameters useful?*

A:

Example: Determining the Scattering Matrix

Example: The Scattering Matrix

The Scattering Matrix

At **"low"** frequencies, we can completely characterize a **linear** device or network using an **impedance** matrix, which relates the currents and voltages at **each** device terminal to the currents and voltages at **all** other terminals.

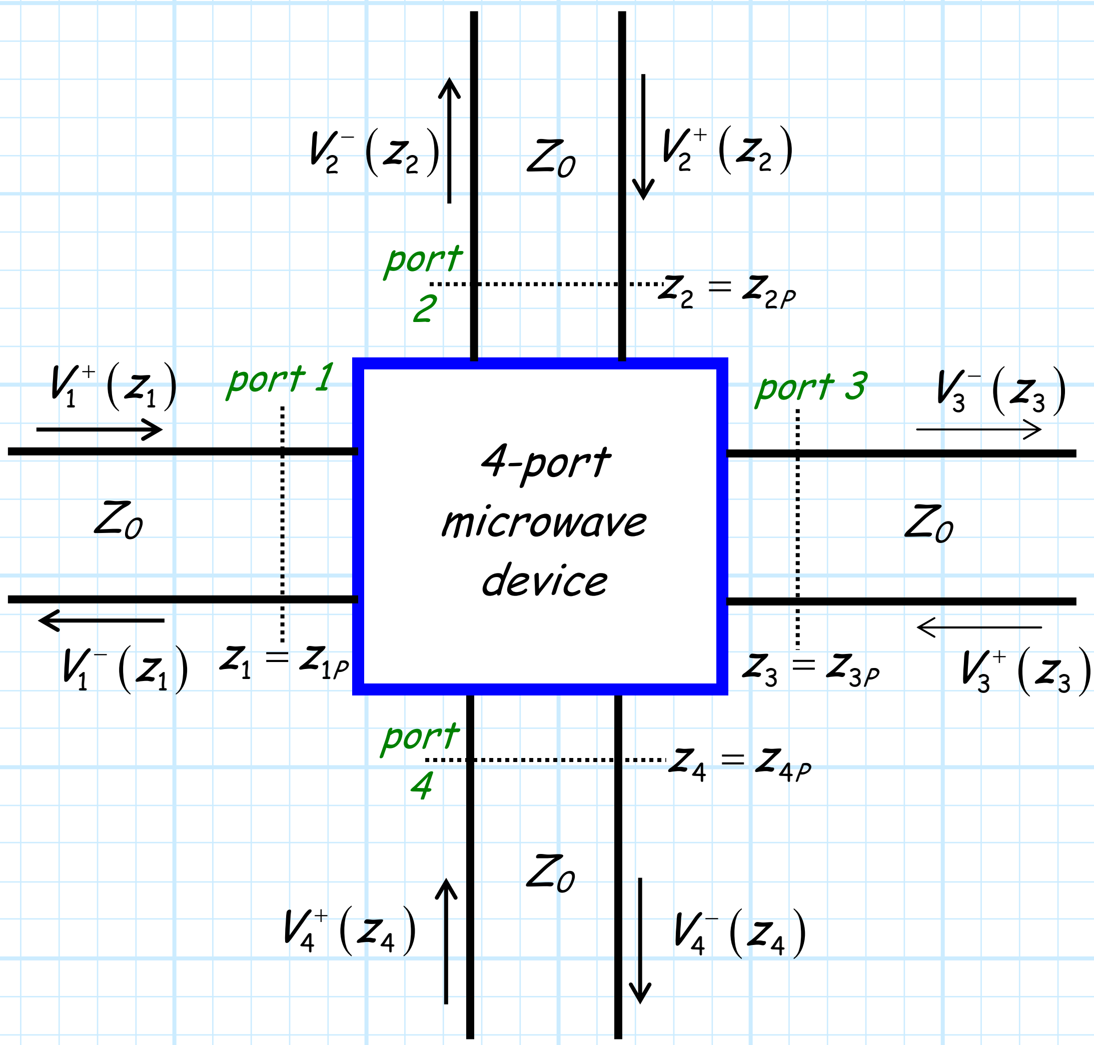
But, at microwave frequencies, it is **difficult** to measure total currents and voltages!

* Instead, we can measure the **magnitude** and **phase** of each of the two transmission line **waves** $V^+(z)$ and $V^-(z)$.

* In other words, we can determine the relationship between the incident and reflected wave at **each** device terminal to the incident and reflected waves at **all** other terminals.

These relationships are completely represented by the **scattering matrix**. It **completely** describes the behavior of a linear, multi-port device at a **given frequency** ω, and a given line impedance Z_0.

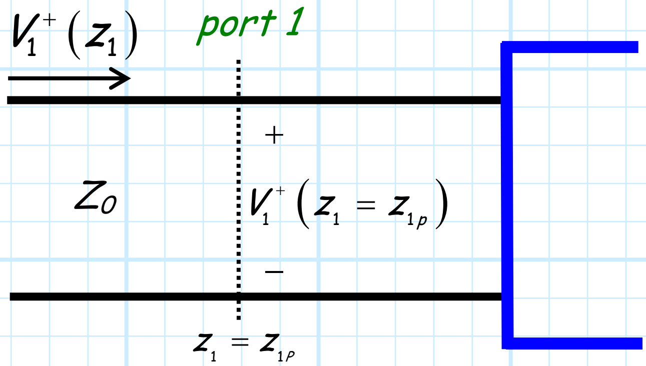
Consider now the **4-port** microwave device shown below:



Note that we have now characterized transmission line activity in terms of incident and "reflected" waves. Note the negative going "reflected" waves can be viewed as the waves **exiting** the multi-port network or device.

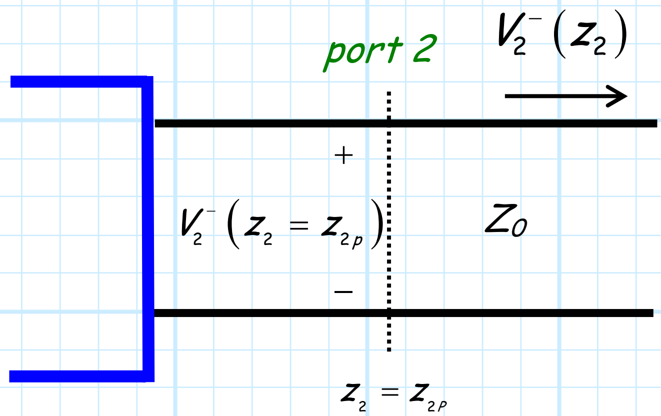
→ Viewing transmission line activity this way, we can fully characterize a multi-port device by its **scattering parameters**!

Say there exists an **incident** wave on **port 1** (i.e., $V_1^+(z_1) \neq 0$), while the incident waves on all other ports are known to be **zero** (i.e., $V_2^+(z_2) = V_3^+(z_3) = V_4^+(z_4) = 0$).



Say we measure/determine the voltage of the wave flowing **into port 1**, at the port 1 **plane** (i.e., determine $V_1^+(z_1 = z_{1P})$).

Say we then measure/determine the voltage of the wave flowing **out** of **port 2**, at the port 2 plane (i.e., determine $V_2^-(z_2 = z_{2P})$).



The complex ratio between $V_1^+(z_1 = z_{1P})$ and $V_2^-(z_2 = z_{2P})$ is know as the **scattering parameter** S_{21}:

$$S_{21} = \frac{V_2^-(z_2 = z_{2P})}{V_1^+(z_1 = z_{1P})} = \frac{V_{02}^- \, e^{+j\beta z_{2P}}}{V_{01}^+ \, e^{-j\beta z_{1P}}} = \frac{V_{02}^-}{V_{01}^+} \; e^{+j\beta(z_{2P}+z_{1P})}$$

Likewise, the scattering parameters S_{31} and S_{41} are:

$$S_{31} = \frac{V_3^-(z_3 = z_{3P})}{V_1^+(z_1 = z_{1P})} \qquad \text{and} \qquad S_{41} = \frac{V_4^-(z_4 = z_{4P})}{V_1^+(z_1 = z_{1P})}$$

We of course could **also** define, say, scattering parameter S_{34} as the ratio between the complex values $V_4^+(z_4 = z_{4P})$ (the wave **into** port 4) and $V_3^-(z_3 = z_{3P})$ (the wave **out of** port 3), given that the input to all other ports (1,2, and 3) are zero.

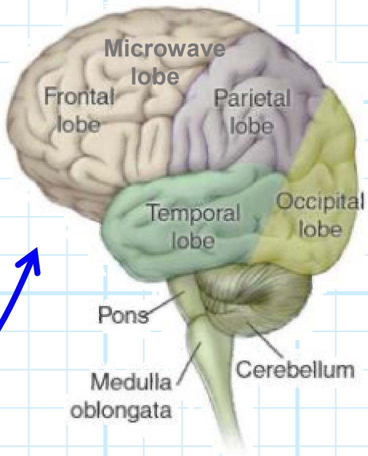
Thus, more **generally**, the ratio of the wave incident on port *n* to the wave emerging from port *m* is:

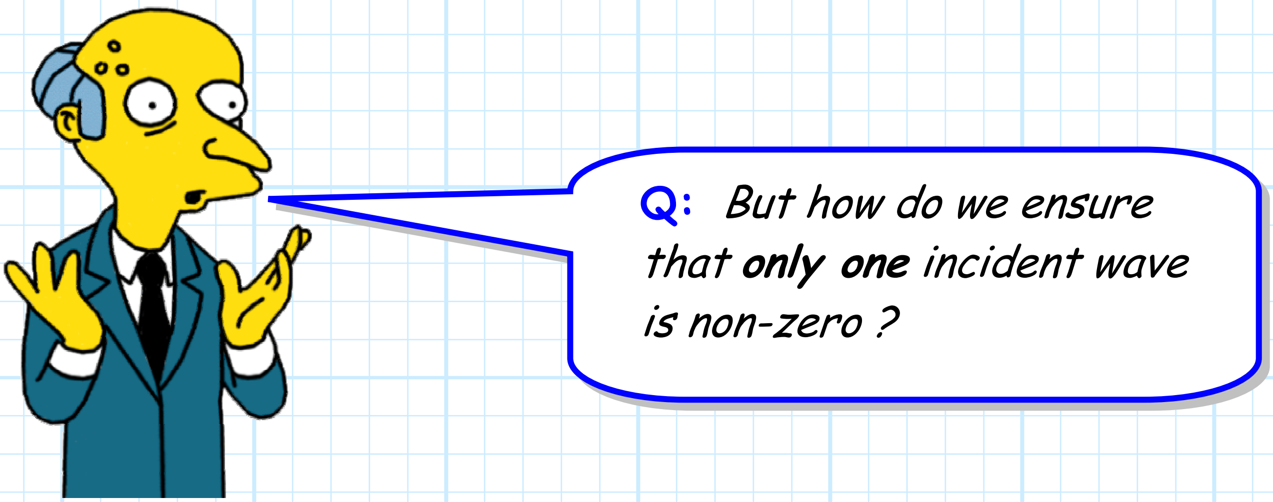
$$S_{mn} = \frac{V_m^-(z_m = z_{mP})}{V_n^+(z_n = z_{nP})} \qquad \text{(given that} \quad V_k^+(z_k) = 0 \;\; \text{for all } k \neq n\text{)}$$

Note that frequently the port positions are assigned a **zero** value (e.g., $z_{1P} = 0$, $z_{2P} = 0$). This of course **simplifies** the scattering parameter calculation:

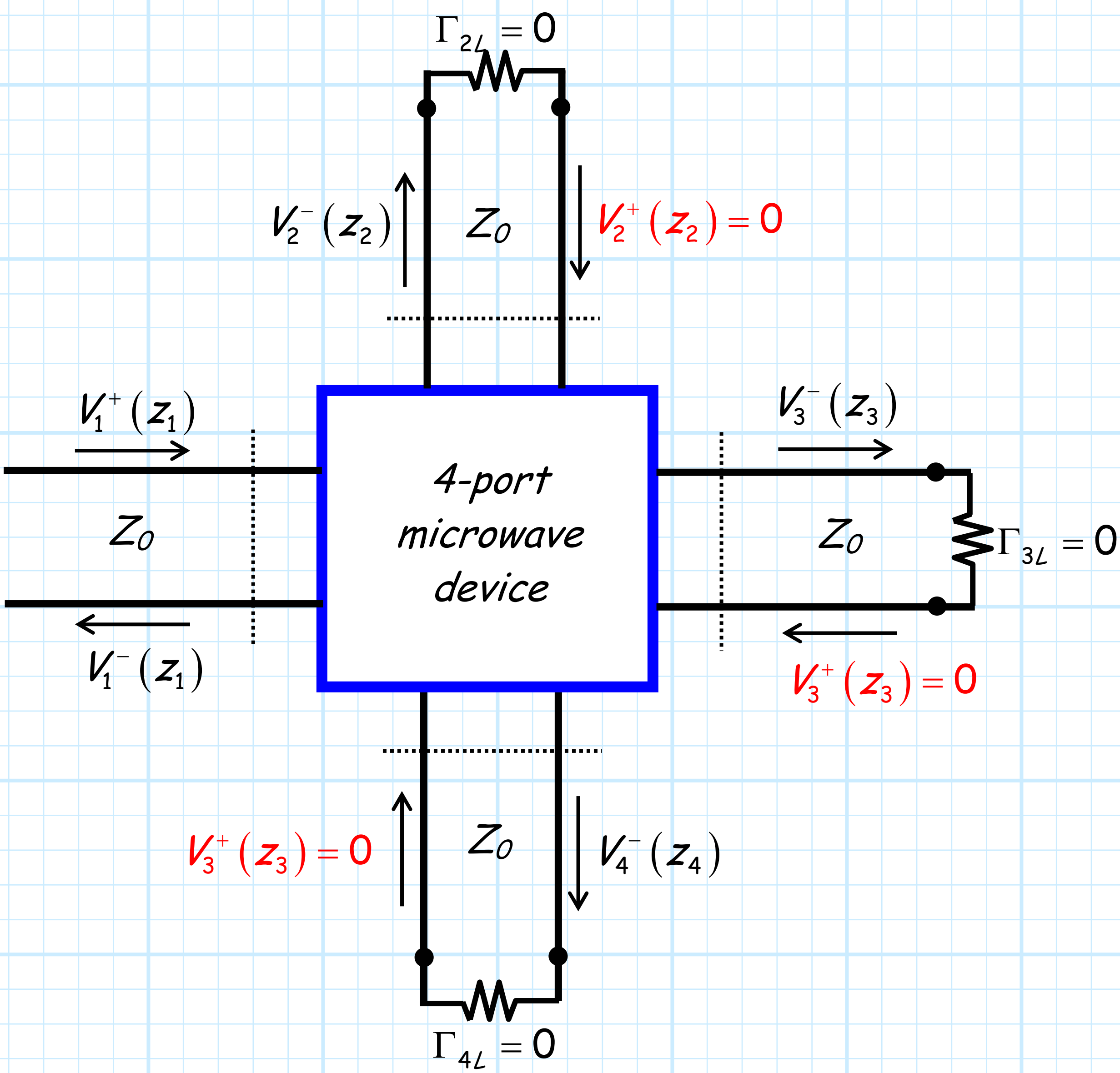
$$S_{mn} = \frac{V_m^-(z_m = 0)}{V_n^+(z_n = 0)} = \frac{V_{0m}^- e^{+j\beta 0}}{V_{0n}^+ e^{-j\beta 0}} = \frac{V_{0m}^-}{V_{0n}^+}$$

We will **generally assume** that the port locations are defined as $z_{nP} = 0$, and thus use the **above** notation. But **remember** where this expression came from!





A: **Terminate** all other ports with a **matched load**!



Note that **if** the ports are terminated in a **matched load** (i.e., $Z_L = Z_0$), then $\Gamma_{nL} = 0$ and therefore:

$$V_n^+(z_n) = 0$$

In other words, terminating a port ensures that there will be **no signal** incident on that port!

Q: *Just between you and me, I think you've messed this up!* ***In all*** *previous handouts you said that if* $\Gamma_L = 0$*, the wave in the* ***minus*** *direction would be zero:*

$$V^-(z) = 0 \quad \text{if} \quad \Gamma_L = 0$$

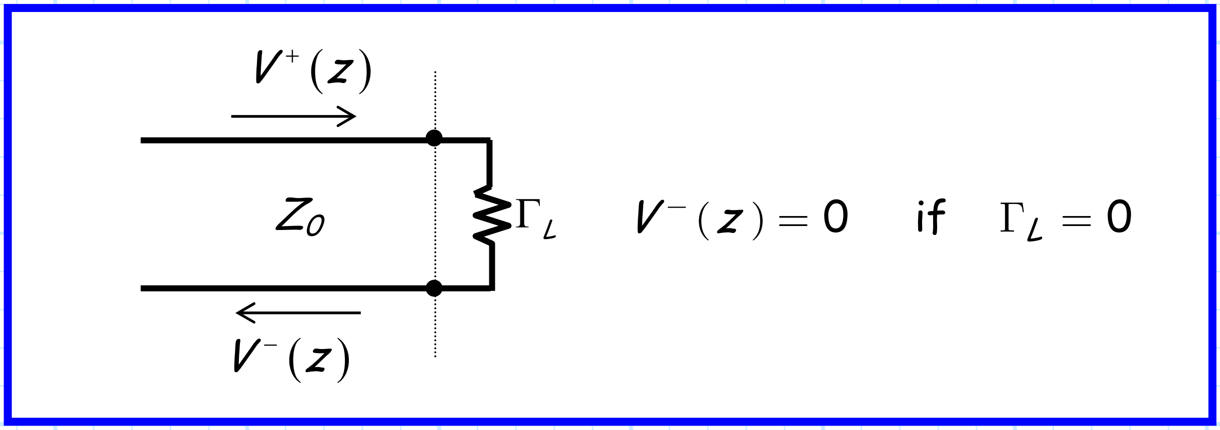
but just ***now*** *you said that the wave in the* ***positive*** *direction would be zero:*

$$V^+(z) = 0 \quad \text{if} \quad \Gamma_L = 0$$

Of course, there is ***no way*** *that* ***both*** *statements can be correct!*

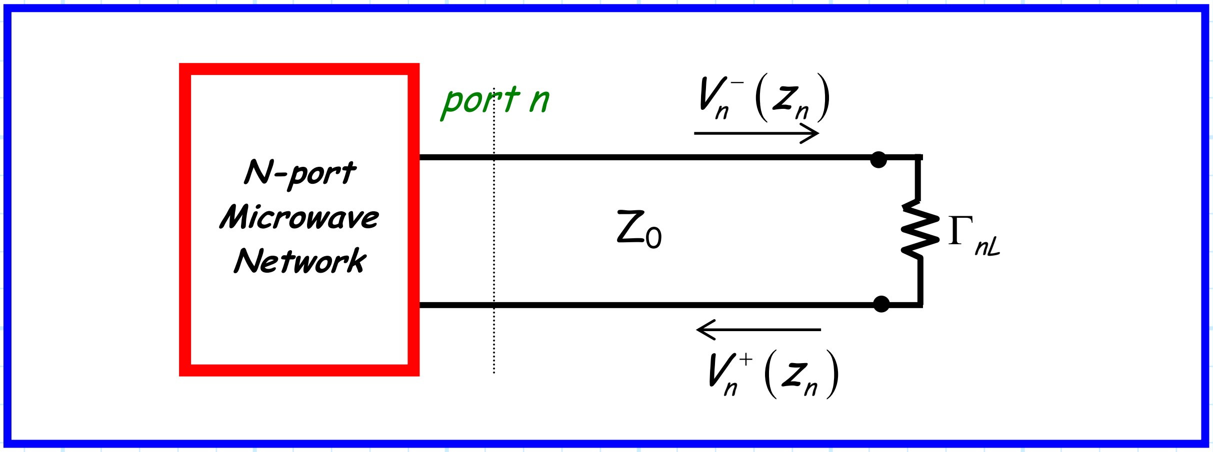
A: Actually, **both** statements are correct! You must be careful to understand the **physical definitions** of the plus and minus directions—in other words, the propagation directions of waves $V_n^+(z_n)$ and $V_n^-(z_n)$!

For example, we **originally** analyzed this case:



In this original case, the wave **incident** on the load is $V^+(z)$ (**plus** direction), while the **reflected** wave is $V^-(z)$ (**minus** direction).

Contrast this with the case we are **now** considering:

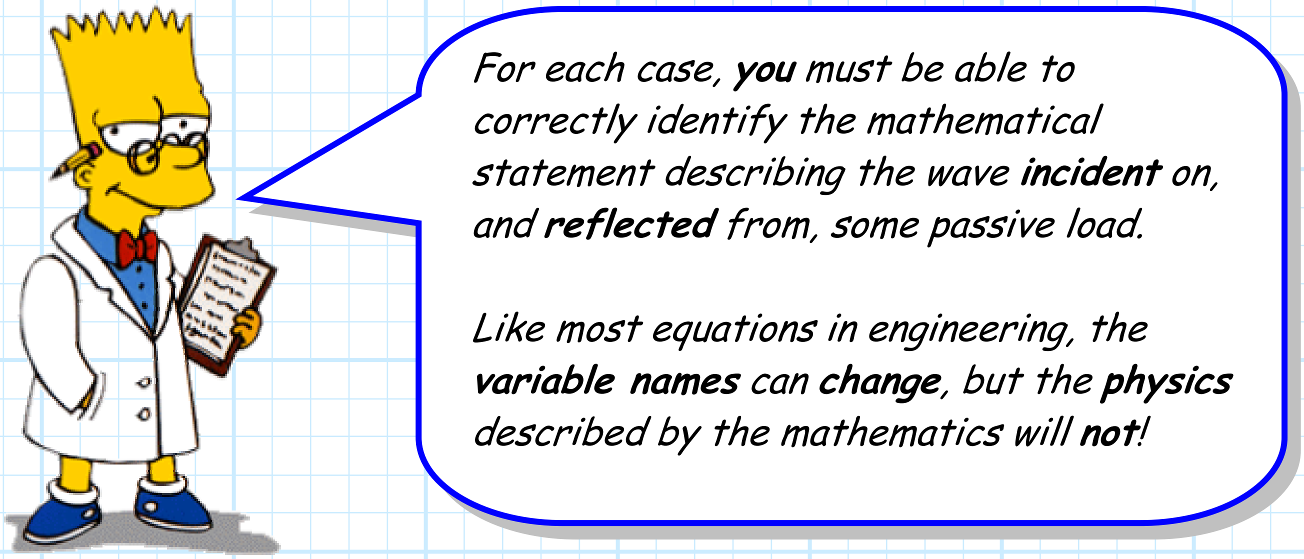


For this current case, the situation is **reversed**. The wave incident on the load is **now** denoted as $V_n^-(z_n)$ (coming **out** of port n), while the wave reflected off the load is **now** denoted as $V_n^+(z_n)$ (going **into** port n).

As a result, $V_n^+(z_n) = 0$ when $\Gamma_{nL} = 0$!

Perhaps we could more **generally** state that for some load Γ_L:

$$V^{reflected}(z = z_L) = \Gamma_L \, V^{incident}(z = z_L)$$



Now, **back** to our discussion of **S-parameters**. We found that **if** $z_{nP} = 0$ for all ports n, the scattering parameters could be directly written in terms of wave **amplitudes** V_{0n}^{+} and V_{0m}^{-}.

$$S_{mn} = \frac{V_{0m}^{-}}{V_{0n}^{+}} \qquad \text{(given that } V_k^{+}(z_k) = 0 \text{ for all } k \neq n\text{)}$$

Which we can now **equivalently** state as:

$$S_{mn} = \frac{V_{0m}^{-}}{V_{0n}^{+}} \qquad \text{(given that all ports, except port } n\text{, are } \textbf{matched}\text{)}$$

One more **important** note—notice that for the **matched** ports (i.e., those ports with **no** incident wave), the voltage of the exiting **wave** is also the **total** voltage!

$$\begin{aligned} V_m(z_m) &= V_{0m}^{+}\, e^{-j\beta z_n} + V_{0m}^{-}\, e^{+j\beta z_n} \\ &= 0 + V_{0m}^{-}\, e^{+j\beta z_m} \\ &= V_{0m}^{-}\, e^{+j\beta z_m} \qquad \text{(for all terminated ports)} \end{aligned}$$

Thus, the value of the exiting wave **at** each terminated **port** is likewise the value of the total voltage **at** those ports:

$$\begin{aligned} V_m(0) &= V_{0m}^{+} + V_{0m}^{-} \\ &= 0 + V_{0m}^{-} \\ &= V_{0m}^{-} \qquad \text{(for all terminated ports)} \end{aligned}$$

And so, we can express **some** of the scattering parameters equivalently as:

$$S_{mn} = \frac{V_m(0)}{V_{0n}^{+}} \qquad \text{(for \textbf{matched} port } m\text{, i.e., for } m \neq n)$$

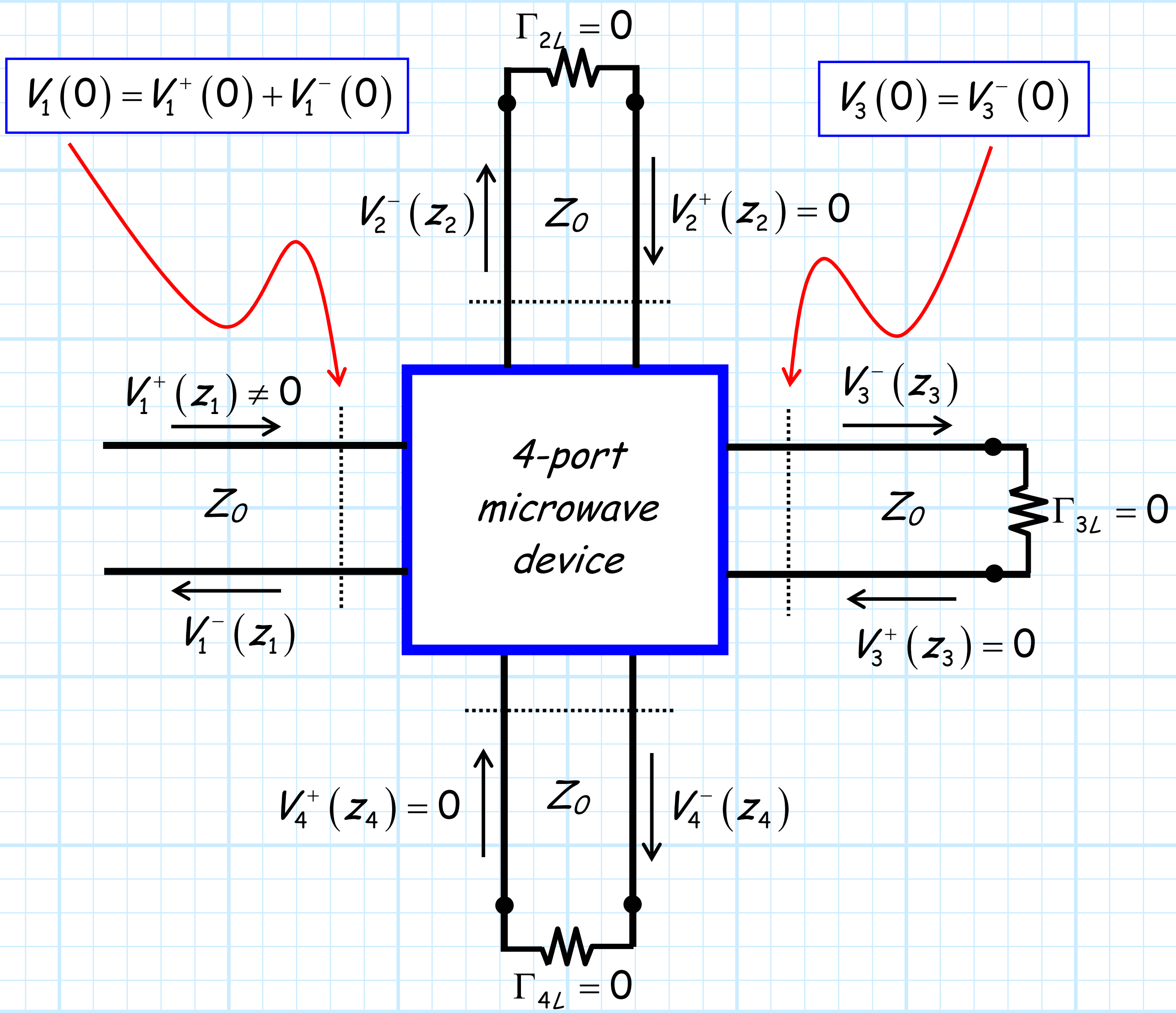
You might find this result **helpful** if attempting to determine scattering parameters where $m \neq n$ (e.g., S_{21}, S_{43}, S_{13}), as we can often use traditional **circuit theory** to easily determine the **total** port voltage $V_m(0)$.

However, we **cannot** use the expression above to determine the scattering parameters when $m=n$ (e.g., S_{11}, S_{22}, S_{33}).

Think about this! The scattering parameters for these cases are:

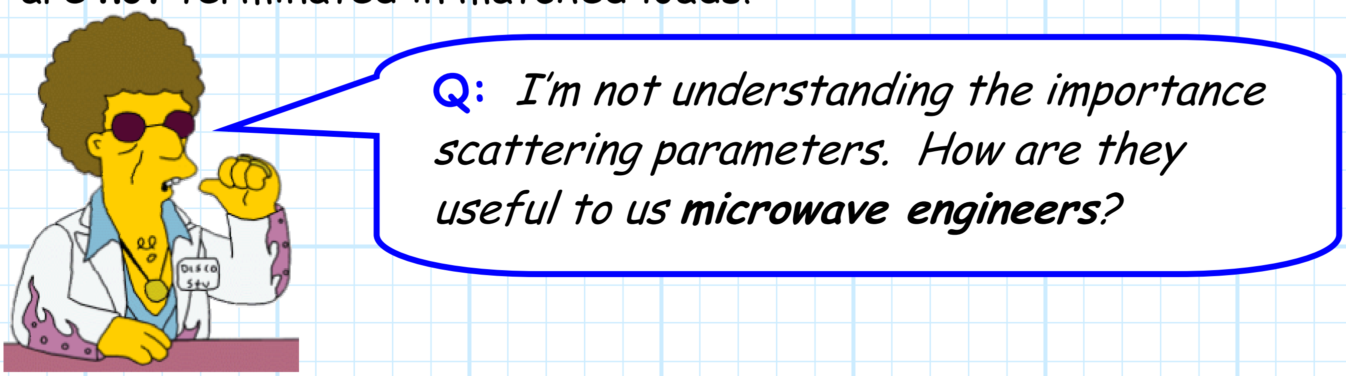
$$S_{nn} = \frac{V_{0n}^-}{V_{0n}^+}$$

Therefore, port n is a port where there actually **is** some incident wave V_{0n}^+ (port n is **not** terminated in a matched load!). And thus, the total voltage is **not** simply the value of the exiting wave, as **both** an incident wave and exiting wave exists at port n.



Typically, it is **much** more difficult to determine/measure the scattering parameters of the form S_{nn}, as opposed to scattering parameters of the form S_{mn} (where $m \neq n$) where there is **only** an **exiting** wave from port m!

We can use the scattering matrix to determine the solution for a more **general** circuit—one where the ports are **not** terminated in matched loads!



A: Since the device is **linear**, we can apply **superposition**. The output at any port due to **all** the incident waves is simply the coherent **sum** of the output at that port due to **each** wave!

For example, the **output** wave at port 3 can be determined by (assuming $z_{nP} = 0$):

$$V_{03}^{-} = S_{34} V_{04}^{+} + S_{33} V_{03}^{+} + S_{32} V_{02}^{+} + S_{31} V_{01}^{+}$$

More **generally**, the output at port m of an N-port device is:

$$V_{0m}^{-} = \sum_{n=1}^{N} S_{mn} V_{0n}^{+} \qquad (z_{nP} = 0)$$

This expression can be written in **matrix** form as:

$$\bar{\mathbf{V}}^- = \bar{\bar{\mathbf{S}}}\,\bar{\mathbf{V}}^+$$

Where $\bar{\mathbf{V}}^-$ is the **vector**:

$$\bar{\mathbf{V}}^- = \left[V_{01}^-, V_{02}^-, V_{03}^-, \ldots, V_{0N}^-\right]^T$$

and $\bar{\mathbf{V}}^+$ is the vector:

$$\bar{\mathbf{V}}^+ = \left[V_{01}^+, V_{02}^+, V_{03}^+, \ldots, V_{0N}^+\right]^T$$

Therefore $\bar{\bar{\mathbf{S}}}$ is the **scattering matrix**:

$$\bar{\bar{\mathbf{S}}} = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{m1} & \cdots & S_{mn} \end{bmatrix}$$

The scattering matrix is a N by N matrix that **completely characterizes** a linear, N-port device. Effectively, the scattering matrix describes a multi-port device the way that Γ_L describes a single-port device (e.g., a load)!

But **beware**! The values of the scattering matrix for a particular device or network, just like Γ_L, are **frequency dependent**! Thus, it may be more instructive to **explicitly** write:

$$\bar{\bar{\mathbf{S}}}(\omega) = \begin{bmatrix} S_{11}(\omega) & \cdots & S_{1n}(\omega) \\ \vdots & \ddots & \vdots \\ S_{m1}(\omega) & \cdots & S_{mn}(\omega) \end{bmatrix}$$

Also realize that—also just like Γ_L—the scattering matrix is dependent on **both** the **device/network** and the Z_0 value of the **transmission lines connected** to it.

Thus, a device connected to transmission lines with $Z_0 = 50\Omega$ will have a **completely different scattering matrix** than that same device connected to transmission lines with $Z_0 = 100\Omega$!!!

Matched, Lossless, Reciprocal Devices

As we discussed earlier, a device can be **lossless** or **reciprocal**. In addition, we can likewise classify at being **matched**.

Let's examine **each** of these three characteristics, and how they relate to the **scattering matrix**.

Matched

A matched device is another way of saying that the **input impedance** at each port is **equal to** Z_0 when **all other** ports are terminated in matched loads. As a result, the **reflection coefficient** of each port is **zero**—no signal will be come out of a port if a signal is incident on that port (and only that port).

In other words, we want:

$$V_m^- = S_{mm} V_m^+ = 0 \quad \text{for all } m$$

a result that occurs when:

$$S_{mm} = 0 \quad \text{for all } m \text{ if matched}$$

We find therefore that a matched device will exhibit a scattering matrix where all **diagonal elements** are **zero**.

Therefore:

$$\bar{\bar{\mathcal{S}}} = \begin{bmatrix} 0 & 0.1 & j0.2 \\ 0.1 & 0 & 0.3 \\ j0.2 & 0.3 & 0 \end{bmatrix}$$

is an example of a scattering matrix for a **matched,** three port device.

Lossless

Recall for a lossless device, all of the power that delivered to each device port must eventually finds its way **out**!

In other words, power is not **absorbed** by the network—no power to be **converted to heat**!

Consider, for example, a **four-port** device. Say a signal is incident on port 1, and that **all** other ports are **terminated**. The power **incident** on port 1 is therefore:

$$P_1^+ = \frac{\left|V_1^+\right|^2}{2Z_0}$$

while the power **leaving** the device at each port is:

$$P_m^- = \frac{|V_m^-|^2}{2Z_0} = \frac{|S_{m1}V_1^-|^2}{2Z_0} = |S_{m1}|^2 P_1^+$$

The **total** power leaving the device is therefore:

$$\begin{aligned} P_{out} &= P_1^- + P_2^- + P_3^- + P_4^- \\ &= |S_{11}|^2 P_1^+ + |S_{21}|^2 P_1^+ + |S_{31}|^2 P_1^+ + |S_{41}|^2 P_1^+ \\ &= \left(|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 + |S_{41}|^2\right) P_1^+ \end{aligned}$$

Note therefore that if the device is **lossless**, the output power will be **equal** to the input power, i.e., $P_{out} = P_1^+$. This is true **only** if:

$$|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 + |S_{41}|^2 = 1$$

If the device is lossless, this will likewise be true for each of the **other** ports:

$$\begin{aligned} |S_{12}|^2 + |S_{22}|^2 + |S_{32}|^2 + |S_{42}|^2 &= 1 \\ |S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 + |S_{43}|^2 &= 1 \\ |S_{14}|^2 + |S_{24}|^2 + |S_{34}|^2 + |S_{44}|^2 &= 1 \end{aligned}$$

We can state in general then:

$$\sum_{m=1}^{N} |S_{mn}|^2 = 1 \quad \text{for all } n$$

In fact, it can be shown that a lossless device will have a **unitary** scattering matrix, i.e.:

$$\bar{\bar{\mathbf{S}}}^H \bar{\bar{\mathbf{S}}} = \bar{\bar{\mathbf{I}}} \qquad \text{if lossless}$$

where H indicates **conjugate transpose** and $\bar{\bar{\mathbf{I}}}$ is the **identity** matrix.

The columns of a unitary matrix form an **orthonormal set**—that is, the **magnitude** of each column is 1 (as shown above) and dissimilar column vector are mutually **orthogonal**. In other words, the inner product (i.e., dot product) of dissimilar vectors is zero:

$$\sum_{n=1}^{N} S_{1i} S_{1j}^* = S_{1i} S_{1j}^* + S_{2i} S_{2j}^* + \cdots + S_{Ni} S_{Nj}^* = 0 \quad \text{for all } i \neq j$$

An **example** of a (unitary) scattering matrix for a **lossless** device is:

$$\bar{\bar{\mathbf{S}}} = \begin{bmatrix} 0 & \frac{1}{2} & j\frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 0 & 0 & j\frac{\sqrt{3}}{2} \\ j\frac{\sqrt{3}}{2} & 0 & 0 & \frac{1}{2} \\ 0 & j\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Reciprocal

Recall **reciprocity** results when we build a **passive** (i.e., unpowered) device with **simple** materials.

For a reciprocal network, we find that the elements of the scattering matrix are **related** as:

$$S_{mn} = S_{nm}$$

For example, a **reciprocal** device will have $S_{21} = S_{12}$ or $S_{32} = S_{23}$. We can write reciprocity in matrix form as:

$$\bar{\bar{\mathbf{S}}}^T = \bar{\bar{\mathbf{S}}} \qquad \text{if reciprocal}$$

where T indicates (non-conjugate) transpose.

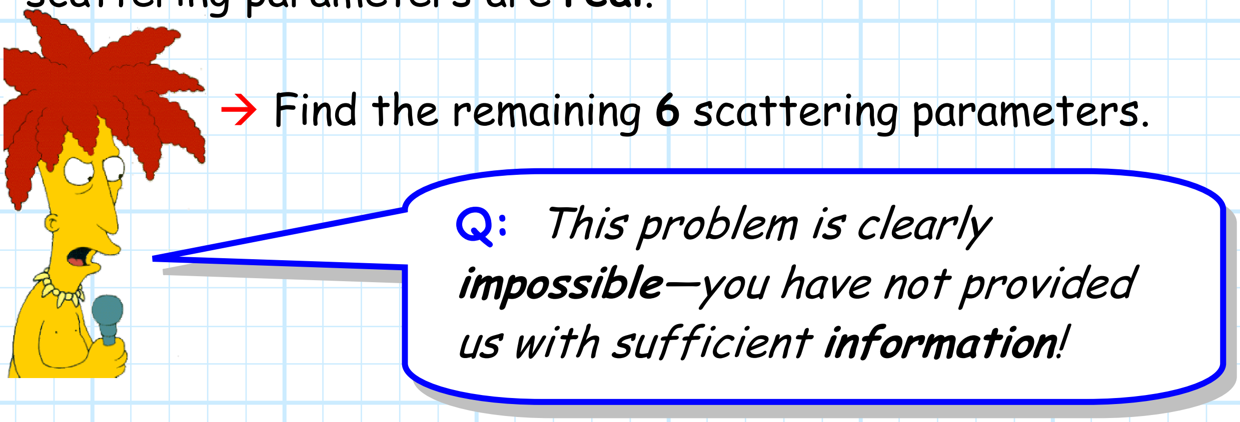
An **example** of a scattering matrix describing a **reciprocal**, but **lossy** and **non-matched** device is:

$$\bar{\bar{\mathbf{S}}} = \begin{bmatrix} 0.10 & -0.40 & -j0.20 & 0.05 \\ -0.40 & j0.20 & 0 & j0.10 \\ -j0.20 & 0 & 0.10 - j0.30 & -0.12 \\ 0.05 & j0.10 & -0.12 & 0 \end{bmatrix}$$

Example: A Lossless, Reciprocal Network

A **lossless**, **reciprocal** 3-port device has $\mathcal{S}$-parameters of $S_{11} = 1/2$, $S_{31} = 1/\sqrt{2}$, and $S_{33} = 0$. It is likewise known that all scattering parameters are **real**.

→ Find the remaining **6** scattering parameters.



A: Yes I have! Note I said the device was **lossless** and **reciprocal**!

Start with what we **currently** know:

$$\bar{\bar{\mathbf{S}}} = \begin{bmatrix} \frac{1}{2} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ \frac{1}{\sqrt{2}} & S_{32} & 0 \end{bmatrix}$$

Because the device is **reciprocal**, we then also know:

$$S_{21} = S_{12} \qquad S_{13} = S_{31} = \frac{1}{\sqrt{2}} \qquad S_{32} = S_{23}$$

And therefore:

$$\bar{\bar{\mathbf{S}}} = \begin{bmatrix} 1/2 & S_{21} & 1/\sqrt{2} \\ S_{21} & S_{22} & S_{32} \\ 1/\sqrt{2} & S_{32} & 0 \end{bmatrix}$$

Now, since the device is **lossless**, we know that:

$$\begin{aligned} 1 &= |S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 \\ &= (1/2)^2 + |S_{21}|^2 + \left(1/\sqrt{2}\right)^2 \end{aligned}$$

Columns have **unit** magnitude.

$$\begin{aligned} 1 &= |S_{12}|^2 + |S_{22}|^2 + |S_{32}|^2 \\ &= |S_{21}|^2 + |S_{22}|^2 + |S_{32}|^2 \end{aligned}$$

$$\begin{aligned} 1 &= |S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 \\ &= (1/2)^2 + |S_{32}|^2 + \left(1/\sqrt{2}\right)^2 \end{aligned}$$

and:

$$\begin{aligned} 0 &= S_{11}S_{12}^* + S_{21}S_{22}^* + S_{31}S_{32}^* \\ &= 1/2\, S_{21}^* + S_{21}S_{22}^* + 1/\sqrt{2}\, S_{32}^* \end{aligned}$$

Columns are **orthogonal**.

$$\begin{aligned} 0 &= S_{11}S_{13}^* + S_{21}S_{23}^* + S_{31}S_{33}^* \\ &= 1/2\left(1/\sqrt{2}\right) + S_{21}S_{32}^* + 1/\sqrt{2}\,(0) \end{aligned}$$

$$\begin{aligned} 0 &= S_{12}S_{13}^* + S_{22}S_{23}^* + S_{32}S_{33}^* \\ &= S_{21}\left(1/\sqrt{2}\right) + S_{22}S_{32}^* + S_{32}\,(0) \end{aligned}$$

These **six** expressions simplify to:

$$\left|S_{21}\right| = \tfrac{1}{2}$$

$$1 = \left|S_{21}\right|^2 + \left|S_{22}\right|^2 + \left|S_{32}\right|^2$$

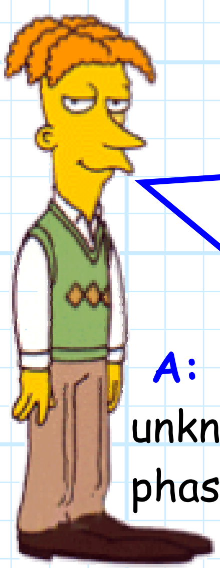
$$\left|S_{32}\right| = \tfrac{1}{\sqrt{2}}$$

$$0 = \tfrac{1}{2} S_{21} + S_{21} S_{22} + \tfrac{1}{\sqrt{2}} S_{32}$$

$$0 = \tfrac{1}{\left(2\sqrt{2}\right)} + S_{21} S_{32}$$

$$0 = S_{21}\left(\tfrac{1}{\sqrt{2}}\right) + S_{22} S_{32}$$

where we have used the fact that since the elements are all **real**, then $S_{21}^* = S_{21}$ (etc.).



Q: *I count the about expressions and find 6 equations yet only a paltry 3 unknowns. Your typical buffoonery appears to have led to an over-constrained condition for which there is* ***no*** *solution!*

A: Actually, we have **six** real equations and **six** real unknowns, since scattering element has a magnitude and phase. In this case we know the values are **real**, and thus the phase is either 0° or 180° (i.e., $e^{j0} = 1$ or $e^{j\pi} = -1$); however, we do not know which one!

From the first three equations, we can find the **magnitudes**:

$$|S_{21}| = \tfrac{1}{2} \qquad |S_{22}| = \tfrac{1}{2} \qquad |S_{32}| = \tfrac{1}{\sqrt{2}}$$

and from the last three equations we find the **phase**:

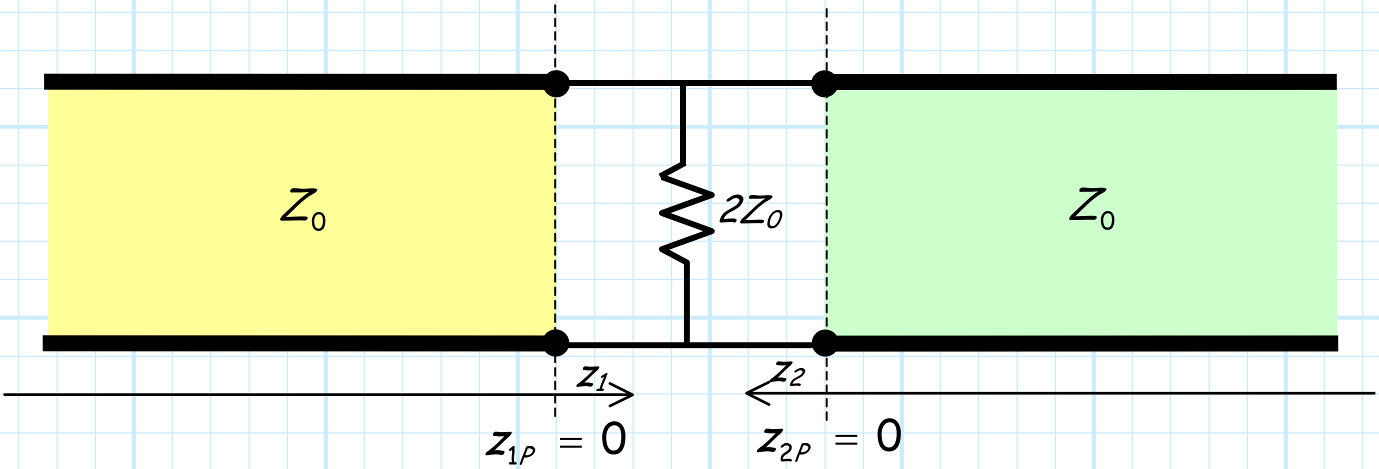
$$S_{21} = \tfrac{1}{2} \qquad S_{22} = \tfrac{1}{2} \qquad S_{32} = -\tfrac{1}{\sqrt{2}}$$

Thus, the scattering matrix for this **lossless**, **reciprocal** device is:

$$\bar{\bar{\mathbf{S}}} = \begin{bmatrix} \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{\sqrt{2}} \\ \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{-1}{\sqrt{2}} \\ \tfrac{1}{\sqrt{2}} & \tfrac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$

Example: Determining the Scattering Matrix

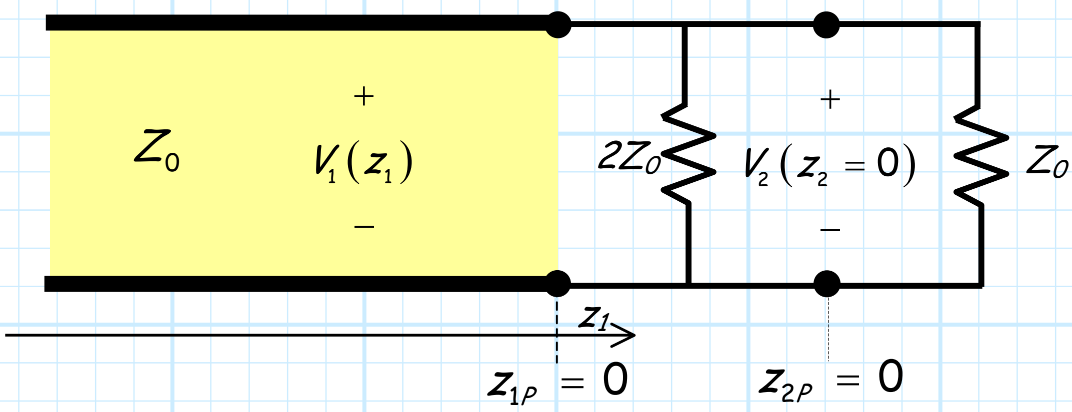
Let's determine the **scattering matrix** of this two-port device:



The **first** step is to terminate port **2** with a **matched** load, and then determine the values:

$$V_1^-(z_1 = z_{P1}) \quad \text{and} \quad V_2^-(z_2 = z_{P2})$$

in terms of $V_1^+(z_1 = z_{P1})$.



Recall that since port 2 is matched, we know that:

$$V_2^+(z_2 = z_{2P}) = 0$$

And thus:

$$\begin{aligned} V_2(z_2 = 0) &= V_2^+(z_2 = 0) + V_2^-(z_2 = 0) \\ &= 0 + V_2^-(z_2 = 0) \\ &= V_2^-(z_2 = 0) \end{aligned}$$

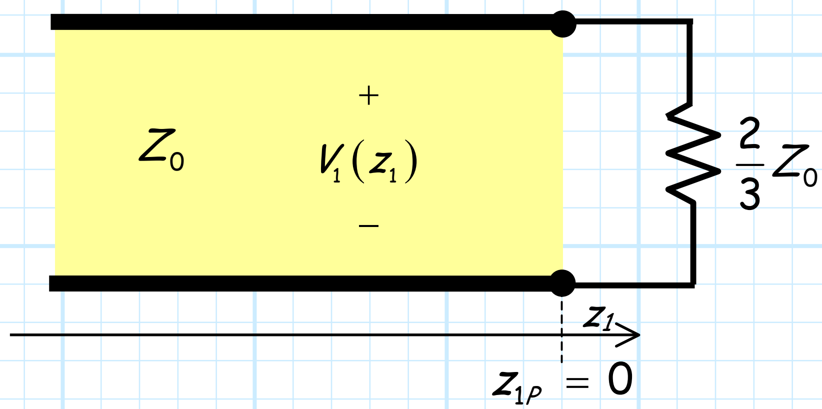
In other words, we **simply** need to determine $V_2(z_2 = 0)$ in order to find $V_2^-(z_2 = 0)$!

However, determining $V_1^-(z_1 = 0)$ is a bit **trickier**. Recall that:

$$V_1(z_1) = V_1^+(z_1) + V_1^-(z_1)$$

Therefore we find $V_1(z_1 = 0) \neq V_1^-(z_1 = 0)$!

Now, we can **simplify** this circuit:



And we know from the **telegraphers equations**:

$$V_1(z_1) = V_1^+(z_1) + V_1^-(z_1)$$
$$= V_{01}^+ e^{-j\beta z_1} + V_{01}^- e^{+j\beta z_1}$$

Since the load $2Z_0/3$ is located at $z_1 = 0$, we know that the **boundary condition** leads to:

$$V_1(z_1) = V_{01}^+ \left(e^{-j\beta z_1} + \Gamma_L e^{+j\beta z_1} \right)$$

where:

$$\begin{aligned}\Gamma_L &= \frac{(2/3) Z_0 - Z_0}{(2/3) Z_0 + Z_0} \\ &= \frac{(2/3) - 1}{(2/3) + 1} \\ &= \frac{-1/3}{5/3} \\ &= -0.2\end{aligned}$$

Therefore:

$$V_1^+(z_1) = V_{01}^+ e^{-j\beta z_1} \quad \text{and} \quad V_1^-(z_1) = V_{01}^+ (-0.2) e^{+j\beta z_1}$$

and thus:

$$V_1^+(z_1 = 0) = V_{01}^+ e^{-j\beta(0)} = V_{01}^+$$

$$V_1^-(z_1 = 0) = V_{01}^+ (-0.2) e^{+j\beta(0)} = -0.2 V_{01}^+$$

We can now determine S_{11}!

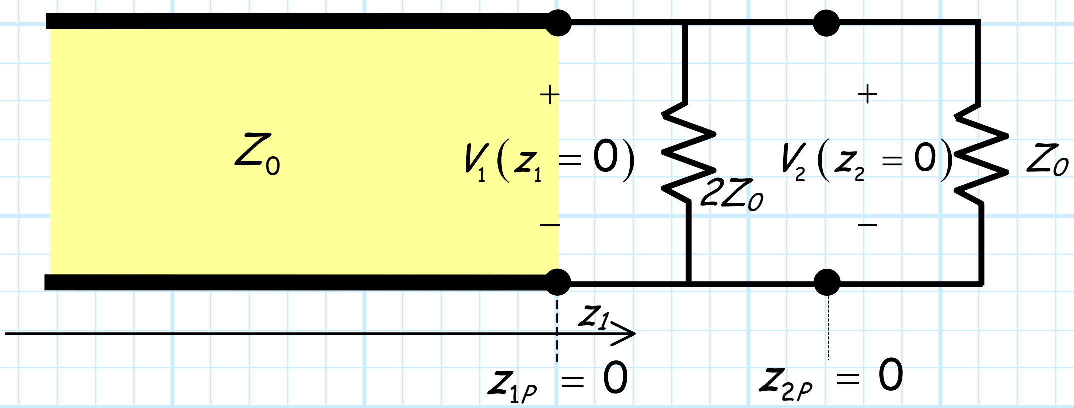
$$S_{11} = \frac{V_1^-(z_1 = 0)}{V_1^+(z_1 = 0)} = \frac{-0.2\, V_{01}^+}{V_{01}^+} = -0.2$$

Now its time to find $V_2^-(z_2=0)$!

Again, since port 2 is terminated, the **incident** wave on port 2 must be **zero**, and thus the value of the **exiting** wave at port 2 is equal to the **total** voltage at port 2:

$$V_2^-(z_2=0)=V_2\ (z_2=0)$$

This **total** voltage is relatively **easy** to determine. Examining the circuit, it is evident that $V_1(z_1=0)=V_2(z_2=0)$.



Therefore:

$$\begin{aligned}V_2(z_2=0)&=V_1(z_1=0)\\&=V_{01}^+\left(e^{-j\beta(0)}-0.2\,e^{+j\beta(0)}\right)\\&=V_{01}^+(1-0.2)\\&=V_{01}^+(0.8)\end{aligned}$$

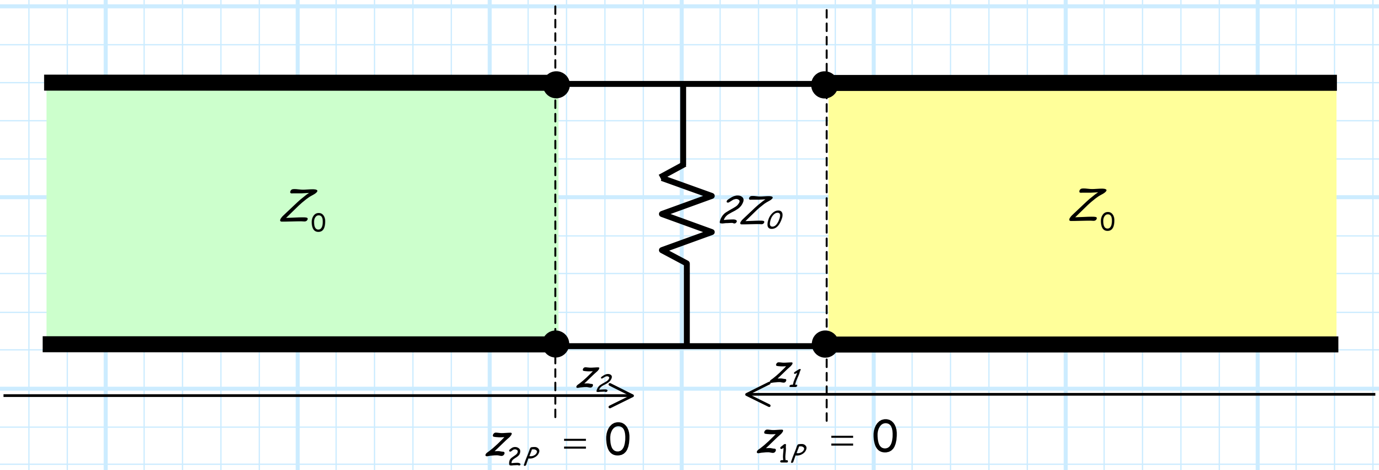
And thus the scattering parameter S_{21} is:

$$S_{21}=\frac{V_2^-(z_2=0)}{V_1^+(z_1=0)}=\frac{0.8\,V_{01}^+}{V_{01}^+}=0.8$$

Now we **just** need to find S_{12} and S_{22}.

Q: *Yikes! This has been an awful lot of work, and you mean that we are only **half-way** done!?*

A: Actually, we are nearly finished! Note that this circuit is **symmetric**—there is really **no** difference between port 1 and port 2. If we "flip" the circuit, it remains **unchanged**!



Thus, we can conclude due to this **symmetry** that:

$$S_{11} = S_{22} = -0.2$$

and:

$$S_{21} = S_{12} = 0.8$$

Note this last equation is **likewise** a result of **reciprocity**.

Thus, the **scattering matrix** for this two port network is:

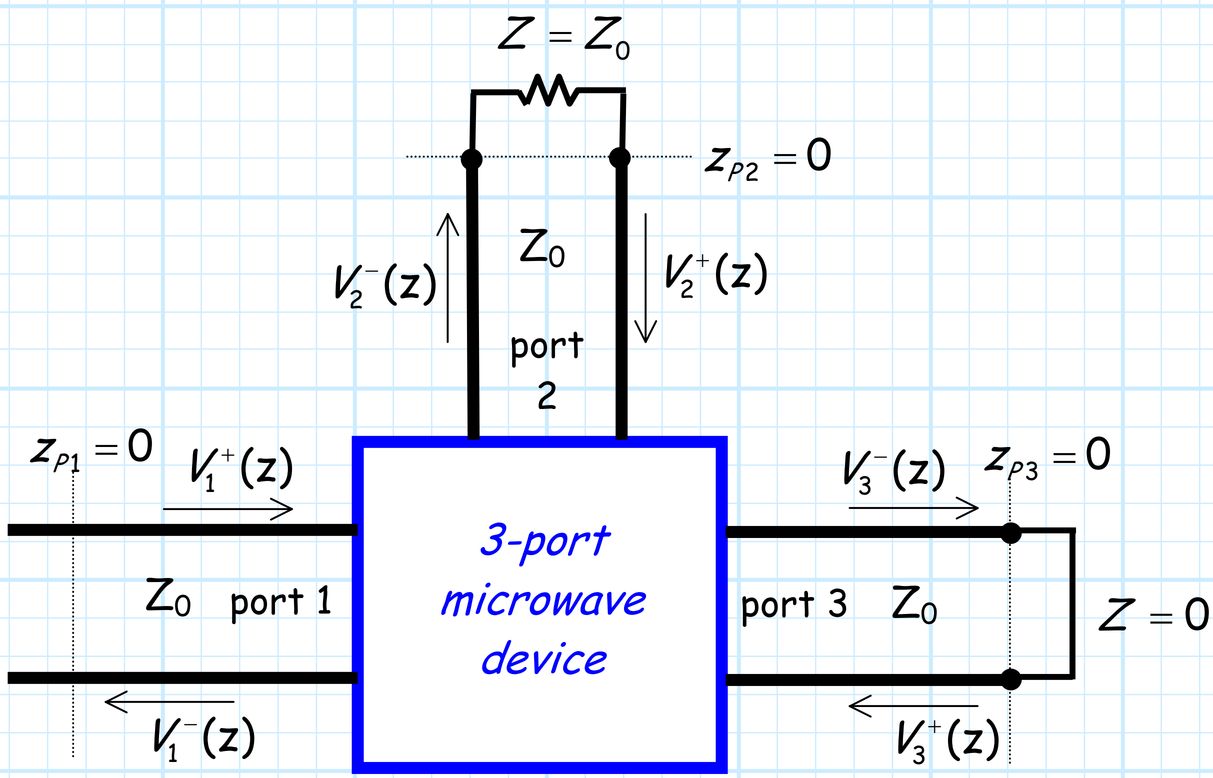
$$\bar{\bar{\mathbf{S}}} = \begin{bmatrix} -0.2 & 0.8 \\ 0.8 & -0.2 \end{bmatrix}$$

Example: The Scattering Matrix

Say we have a 3-port network that is completely characterized at some frequency ω by the **scattering matrix**:

$$\overline{\overline{S}} = \begin{bmatrix} 0.0 & 0.2 & 0.5 \\ 0.5 & 0.0 & 0.2 \\ 0.5 & 0.5 & 0.0 \end{bmatrix}$$

A **matched load** is attached to port 2, while a **short circuit** has been placed at port 3:

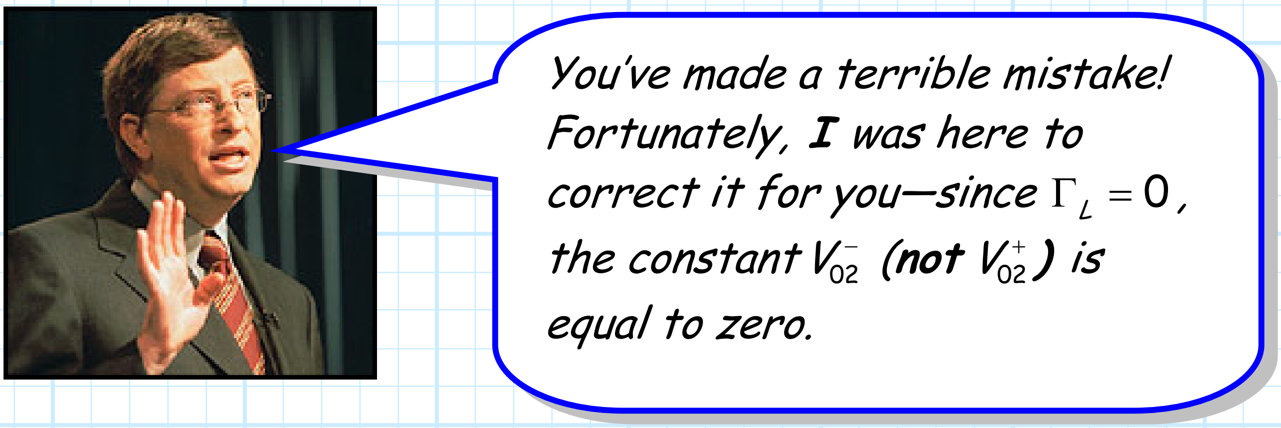


Because of the **matched** load at port 2 (i.e., $\Gamma_L = 0$), we know that:

$$\frac{V_2^+(z_2=0)}{V_2^-(z_2=0)} = \frac{V_{02}^+}{V_{02}^-} = 0$$

and therefore:

$$V_{02}^+ = 0$$



NO!! Remember, the signal $V_2^-(z)$ is **incident** on the matched load, and $V_2^+(z)$ is the **reflected** wave from the load (i.e., $V_2^+(z)$ is incident on **port 2**). Therefore, $V_{02}^+ = 0$ **is correct**!

Likewise, because of the **short** circuit at port 3 ($\Gamma_L = -1$):

$$\frac{V_3^+(z_3=0)}{V_3^-(z_3=0)} = \frac{V_{03}^+}{V_{03}^-} = -1$$

and therefore:

$$V_{03}^+ = -V_{03}^-$$

Problem:

a) Find the **reflection** coefficient at port 1, i.e.:

$$\Gamma_1 \doteq \frac{V_{01}^-}{V_{01}^+}$$

b) Find the **transmission** coefficient from port 1 to port 2, i.e.,

$$T_{21} \doteq \frac{V_{02}^-}{V_{01}^+}$$

*I am amused by the trivial problems that **you** apparently find so difficult. **I** know that:*

$$\Gamma_1 = \frac{V_{01}^-}{V_{01}^+} = S_{11} = 0.0$$

and

$$T_{21} = \frac{V_{02}^-}{V_{01}^+} = S_{21} = 0.5$$

NO!!! The above statement is **not correct**!

Remember, $V_1^- / V_1^+ = S_{11}$ **only** if ports 2 and 3 are terminated in **matched** loads! In this problem port 3 is terminated with a **short circuit**.

Therefore:

$$\Gamma_1 = \frac{V_{01}^-}{V_{01}^+} \neq S_{11}$$

and similarly:

$$T_{21} = \frac{V_{02}^-}{V_{01}^+} \neq S_{21}$$

To determine the values T_{21} and Γ_1, we must start with the **three** equations provided by the **scattering matrix**:

$$V_{01}^- = \qquad 0.2\,V_{02}^+ + 0.5\,V_{03}^+$$

$$V_{02}^- = 0.5\,V_{01}^+ \qquad + 0.2\,V_{03}^+$$

$$V_{03}^- = 0.5\,V_{01}^+ + 0.5\,V_{02}^+$$

and the **two** equations provided by the **attached loads**:

$$V_{02}^+ = 0$$

$$V_{03}^+ = -V_{03}^-$$

We can divide all of these equations by V_{01}^{+}, resulting in:

$$\Gamma_1 = \frac{V_{01}^{-}}{V_{01}^{+}} = \qquad 0.2\,\frac{V_{02}^{+}}{V_{01}^{+}} + 0.5\,\frac{V_{03}^{+}}{V_{01}^{+}}$$

$$T_{21} = \frac{V_{02}^{-}}{V_{01}^{+}} = 0.5 \qquad + 0.2\,\frac{V_{03}^{+}}{V_{01}^{+}}$$

$$\frac{V_{03}^{-}}{V_{01}^{+}} = 0.5 + 0.5\,\frac{V_{02}^{+}}{V_{01}^{+}}$$

$$\frac{V_{02}^{+}}{V_{01}^{+}} = 0$$

$$\frac{V_{03}^{+}}{V_{01}^{+}} = -\frac{V_{03}^{-}}{V_{01}^{+}}$$

Look what we have—**5** equations and **5** unknowns! Inserting equations 4 and 5 into equations 1 through 3, we get:

$$\Gamma_1 = \frac{V_{01}^{-}}{V_{01}^{+}} = -0.5\,\frac{V_{03}^{+}}{V_{01}^{+}}$$

$$T_{21} = \frac{V_{02}^{-}}{V_{01}^{+}} = 0.5 - 0.2\,\frac{V_{03}^{+}}{V_{01}^{+}}$$

$$\frac{V_{03}^{-}}{V_{01}^{+}} = 0.5$$

Solving, we find:

$$\Gamma_1 = -0.5(0.5) = -0.25$$

$$T_{21} = 0.5 - 0.2(0.5) = 0.4$$