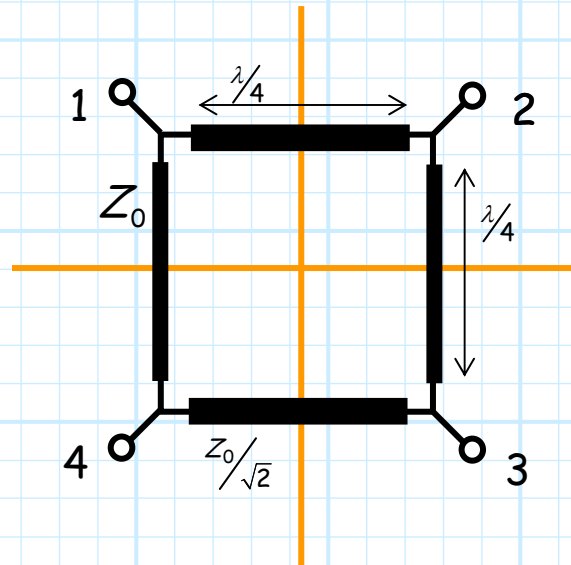
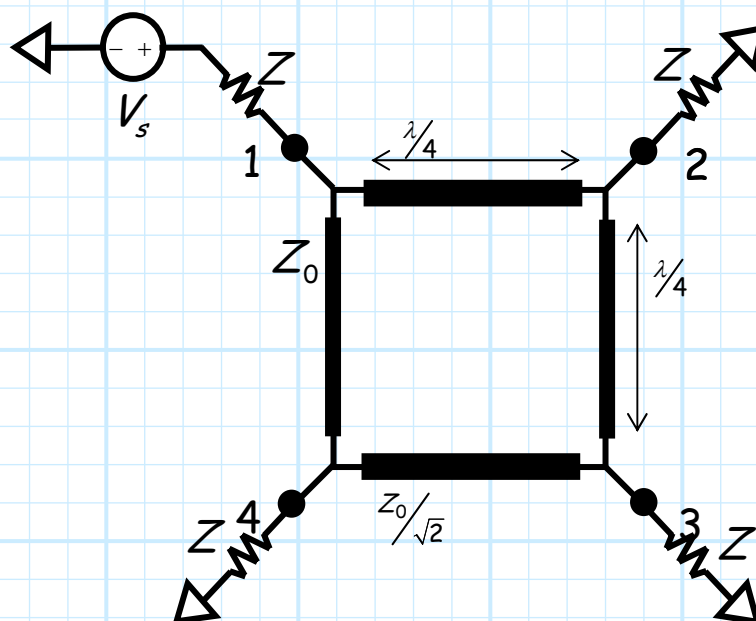


# A Quad-Mode Analysis of the Quadrature Hybrid

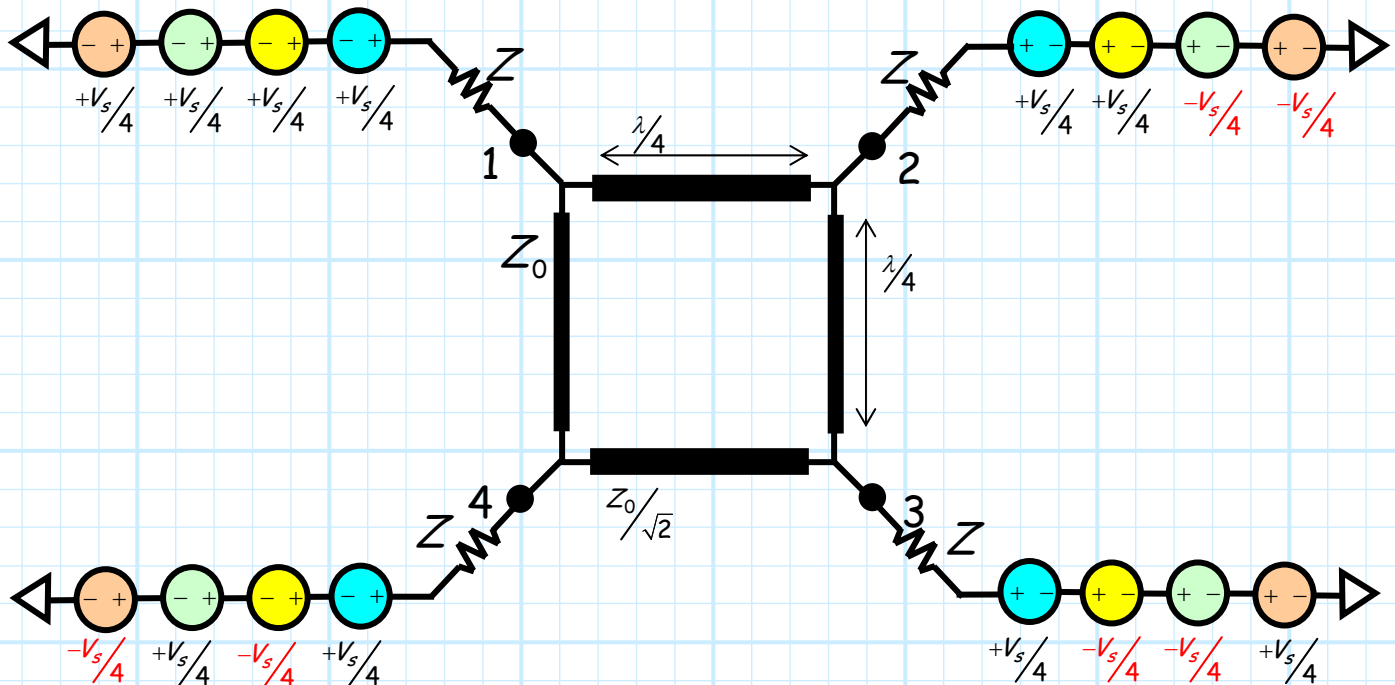
The quadrature hybrid is a matched, lossless, reciprocal four-port network that possesses two planes of **bilateral symmetry** (i.e.  $D_4$  symmetry):



To determine the scattering parameters  $S_{11}, S_{21}, S_{31}, S_{41}$ , of this network, a matched source is placed on port 1, while matched loads terminate the other 3 ports.



This source destroys both planes of bilateral symmetry in the circuit. We can however recast the circuit above with a precisely equivalent circuit:



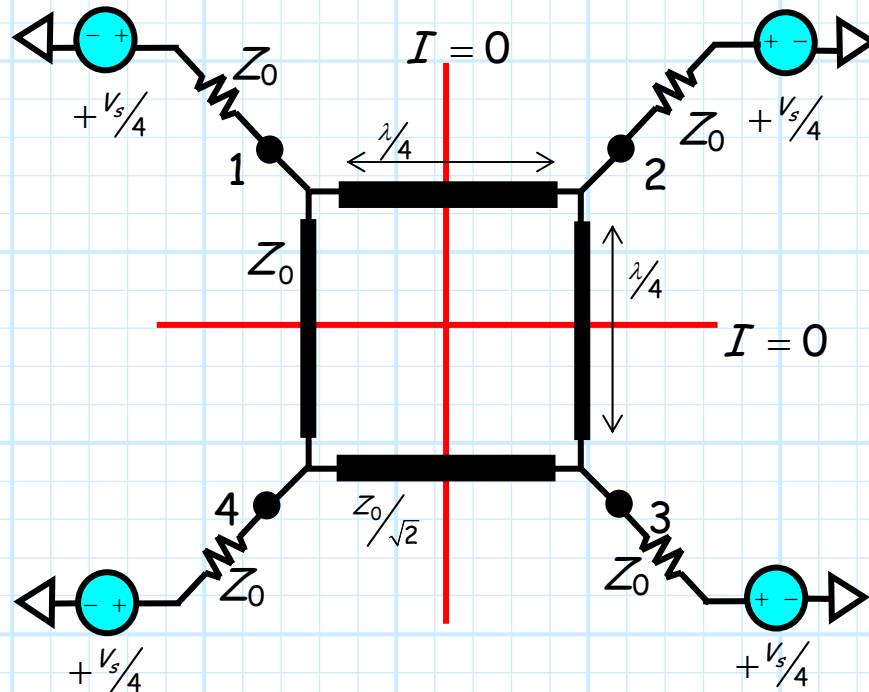
Note that the four series voltage sources on port 1 add to the original value of  $V_s$ , while the series source at the other four ports add to a value of **zero**—thus providing short circuit from the passive load  $Z_0$  to ground.

To analyze this circuit, we can apply **superposition**.

Sequentially turning off all but one source at each of the 4 ports provides us with **four "modes"**. Each of these four modes can be analyzed, and the resulting circuit response is simply a coherent summation of the results of each of the four modes!

The benefit of this procedure is that each of the four modes preserve circuit symmetry. As a result, the planes of bilateral symmetry become virtual shorts and/or virtual opens.

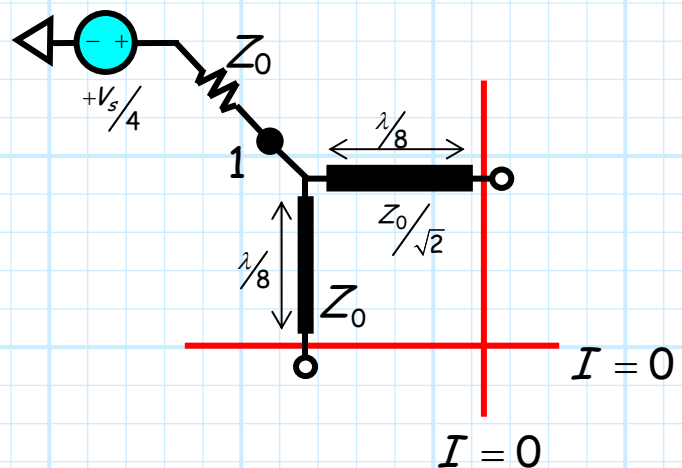
### Mode A



The even symmetry of this circuit is now restored, and so the voltages at each port are identical:

$$V_{1a}^+ = V_{2a}^+ = V_{3a}^+ = V_{4a}^+ = \frac{V_s}{8} \quad \text{and} \quad V_{1a}^- = V_{2a}^- = V_{3a}^- = V_{4a}^-$$

The two virtual opens segment this circuit into 4 identical sections. To determine the amplitude  $V_{1a}^-$ , we need only analyze one of these sections:



The circuit has simplified to a 1-port device consisting of the parallel combination of two  $\lambda/8$  open-circuited stubs. The admittance of a  $\lambda/8$  open-circuit stub is:

$$\begin{aligned} Y_{stub}^{oc} &= jY_0 \cot \beta l \\ &= jY_0 \cot \lambda/8 \\ &= jY_0 \end{aligned}$$

As a result, the input admittance is:

$$\begin{aligned} Y_{in}^d &= j\sqrt{2} Y_0 + j Y_0 \\ &= jY_0 (\sqrt{2} + 1) \end{aligned}$$

The corresponding reflection coefficient is:

$$\begin{aligned} \Gamma_a &= \frac{Y_0 - Y_{in}^a}{Y_0 + Y_{in}^a} \\ &= \frac{Y_0 - jY_0 (\sqrt{2} + 1)}{Y_0 + jY_0 (\sqrt{2} + 1)} \\ &= \frac{1 - j(\sqrt{2} + 1)}{1 + j(\sqrt{2} + 1)} \end{aligned}$$

Since the input admittance is purely reactive, the magnitude of this reflection coefficient is  $|\Gamma_a| = 1.0$ . The phase of this complex value can be determined from its real and imaginary part:

$$\begin{aligned}
 \Gamma_a &= \frac{1 - j(\sqrt{2} + 1)}{1 + j(\sqrt{2} + 1)} \left( \frac{1 - j(\sqrt{2} + 1)}{1 - j(\sqrt{2} + 1)} \right) \\
 &= \frac{1 - j2(\sqrt{2} + 1) - (\sqrt{2} + 1)^2}{1 + (\sqrt{2} + 1)^2} \\
 &= \frac{-2(\sqrt{2} + 1) - j2(\sqrt{2} + 1)}{4 + 2\sqrt{2}} \\
 &= \frac{-(\sqrt{2} + 1) - j(\sqrt{2} + 1)}{\sqrt{2}(\sqrt{2} + 1)} \\
 &= \frac{-1 - j}{\sqrt{2}}
 \end{aligned}$$

So that:

$$\operatorname{Re}\{\Gamma_a\} = \frac{-1}{\sqrt{2}} \quad \text{and} \quad \operatorname{Im}\{\Gamma_a\} = \frac{-1}{\sqrt{2}}$$

Thus the reflection coefficient for mode A is:

$$\Gamma_a = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = e^{-j^{3\pi/4}}$$

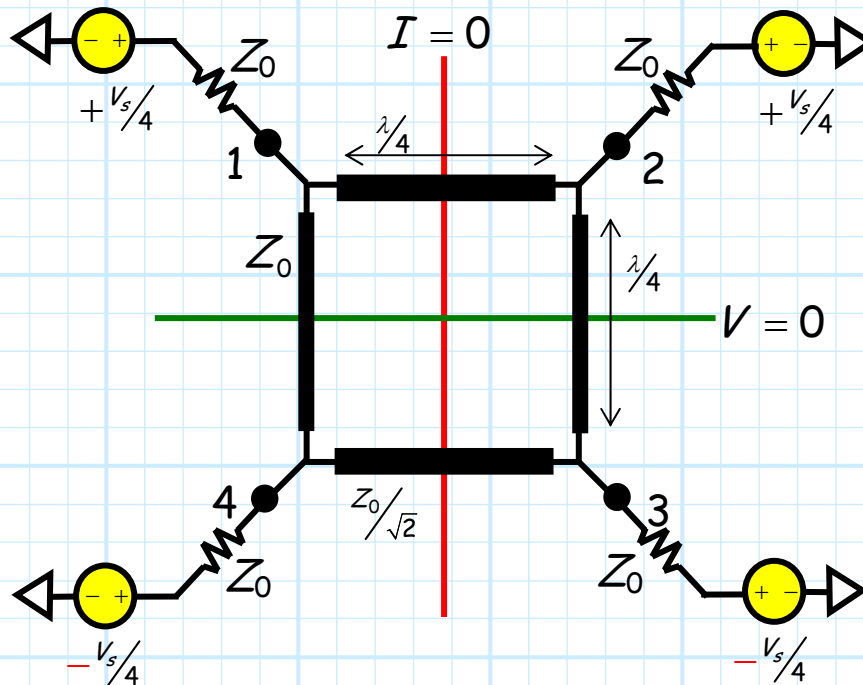
And thus the amplitude of the reflected wave at port 1 is:

$$V_{1a}^- = \Gamma_a V_{1a}^+ = V_s \frac{e^{-j^{3\pi/4}}}{8}$$

And so from the even symmetry of mode A we conclude:

$$V_{1a}^- = V_{2a}^- = V_{3a}^- = V_{4a}^- = V_s \frac{e^{-j3\pi/4}}{8}$$

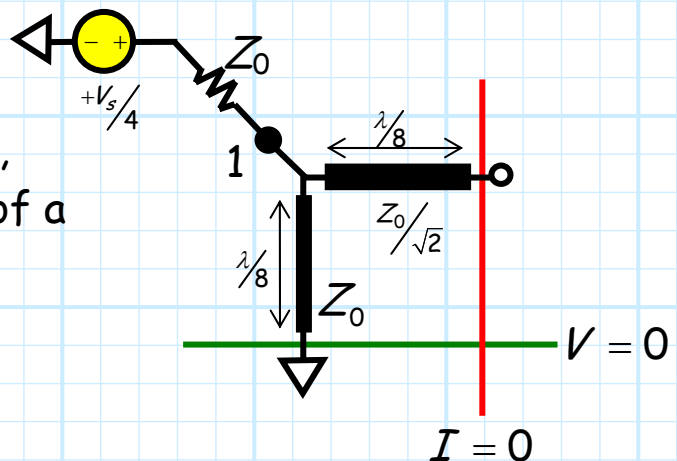
## Mode B



For mode B, the even symmetry exists about the vertical circuit plane, while odd symmetry occurs across the horizontal plane.

$$V_{1b}^+ = V_{2b}^+ = -V_{3b}^+ = -V_{4b}^+ = \frac{V_s}{8} \quad \text{and} \quad V_{1b}^- = V_{2b}^- = -V_{3b}^- = -V_{4b}^-$$

The circuit can again be segmented into four sections, with each section consisting of a shorted  $\lambda/8$  stub and an open-circuited  $\lambda/8$  stub in parallel.



The admittance of a  $\lambda/8$  short-circuit stub is:

$$\begin{aligned} Y_{stub}^{sc} &= -jY_0 \tan \beta l \\ &= -jY_0 \tan \lambda/8 \\ &= -jY_0 \end{aligned}$$

As a result, the input admittance is:

$$Y_{in}^b = j\sqrt{2} Y_0 - j Y_0 = jY_0(\sqrt{2} - 1)$$

The corresponding reflection coefficient is:

$$\begin{aligned} \Gamma_b &= \frac{1 - j(\sqrt{2} - 1)}{1 + j(\sqrt{2} - 1)} \\ &= \frac{1 - j2(\sqrt{2} - 1) - (\sqrt{2} - 1)^2}{1 + (\sqrt{2} - 1)^2} \\ &= \frac{2(\sqrt{2} - 1) - j2(\sqrt{2} - 1)}{2\sqrt{2}(\sqrt{2} - 1)} \\ &= \frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}} \\ &= e^{-j\pi/4} \end{aligned}$$

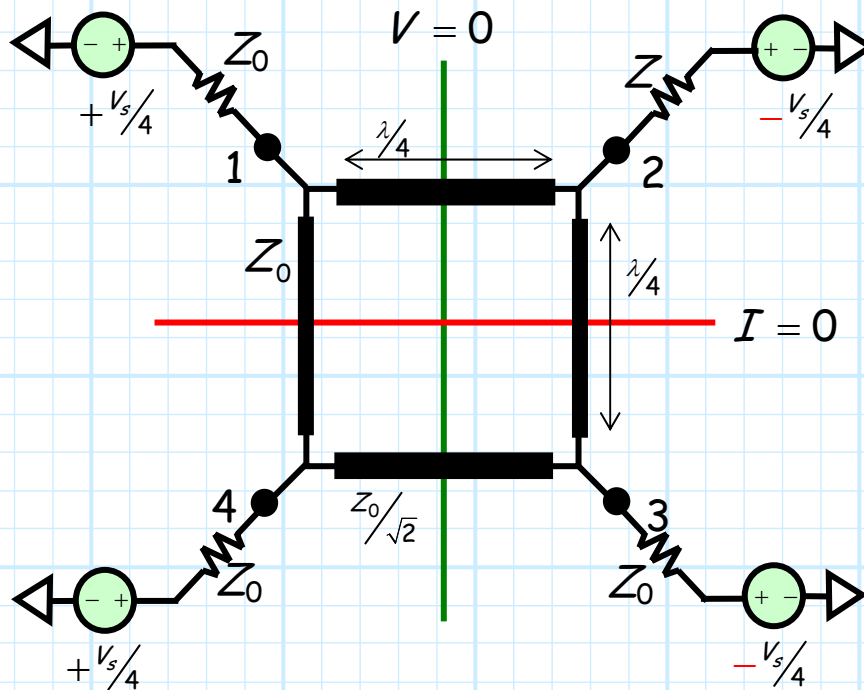
And thus the amplitude of the reflected wave at port 1 is:

$$V_{1b}^- = \Gamma_b V_{1b}^+ = V_s \frac{e^{-j\pi/4}}{8}$$

And so from the symmetry of mode B we conclude:

$$V_{1b}^- = V_{2b}^- = -V_{3b}^- = -V_{4b}^- = V_s \frac{e^{-j\pi/4}}{8}$$

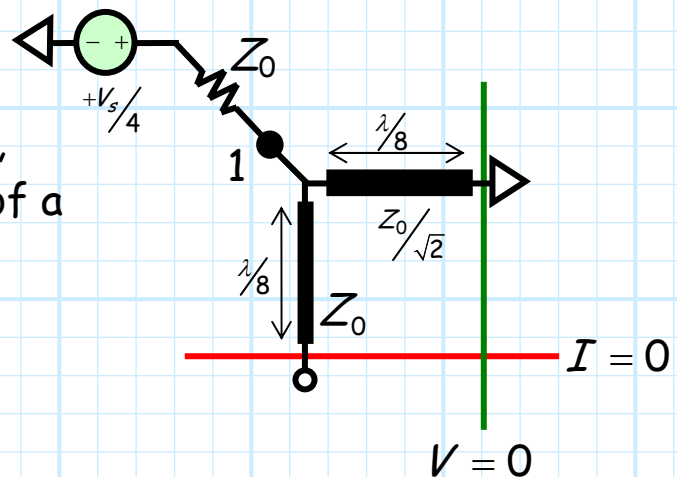
### Mode C



For mode C, odd symmetry exists about the vertical circuit plane, while even symmetry occurs across the horizontal plane.

$$V_{1c}^+ = -V_{2c}^+ = -V_{3c}^+ = V_{4c}^+ = \frac{V_s}{8} \quad \text{and} \quad V_{1c}^- = -V_{2c}^- = -V_{3c}^- = V_{4c}^-$$

The circuit can again be segmented into four sections, with each section consisting of a shorted \$\lambda/8\$ stub and an open-circuited \$\lambda/8\$ stub in parallel.





As a result, the input admittance is:

$$Y_{in}^c = j\sqrt{2} Y_0 + j Y_0 = jY_0(\sqrt{2} - 1)$$

Note that this result is simply the complex conjugate of  $Y_{in}^b$ , and so we can immediately conclude:

$$\Gamma_c = \Gamma_b^* = e^{+j\pi/4}$$

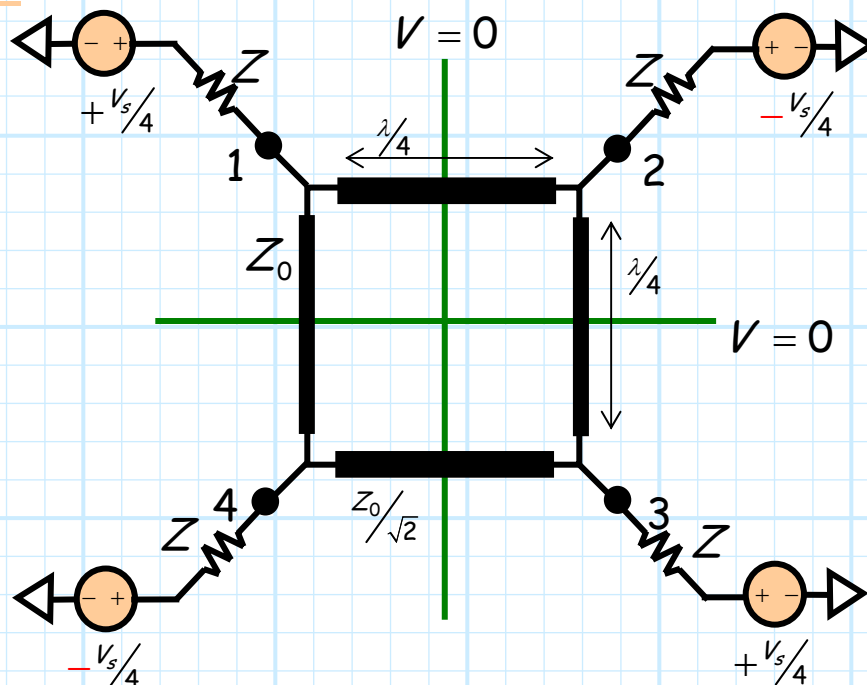
And thus the amplitude of the reflected wave at port 1 is:

$$V_{1c}^- = \Gamma_c V_{1c}^+ = V_s \frac{e^{+j\pi/4}}{8}$$

And so from the symmetry of mode C we conclude:

$$V_{1c}^- = -V_{2c}^- = -V_{3c}^- = V_{4c}^- = V_s \frac{e^{+j\pi/4}}{8}$$

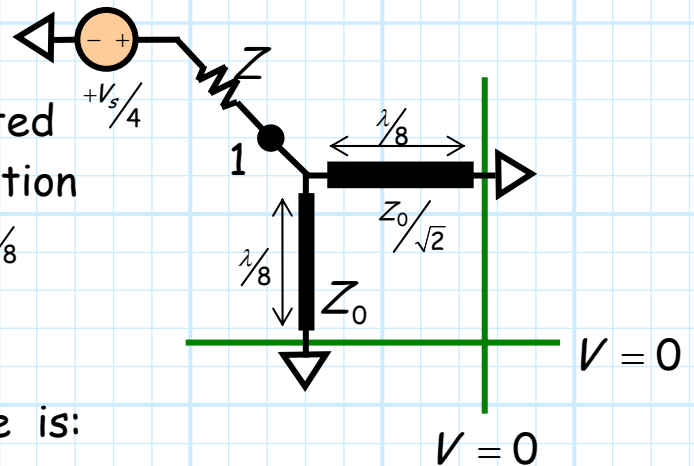
### Mode D



For mode D, odd symmetry exists about both planes of circuit symmetry.

$$+V_{1c}^+ = -V_{2c}^+ = +V_{3c}^+ = -V_{4c}^+ = \frac{V_s}{8} \quad \text{and} \quad +V_{1c}^- = -V_{2c}^- = +V_{3c}^- = -V_{4c}^-$$

The circuit can again be segmented into four sections, with each section consisting two short-circuited  $\lambda/8$  stubs in parallel.



As a result, the input admittance is:

$$Y_{in}^d = -j\sqrt{2} Y_0 - j Y_0 = -j Y_0 (\sqrt{2} + 1)$$

Note that this result is simply the complex conjugate of  $Y_{in}^a$ , and so we can immediately conclude:

$$\Gamma_d = \Gamma_a^* = e^{+j3\pi/4}$$

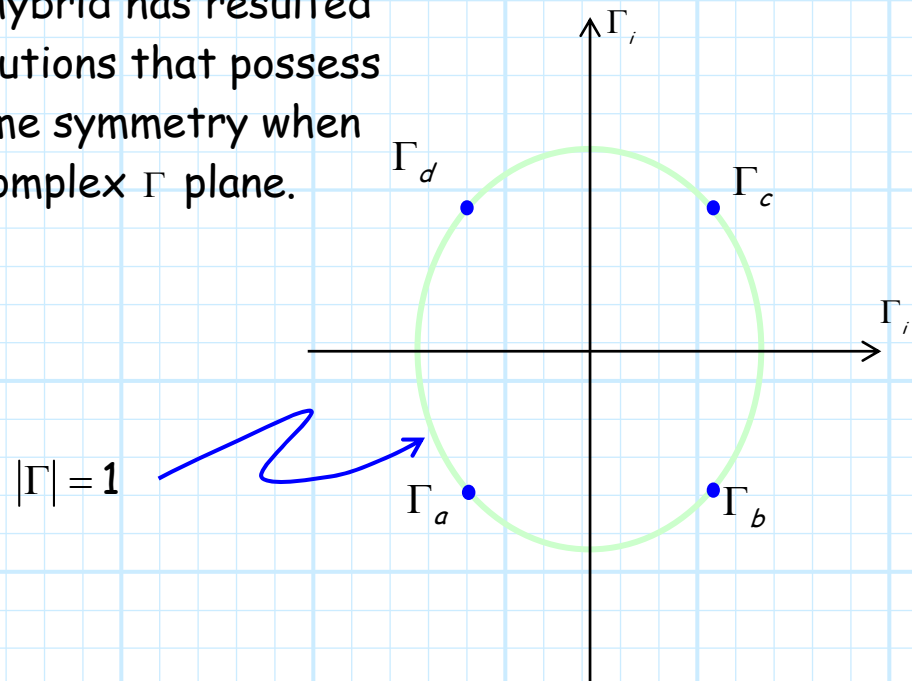
And thus the amplitude of the reflected wave at port 1 is:

$$V_{1d}^- = \Gamma_d V_{1d}^+ = V_s \frac{e^{+j3\pi/4}}{8}$$

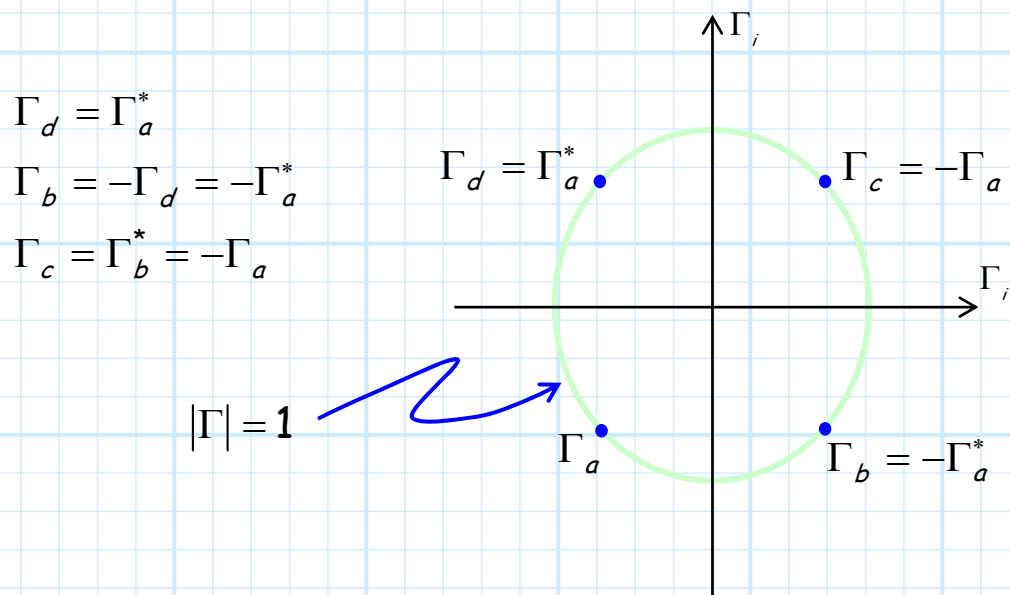
And so from the symmetry of mode D we find:

$$+V_{1c}^- = -V_{2c}^- = +V_{3c}^- = -V_{4c}^- = V_s \frac{e^{+j3\pi/4}}{8}$$

Not surprisingly, the symmetry of the quadrature hybrid has resulted in four modal solutions that possess precisely the same symmetry when plotted on the complex  $\Gamma$  plane.



The modal solutions associated with the other three ports are simply symmetric permutations of the port 1 solutions:



Since our circuit is linear, we can determine the solution to our original circuit as a superposition of our four modal solutions:

$$\begin{aligned}
 V_{01}^- &= V_{1a}^- + V_{1b}^- + V_{1c}^- + V_{1d}^- \\
 &= \frac{V_s}{8} e^{-j3\pi/4} + \frac{V_s}{8} e^{-j\pi/4} + \frac{V_s}{8} e^{+j\pi/4} + \frac{V_s}{8} e^{+j3\pi/4} \\
 &= \left( e^{-j3\pi/4} + e^{-j\pi/4} + e^{+j\pi/4} + e^{+j3\pi/4} \right) \frac{V_s}{8} \\
 &= \left( 2 \cos 3\pi/4 + 2 \cos \pi/4 \right) \frac{V_s}{8} \\
 &= \left( -2 \cos \pi/4 + 2 \cos \pi/4 \right) \frac{V_s}{8} \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 V_{02}^- &= V_{2a}^- + V_{2b}^- + V_{2c}^- + V_{2d}^- \\
 &= \frac{V_s}{8} e^{-j3\pi/4} + \frac{V_s}{8} e^{-j\pi/4} - \frac{V_s}{8} e^{+j\pi/4} - \frac{V_s}{8} e^{+j3\pi/4} \\
 &= \left( e^{-j3\pi/4} + e^{-j\pi/4} - e^{+j\pi/4} - e^{+j3\pi/4} \right) \frac{V_s}{8} \\
 &= \left( -j2 \sin 3\pi/4 - j2 \sin \pi/4 \right) \frac{V_s}{8} \\
 &= \left( -j2 \cos \pi/4 - j2 \cos \pi/4 \right) \frac{V_s}{8} \\
 &= V_s \frac{-j}{2\sqrt{2}}
 \end{aligned}$$

$$\begin{aligned}
 V_{03}^- &= V_{3a}^- + V_{3b}^- + V_{3c}^- + V_{3d}^- \\
 &= \frac{V_s}{8} e^{-j^{3\pi/4}} - \frac{V_s}{8} e^{j\pi/4} - \frac{V_s}{8} e^{-j\pi/4} + \frac{V_s}{8} e^{-j^{3\pi/4}} \\
 &= \left( e^{-j^{3\pi/4}} - e^{-j\pi/4} - e^{+j\pi/4} + e^{+j^{3\pi/4}} \right) \frac{V_s}{8} \\
 &= \left( 2 \cos^{3\pi/4} - 2 \cos \pi/4 \right) \frac{V_s}{8} \\
 &= \left( -2 \cos \pi/4 - 2 \cos \pi/4 \right) \frac{V_s}{8} \\
 &= V_s \frac{-1}{2\sqrt{2}}
 \end{aligned}$$

$$\begin{aligned}
 V_{04}^- &= V_{4a}^- + V_{4b}^- + V_{4c}^- + V_{4d}^- \\
 &= \frac{V_s}{8} e^{-j^{3\pi/4}} - \frac{V_s}{8} e^{j\pi/4} + \frac{V_s}{8} e^{-j\pi/4} - \frac{V_s}{8} e^{-j^{3\pi/4}} \\
 &= \left( e^{-j^{3\pi/4}} - e^{j\pi/4} + e^{-j\pi/4} - e^{-j^{3\pi/4}} \right) \frac{V_s}{8} \\
 &= \left( j2 \sin^{3\pi/4} - j2 \sin \pi/4 \right) \frac{V_s}{8} \\
 &= \left( j2 \sin \pi/4 - j2 \sin \pi/4 \right) \frac{V_s}{8} \\
 &= 0
 \end{aligned}$$

From these results we can determine the scattering parameters  $S_{11}, S_{21}, S_{31}, S_{41}$ :

$$S_{11} = \frac{V_{01}^-}{V_{01}^+} = \frac{2}{V_s} 0 = 0$$

$$S_{21} = \frac{V_{02}^-}{V_{01}^+} = \frac{2}{V_s} \frac{-jV_s}{2\sqrt{2}} = \frac{-j}{\sqrt{2}}$$

$$S_{31} = \frac{V_{03}^-}{V_{01}^+} = \frac{2}{V_s} \frac{-V_s}{2\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

$$S_{41} = \frac{V_{04}^-}{V_{01}^+} = \frac{2}{V_s} 0 = 0$$

Given the symmetry of the device, we can extend these four results to determine the entire scattering matrix:

$$\mathbf{S} = \begin{bmatrix} 0 & -j/\sqrt{2} & -1/\sqrt{2} & 0 \\ -j/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 0 & -j/\sqrt{2} \\ 0 & -1/\sqrt{2} & -j/\sqrt{2} & 0 \end{bmatrix}$$

Precisely the **correct result!**