The Characteristic Impedance of a Transmission Line

So, from the **telegrapher's** differential equations, we know that the complex current $I(z)$ and voltage $V(z)$ **must** have the form:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \qquad I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$$

Let's insert the expression for $V(z)$ into the first telegrapher's equation, and **see what happens**!

$$\frac{d\,V(z)}{dz} = -\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{+\gamma z} = -(R + j\omega L)\,I(z)$$

Therefore, rearranging, current $I(z)$ must be:

$$I(z) = \frac{\gamma}{R + j\omega L}\left(V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}\right)$$

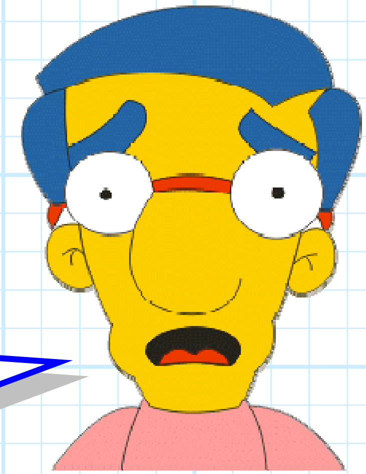
I thought we knew this?!

Q: ***But wait !*** *I thought we already knew current $I(z)$.*

Isn't it:

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} \quad ??$$

How can ***both*** *expressions for $I(z)$ be true??*



A: Easy ! Both expressions for current are **equal** to each other.

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} = \frac{\gamma}{R + j\omega L}\left(V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}\right)$$

For the above equation to be true for **all** z, I_0 and V_0 must be related as:

$$I_0^+ e^{-\gamma z} = \left(\frac{\gamma}{R + j\omega L}\right) V_0^+ e^{-\gamma z} \qquad \text{and} \qquad I_0^- e^{+\gamma z} = \left(\frac{-\gamma}{R + j\omega L}\right) V_0^- e^{+\gamma z}$$

A startling conclusion

Or—recalling that $V_0^+ e^{-\gamma z} = V^+(z)$ (etc.)—we can express this in terms of the **two propagating waves**:

$$I^+(z) = \left(\frac{+\gamma}{R + j\omega L}\right) V^+(z) \quad \text{and} \quad I^-(z) = \left(\frac{-\gamma}{R + j\omega L}\right) V^-(z)$$

Now, we note that since:

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

We find that:

$$\frac{\gamma}{R + j\omega L} = \frac{\sqrt{(R + j\omega L)(G + j\omega C)}}{R + j\omega L} = \sqrt{\frac{G + j\omega C}{R + j\omega L}}$$

Thus, we come to the **startling** conclusion that:

$$\frac{V^+(z)}{I^+(z)} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \quad \text{and} \quad \frac{-V^-(z)}{I^-(z)} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Characteristic Impedance

Q: *What's so startling about **this** conclusion?*

A: Note that although each propagating wave is a **function** of transmission line **position** z (e.g., $V^+(z)$ and $I^+(z)$), the **ratio** of the voltage and current of **each wave** is independent of position—a **constant** with respect to position z!

Although $V_0^\pm$ and $I_0^\pm$ are determined by **boundary conditions** (i.e., what's **connected** to either end of the transmission line), the **ratio** $V_0^\pm / I_0^\pm$ is determined by the parameters of the **transmission line only** (i.e., R, L, G, C).

→ This ratio is an important **characteristic of a transmission line**, called its **Characteristic Impedance** Z_0.

$$Z_0 \doteq \frac{V^+(z)}{I^+(z)} = \frac{-V_0^-(z)}{I_0^-(z)} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

An alternative transmission line description

We can therefore describe the **current and voltage** along a transmission line as:

$$V(z) = V_0^+ \, e^{-\gamma z} + V_0^- \, e^{+\gamma z}$$

$$I(z) = \frac{V_0^+}{Z_0} \, e^{-\gamma z} - \frac{V_0^-}{Z_0} \, e^{+\gamma z}$$

or equivalently:

$$V(z) = Z_0 \, I_0^+ \, e^{-\gamma z} - Z_0 \, I_0^- \, e^{+\gamma z}$$

$$I(z) = I_0^+ \, e^{-\gamma z} + I_0^- \, e^{+\gamma z}$$

→ Note that instead of characterizing a transmission line with **real** parameters R, G, L, and C, we can (and typically do!) describe a transmission line using **complex** parameters Z_0 and γ.