Circuit Symmetry

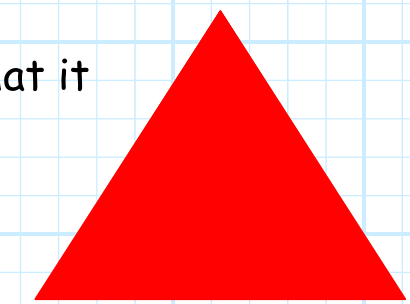
One of the most powerful concepts in for evaluating circuits is that of symmetry. **Normal** humans have a **conceptual** understanding of symmetry, based on an **esthetic** perception of structures and figures.

Évariste Galois

On the other hand, **mathematicians** (as they are wont to do) have defined symmetry in a very precise and unambiguous way. Using a branch of mathematics called **Group Theory**, first developed by the young genius **Évariste Galois** (1811-1832), **symmetry** is defined by a set of operations (a group) that leaves an object **unchanged**.

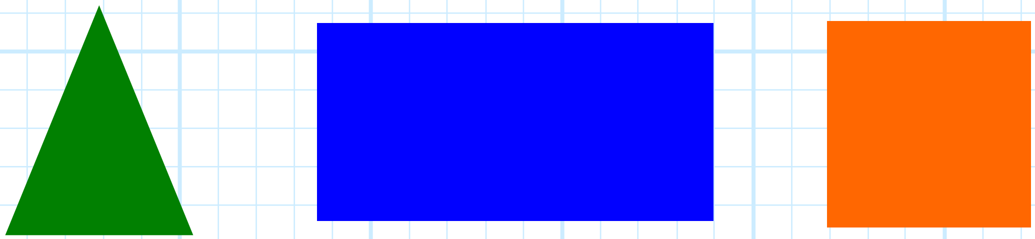
Initially, the symmetric "objects" under consideration by Galois were **polynomial functions**, but group theory can likewise be applied to evaluate the symmetry of **structures**.

For example, consider an ordinary **equilateral triangle**; we find that it is a highly **symmetric** object!

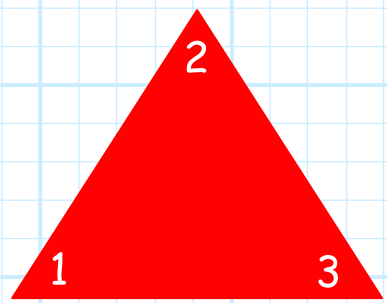


Q: ***Obviously*** *this is true. We don't need a mathematician to tell us that!*

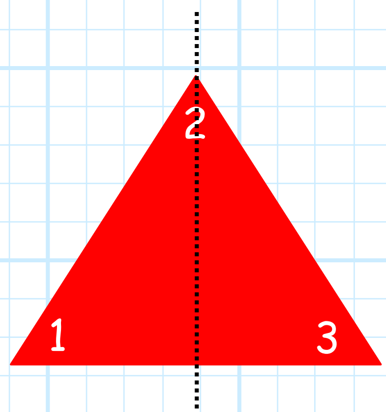
A: Yes, but **how** symmetric is it? How does the symmetry of an equilateral triangle **compare** to that of an isosceles triangle, a rectangle, or a square?



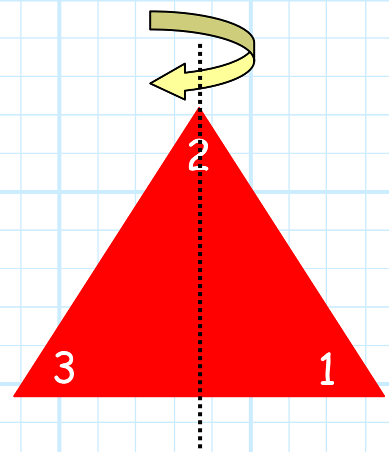
To determine its level of symmetry, let's first label each corner as corner **1**, corner **2**, and corner **3**.



First, we note that the triangle exhibits a plane of **reflection symmetry**:



Thus, if we "**reflect**" the triangle across this plane we get:



Note that although corners 1 and 3 have changed places, the triangle itself remains **unchanged**—that is, it has the same **shape**, same **size**, and same **orientation** after reflecting across the symmetric plane!

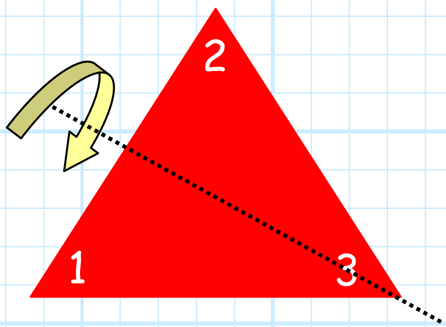
Mathematicians say that these two triangles are **congruent**.

Note that we can write this reflection operation as a **permutation** (an exchange of position) of the corners, defined as:

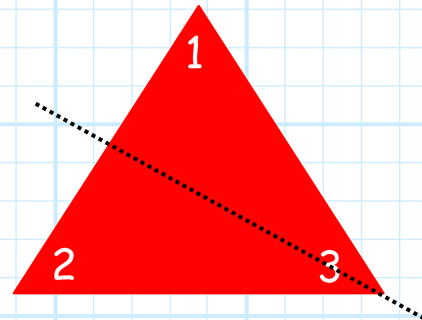
$$1 \rightarrow 3$$
$$2 \rightarrow 2$$
$$3 \rightarrow 1$$

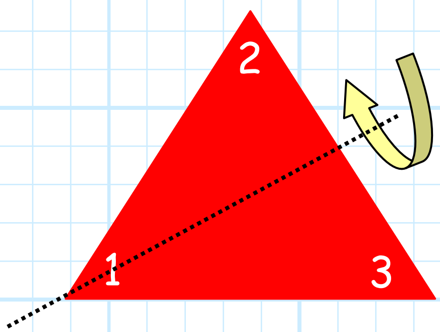
Q: *But wait! Isn't there is* ***more*** *than just* ***one*** *plane of reflection symmetry?*

A: Definitely! There are **two more**:

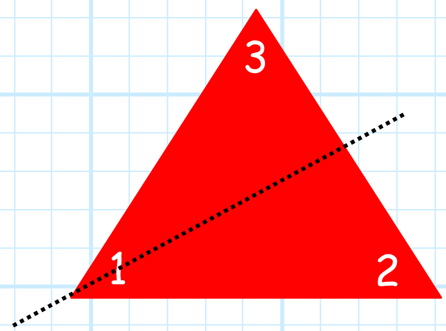


$$1 \rightarrow 2$$
$$2 \rightarrow 1$$
$$3 \rightarrow 3$$

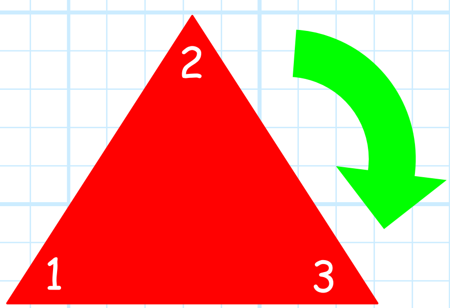




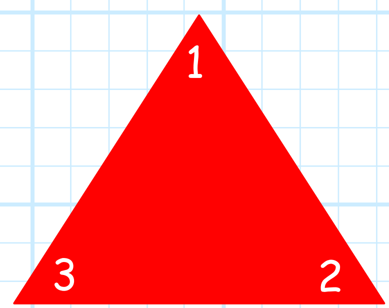
$$1 \rightarrow 1$$
$$2 \rightarrow 3$$
$$3 \rightarrow 2$$



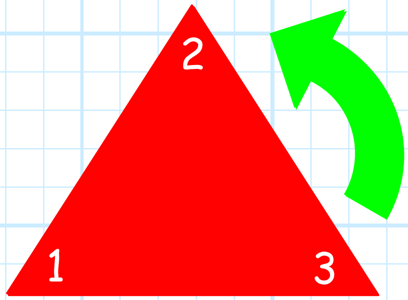
In addition, an equilateral triangle exhibits **rotation symmetry! Rotating** the triangle 120° clockwise also results in a **congruent** triangle:



$$1 \rightarrow 2$$
$$2 \rightarrow 3$$
$$3 \rightarrow 1$$



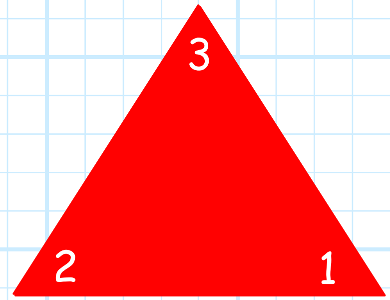
Likewise, rotating the triangle 120° **counter-clockwise** results in a congruent triangle:



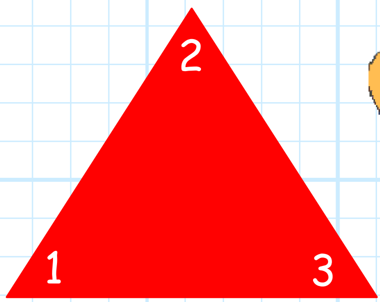
$1 \to 3$

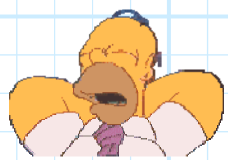
$2 \to 1$

$3 \to 2$



Additionally, there is **one more** operation that will result in a congruent triangle—do **nothing**!

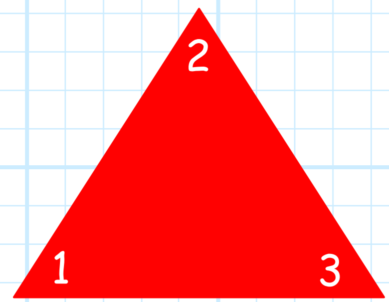




$1 \to 1$

$2 \to 2$

$3 \to 3$

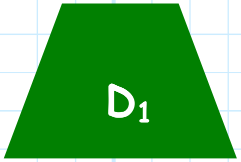


This seemingly **trivial** operation is known as the **identity operation**, and is an element of **every** symmetry group.

These 6 operations form the **dihedral symmetry group D_3** which has **order six** (i.e., it consists of six operations). An object that remains **congruent** when operated on by any and all of these six operations is said to have **D_3** symmetry.

An equilateral triangle has **D_3** symmetry!

By applying a similar analysis to a isosceles triangle, rectangle, and square, we find that:



An isosceles trapezoid has D_1 symmetry, a dihedral group of **order 2**.

A rectangle has D_2 symmetry, a dihedral group of **order 4**.

A square has D_4 symmetry, a dihedral group of **order 8**.

Thus, a square is the **most** symmetric object of the four we have discussed; the isosceles trapezoid is the **least**.

Q: *Well that's all just fascinating—but just what the heck does this have to do with* ***microwave circuits!?!***

A: Plenty! **Useful circuits** often display high levels of **symmetry**.

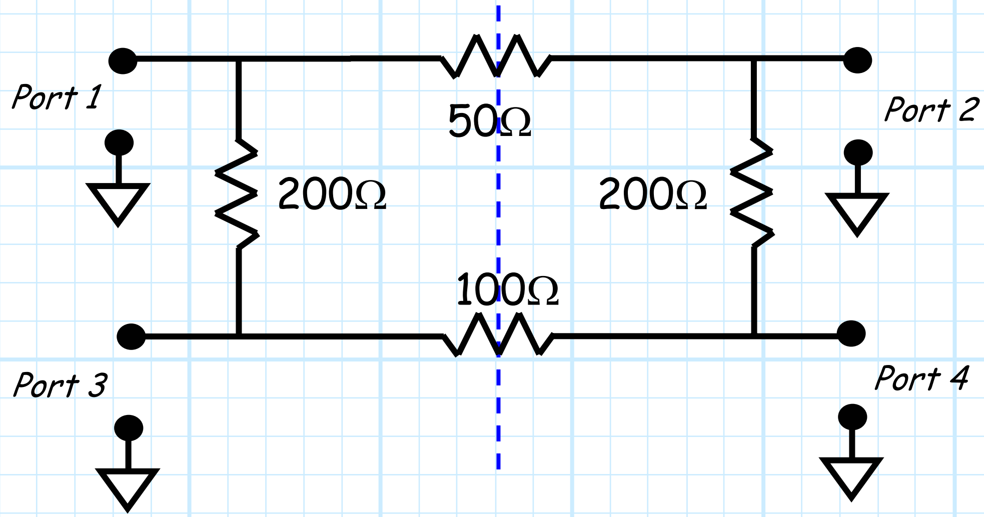
For example, consider these **D₁** symmetric multi-port circuits:

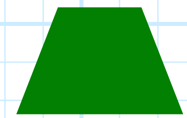
$1 \rightarrow 2$

$2 \rightarrow 1$

$3 \rightarrow 4$

$4 \rightarrow 3$



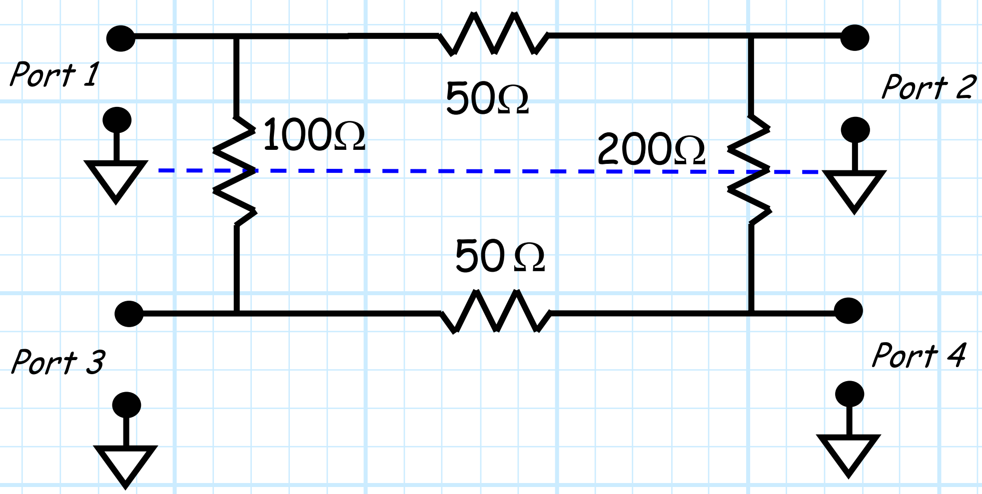


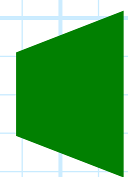
$1 \rightarrow 3$

$2 \rightarrow 4$

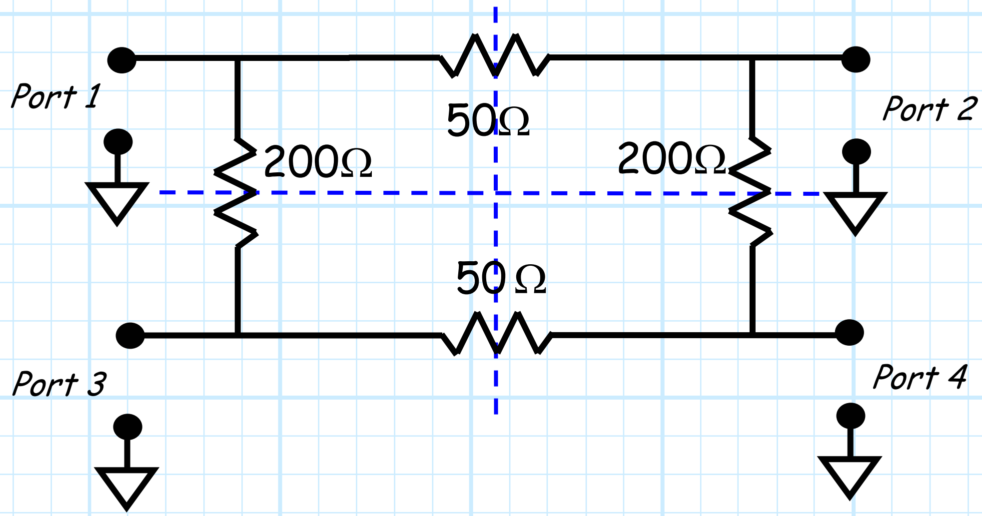
$3 \rightarrow 1$

$4 \rightarrow 2$





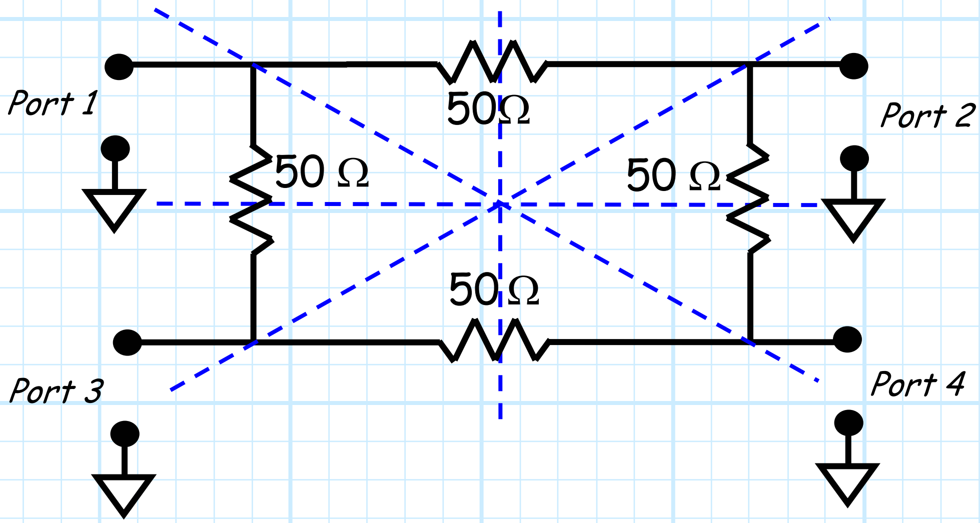
Or this circuit with D_2 symmetry:



which is **congruent** under these permutations:

$1 \rightarrow 3$	$1 \rightarrow 2$	$1 \rightarrow 4$
$2 \rightarrow 4$	$2 \rightarrow 1$	$2 \rightarrow 3$
$3 \rightarrow 1$	$3 \rightarrow 4$	$3 \rightarrow 2$
$4 \rightarrow 2$	$4 \rightarrow 3$	$4 \rightarrow 1$

Or this circuit with D_4 symmetry:

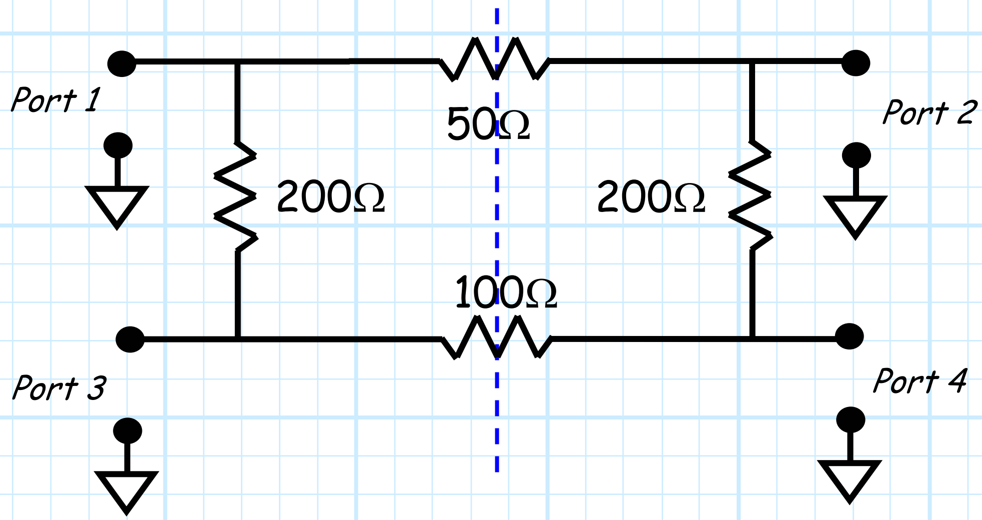


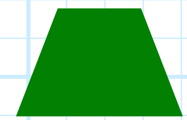
which is **congruent** under these permutations:

$1 \rightarrow 3$	$1 \rightarrow 2$	$1 \rightarrow 4$	$1 \rightarrow 4$	$1 \rightarrow 1$
$2 \rightarrow 4$	$2 \rightarrow 1$	$2 \rightarrow 3$	$2 \rightarrow 2$	$2 \rightarrow 3$
$3 \rightarrow 1$	$3 \rightarrow 4$	$3 \rightarrow 2$	$3 \rightarrow 3$	$3 \rightarrow 2$
$4 \rightarrow 2$	$4 \rightarrow 3$	$4 \rightarrow 1$	$4 \rightarrow 1$	$4 \rightarrow 4$

The **importance** of this can be seen when considering the scattering matrix, impedance matrix, or admittance matrix of these networks.

For **example**, consider again this **symmetric circuit**:





This four-port network has a single plane of **reflection symmetry** (i.e., D_1 symmetry), and thus is congruent under the permutation:

$$
\begin{aligned}
1 &\rightarrow 2 \\
2 &\rightarrow 1 \\
3 &\rightarrow 4 \\
4 &\rightarrow 3
\end{aligned}
$$

So, since (for example) $1 \rightarrow 2$, we find that for this circuit:

$$S_{11} = S_{22} \quad Z_{11} = Z_{22} \quad Y_{11} = Y_{22}$$

must be true!

Or, since $1 \rightarrow 2$ and $3 \rightarrow 4$ we find:

$$S_{13} = S_{24} \qquad Z_{13} = Z_{24} \qquad Y_{13} = Y_{24}$$

$$S_{31} = S_{42} \qquad Z_{31} = Z_{42} \qquad Y_{31} = Y_{42}$$

Continuing for **all** elements of the permutation, we find that for this symmetric circuit, the scattering matrix **must** have **this** form:

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{21} & S_{13} & S_{14} \\ S_{21} & S_{11} & S_{14} & S_{13} \\ S_{31} & S_{41} & S_{33} & S_{43} \\ S_{41} & S_{31} & S_{43} & S_{33} \end{bmatrix}$$

and the **impedance** and **admittance** matrices would likewise have this same form.

Note there are just **8** independent elements in this matrix. If we also consider **reciprocity** (a constraint independent of symmetry) we find that $S_{31} = S_{13}$ and $S_{41} = S_{14}$, and the matrix reduces further to one with just **6** independent elements:

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{21} & S_{31} & S_{41} \\ S_{21} & S_{11} & S_{41} & S_{31} \\ S_{31} & S_{41} & S_{33} & S_{43} \\ S_{41} & S_{31} & S_{43} & S_{33} \end{bmatrix}$$

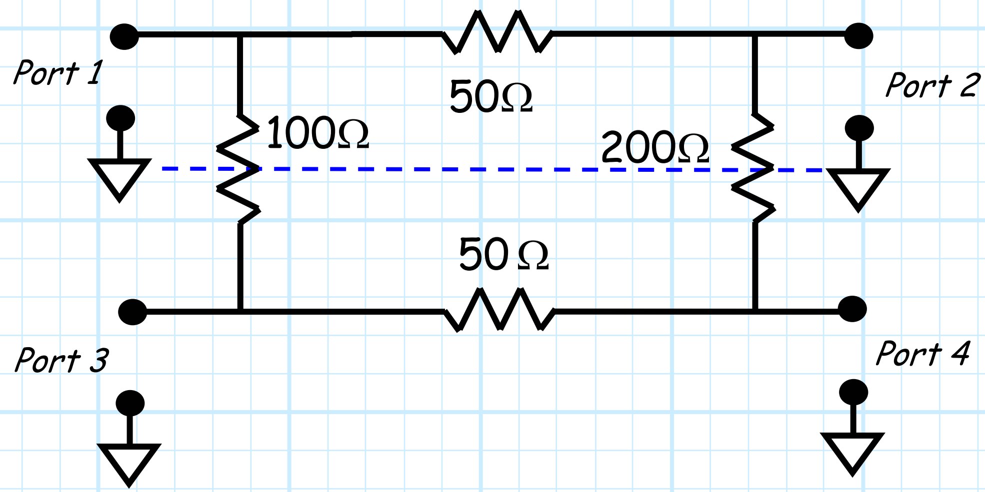
Or, for circuits with **this** D_1 symmetry:

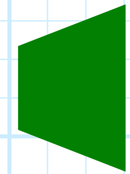
$1 \rightarrow 3$

$2 \rightarrow 4$

$3 \rightarrow 1$

$4 \rightarrow 2$



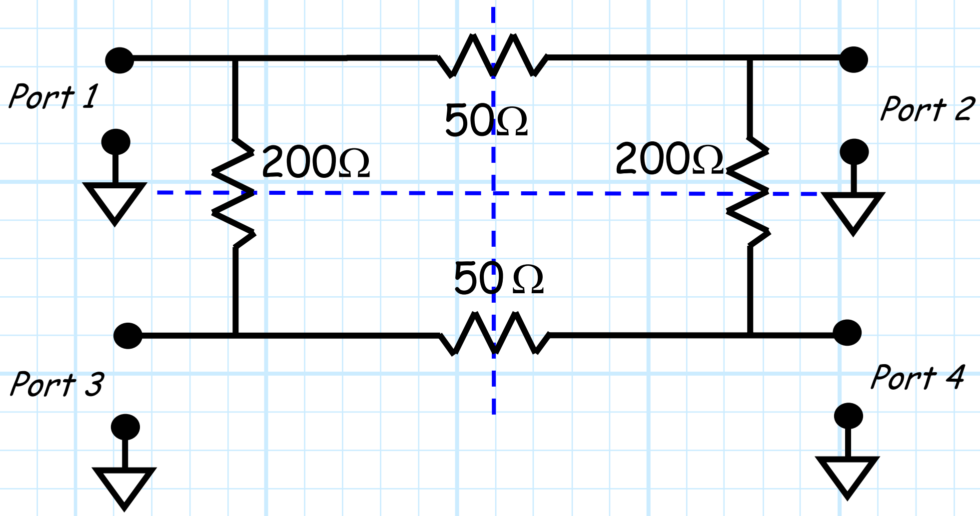


$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{21} & S_{31} & S_{41} \\ S_{21} & S_{22} & S_{41} & S_{31} \\ S_{31} & S_{41} & S_{11} & S_{21} \\ S_{41} & S_{31} & S_{21} & S_{22} \end{bmatrix}$$

Q: *Interesting. But* ***why do we care?***

A: This will greatly **simplify** the analysis of this symmetric circuit, as we need to determine **only** six matrix elements!

For a circuit with $\mathbf{D_2}$ symmetry:

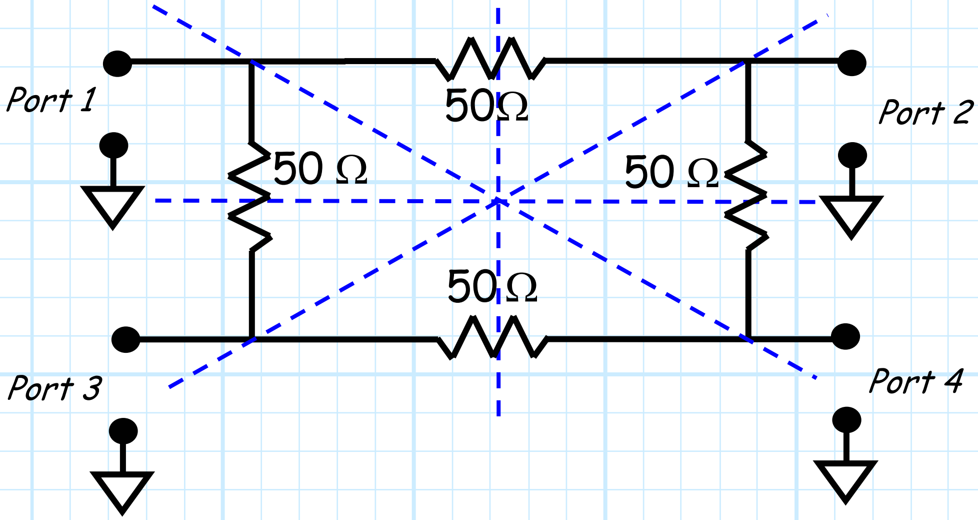


we find that the impedance (or scattering, or admittance) matrix has the form:

$$\mathcal{Z} = \begin{bmatrix} Z_{11} & Z_{21} & Z_{31} & Z_{41} \\ Z_{21} & Z_{11} & Z_{41} & Z_{31} \\ Z_{31} & Z_{41} & Z_{11} & Z_{21} \\ Z_{41} & Z_{31} & Z_{21} & Z_{11} \end{bmatrix}$$

Note here that there are just **four** independent values!

For a circuit with D_4 symmetry:



we find that the admittance (or scattering, or impedance) matrix has the form:

$$\mathcal{Y} = \begin{bmatrix} y_{11} & y_{21} & y_{21} & y_{41} \\ y_{21} & y_{11} & y_{41} & y_{21} \\ y_{21} & y_{41} & y_{11} & y_{21} \\ y_{41} & y_{21} & y_{21} & y_{11} \end{bmatrix}$$

Note here that there are just **three** independent values!

One more interesting thing (yet **another** one!); recall that we earlier found that a matched, lossless, reciprocal **4-port** device must have a scattering matrix with one of **two forms**:

$$\mathcal{S} = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & j\beta \\ j\beta & 0 & 0 & \alpha \\ 0 & j\beta & \alpha & 0 \end{bmatrix}$$

The "symmetric" solution

$$\mathcal{S} = \begin{bmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & -\beta \\ \beta & 0 & 0 & \alpha \\ 0 & -\beta & \alpha & 0 \end{bmatrix}$$

The "anti-symmetric" solution

Compare these to the matrix forms above. The "symmetric solution" has the **same form** as the scattering matrix of a circuit with **D_2** symmetry!

$$\mathcal{S} = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & j\beta \\ j\beta & 0 & 0 & \alpha \\ 0 & j\beta & \alpha & 0 \end{bmatrix}$$

Q: *Does this mean that a matched, lossless, reciprocal four-port device with the "symmetric" scattering matrix **must** exhibit D_2 symmetry?*

A: That's **exactly** what it means!

Not only can we determine from the **form** of the scattering matrix **whether** a particular design is possible (e.g., a matched, lossless, reciprocal 3-port device is impossible), we can also determine the **general structure** of a possible solutions (e.g. the circuit must have D_2 symmetry).

Likewise, the "anti-symmetric" matched, lossless, reciprocal four-port network **must** exhibit D_1 symmetry!

$$\mathcal{S} = \begin{bmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & -\beta \\ \beta & 0 & 0 & \alpha \\ 0 & -\beta & \alpha & 0 \end{bmatrix}$$

We'll see just what these symmetric, matched, lossless, reciprocal four-port circuits actually **are** later in the course!