Example: A Lossless, Reciprocal Network

A **lossless**, **reciprocal** 3-port device has $\mathcal{S}$-parameters of $S_{11} = 1/2$, $S_{31} = 1/\sqrt{2}$, and $S_{33} = 0$. It is likewise known that all scattering parameters are **real**.

→ Find the remaining **6** scattering parameters.

Q: *This problem is clearly* ***impossible****—you have not provided us with sufficient* ***information****!*

A: Yes I have! Note I said the device was **lossless** and **reciprocal**!

Start with what we **currently** know:

$$\mathcal{S} = \begin{bmatrix} \frac{1}{2} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ \frac{1}{\sqrt{2}} & S_{32} & 0 \end{bmatrix}$$

Because the device is **reciprocal**, we then also know:

$$S_{21} = S_{12} \qquad S_{13} = S_{31} = \frac{1}{\sqrt{2}} \qquad S_{32} = S_{23}$$

And therefore:

$$\mathcal{S} = \begin{bmatrix} \frac{1}{2} & S_{21} & \frac{1}{\sqrt{2}} \\ S_{21} & S_{22} & S_{32} \\ \frac{1}{\sqrt{2}} & S_{32} & 0 \end{bmatrix}$$

Now, since the device is **lossless**, we know that:

$$\begin{aligned} 1 &= |S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 \\ &= \left(\tfrac{1}{2}\right)^2 + |S_{21}|^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2 \end{aligned}$$

Columns have **unit** magnitude.

$$\begin{aligned} 1 &= |S_{12}|^2 + |S_{22}|^2 + |S_{32}|^2 \\ &= |S_{21}|^2 + |S_{22}|^2 + |S_{32}|^2 \end{aligned}$$

$$\begin{aligned} 1 &= |S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 \\ &= \left(\tfrac{1}{2}\right)^2 + |S_{32}|^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2 \end{aligned}$$

and:

$$\begin{aligned} 0 &= S_{11}S_{12}^* + S_{21}S_{22}^* + S_{31}S_{32}^* \\ &= \tfrac{1}{2}\,S_{21}^* + S_{21}S_{22}^* + \tfrac{1}{\sqrt{2}}\,S_{32}^* \end{aligned}$$

Columns are **orthogonal**.

$$\begin{aligned} 0 &= S_{11}S_{13}^* + S_{21}S_{23}^* + S_{31}S_{33}^* \\ &= \tfrac{1}{2}\left(\tfrac{1}{\sqrt{2}}\right) + S_{21}S_{32}^* + \tfrac{1}{\sqrt{2}}\,(0) \end{aligned}$$

$$\begin{aligned} 0 &= S_{12}S_{13}^* + S_{22}S_{23}^* + S_{32}S_{33}^* \\ &= S_{21}\left(\tfrac{1}{\sqrt{2}}\right) + S_{22}S_{32}^* + S_{32}\,(0) \end{aligned}$$

These **six** expressions simplify to:

$$|S_{21}| = \tfrac{1}{2}$$

$$1 = |S_{21}|^2 + |S_{22}|^2 + |S_{32}|^2$$

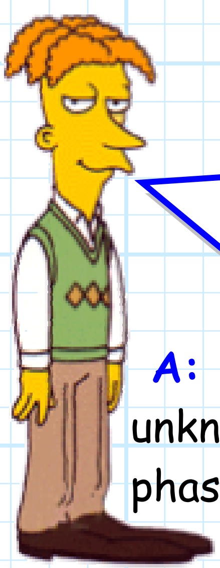
$$|S_{32}| = \tfrac{1}{\sqrt{2}}$$

$$0 = \tfrac{1}{2} S_{21} + S_{21} S_{22} + \tfrac{1}{\sqrt{2}} S_{32}$$

$$0 = \tfrac{1}{(2\sqrt{2})} + S_{21} S_{32}$$

$$0 = S_{21}\left(\tfrac{1}{\sqrt{2}}\right) + S_{22} S_{32}$$

where we have used the fact that since the elements are all **real**, then $S_{21}^* = S_{21}$ (etc.).



Q: *I count the expressions and find 6 equations yet only a paltry 3 unknowns. Your typical buffoonery appears to have led to an over-constrained condition for which there is **no** solution!*

A: Actually, we have **six** real equations and **six** real unknowns, since scattering element has a magnitude and phase. In this case we know the values are **real**, and thus the phase is either 0° or 180° (i.e., $e^{j0} = 1$ or $e^{j\pi} = -1$); however, we do not know which one!

From the first three equations, we can find the **magnitudes**:

$$|S_{21}| = \tfrac{1}{2} \qquad |S_{22}| = \tfrac{1}{2} \qquad |S_{32}| = \tfrac{1}{\sqrt{2}}$$

and from the last three equations we find the **phase**:

$$S_{21} = \tfrac{1}{2} \qquad S_{22} = \tfrac{1}{2} \qquad S_{32} = -\tfrac{1}{\sqrt{2}}$$

Thus, the scattering matrix for this **lossless**, **reciprocal** device is:

$$\mathcal{S} = \begin{bmatrix} \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{\sqrt{2}} \\ \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{-1}{\sqrt{2}} \\ \tfrac{1}{\sqrt{2}} & \tfrac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$