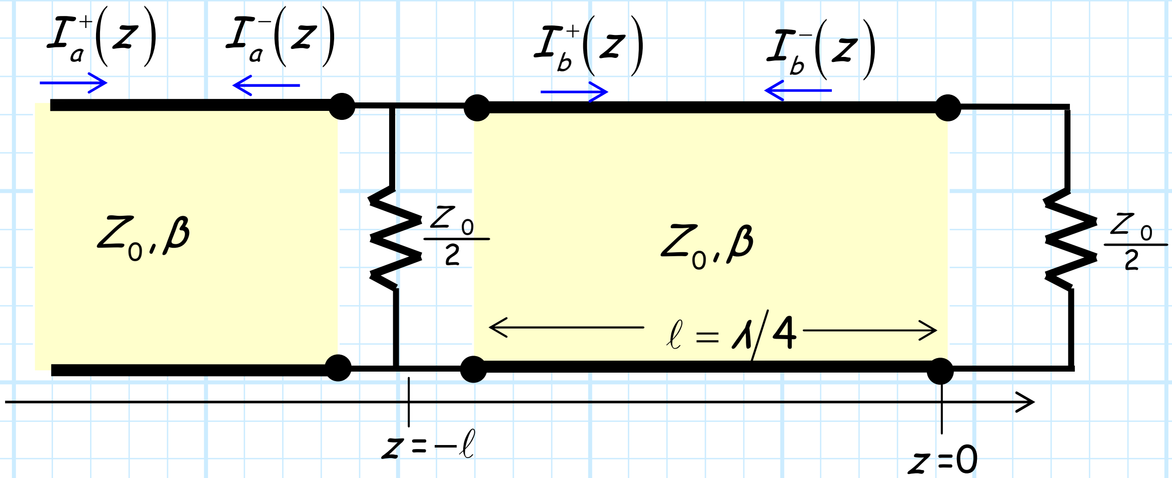
Example: Another Boundary Condition Problem



The **total** voltage along the transmission line shown above is expressed as:

$$V(z) = \begin{cases} V_{0a}^+ e^{-j\beta z} + V_{0a}^- e^{+j\beta z} & z < -\ell \\ V_{0b}^+ e^{-j\beta z} + V_{0b}^- e^{+j\beta z} & -\ell < z < 0 \end{cases}$$

Carefully determine and apply boundary conditions at both $z = 0$ and $z = -\ell$ to **find the three values**:

$$\frac{V_{0a}^-}{V_{0a}^+}, \quad \frac{V_{0b}^+}{V_{0a}^+}, \quad \frac{V_{0b}^-}{V_{0a}^+}$$

Solution

From the telegrapher's equation, we likewise know that the current along the transmission lines is:

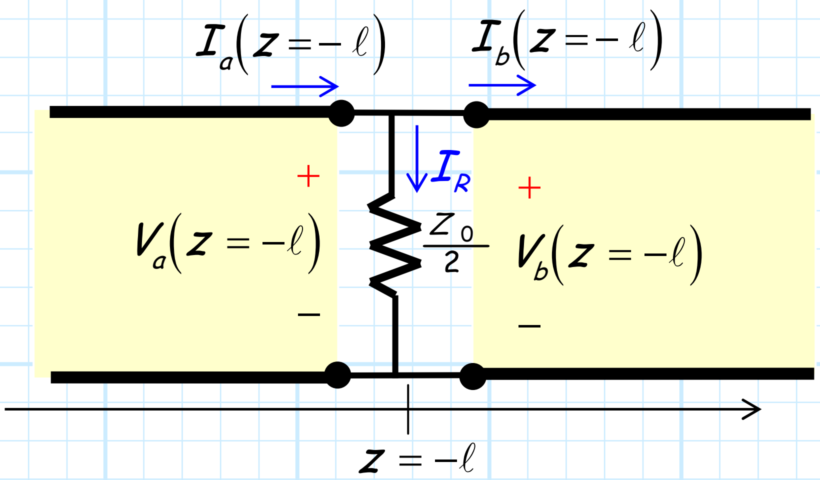
$$I(z) = \begin{cases} \dfrac{V_{0a}^{+}}{Z_0} e^{-j\beta z} - \dfrac{V_{0a}^{-}}{Z_0} e^{+j\beta z} & z < -\ell \\ \\ \dfrac{V_{0b}^{+}}{Z_0} e^{-j\beta z} - \dfrac{V_{0b}^{-}}{Z_0} e^{+j\beta z} & -\ell < z < 0 \end{cases}$$

To find the values:

$$\frac{V_{0a}^{-}}{V_{0a}^{+}}, \quad \frac{V_{0b}^{+}}{V_{0a}^{+}}, \quad \frac{V_{0b}^{-}}{V_{0a}^{+}}$$

We need only to evaluate boundary conditions!

Boundary Conditions at $z = -\ell$



From KVL, we conclude:

$$V_a(z = -\ell) = V_b(z = -\ell)$$

From KCL:

$$I_a(z=-\ell) = I_b(z=-\ell) + I_R$$

And from Ohm's Law:

$$I_R = \frac{V_a(z=-\ell)}{Z_0/2} = \frac{2\,V_a(z=-\ell)}{Z_0} = \frac{2\,V_b(z=-\ell)}{Z_0}$$

We likewise know from the telegrapher's equation that:

$$\begin{aligned} V_a(z=-\ell) &= V_{0a}^{+}\,e^{-j\beta(-\ell)} + V_{0a}^{-}\,e^{+j\beta(-\ell)} \\ &= V_{0a}^{+}\,e^{+j\beta\ell} + V_{0a}^{-}\,e^{-j\beta\ell} \end{aligned}$$

And since $\ell = \lambda/4$, we find:

$$\beta\ell = \left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda}{4}\right) = \frac{\pi}{2}$$

And so:

$$\begin{aligned} V_a(z=-\ell) &= V_{0a}^{+}\,e^{+j\beta\ell} + V_{0a}^{-}\,e^{-j\beta\ell} \\ &= V_{0a}^{+}\,e^{+j(\pi/2)} + V_{0a}^{-}\,e^{-j(\pi/2)} \\ &= V_{0a}^{+}(j) + V_{0a}^{-}(-j) \\ &= j\left(V_{0a}^{+} - V_{0a}^{-}\right) \end{aligned}$$

We similarly find that:

$$V_b(z=-\ell) = j\left(V_{0b}^{+} - V_{0b}^{-}\right)$$

and for currents:

$$I_a(z=-\ell) = j\,\frac{V_{0a}^{+}+V_{0a}^{-}}{Z_0}$$

$$I_b(z=-\ell) = j\,\frac{V_{0b}^{+}+V_{0b}^{-}}{Z_0}$$

Inserting these results into our KVL boundary condition statement:

$$\begin{gathered} V_a(z=-\ell) = V_b(z=-\ell) \\ j\left(V_{0a}^{+}-V_{0a}^{-}\right) = j\left(V_{0b}^{+}-V_{0b}^{-}\right) \\ V_{0a}^{+}-V_{0a}^{-} = V_{0b}^{+}-V_{0b}^{-} \end{gathered}$$

Normalizing to (i.e., dividing by) V_{0a}^{+}, we conclude:

$$1-\frac{V_{0a}^{-}}{V_{0a}^{+}} = \frac{V_{0b}^{+}}{V_{0a}^{+}} - \frac{V_{0b}^{-}}{V_{0a}^{+}}$$

From Ohm's Law:

$$I_R = \frac{2\,V_a(z=-\ell)}{Z_0} = \frac{2\,j\left(V_{0a}^{+}-V_{0a}^{-}\right)}{Z_0}$$

$$I_R = \frac{2\,V_b(z=-\ell)}{Z_0} = \frac{2\,j\left(V_{0b}^{+}-V_{0b}^{-}\right)}{Z_0}$$

And finally from our KCL boundary condition:

$$\begin{gathered} I_a(z=-\ell) = I_b(z=-\ell) + I_R \\ j\,\frac{V_{0a}^{+}+V_{0a}^{-}}{Z_0} = j\,\frac{V_{0b}^{+}+V_{0b}^{-}}{Z_0} + I_R \end{gathered}$$

After an **enjoyable** little bit of algebra, we can thus conclude:

$$V_{0a}^{+} + V_{0a}^{-} = V_{0b}^{+} + V_{0b}^{-} - j\,I_R Z_0$$

And inserting the result from Ohm's Law:

$$\begin{aligned}
V_{0a}^{+} + V_{0a}^{-} &= V_{0b}^{+} + V_{0b}^{-} - jI_R Z_0 \\
&= V_{0b}^{+} + V_{0b}^{-} - j\left(\frac{2\,j\left(V_{0b}^{+} - V_{0b}^{-}\right)}{Z_0}\right)Z_0 \\
&= V_{0b}^{+} + V_{0b}^{-} - 2\,j^2\left(V_{0b}^{+} - V_{0b}^{-}\right)\left(\frac{Z_0}{Z_0}\right) \\
&= V_{0b}^{+} + V_{0b}^{-} - 2\left(-1\right)\left(V_{0b}^{+} - V_{0b}^{-}\right) \\
&= V_{0b}^{+} + V_{0b}^{-} + 2V_{0b}^{+} - 2V_{0b}^{-} \\
&= 3V_{0b}^{+} - V_{0b}^{-}
\end{aligned}$$

Again normalizing to V_{0a}^{+}, we get a second relationship:

$$1 + \frac{V_{0a}^{-}}{V_{0a}^{+}} = 3\frac{V_{0b}^{+}}{V_{0a}^{+}} - \frac{V_{0b}^{-}}{V_{0a}^{+}}$$

Q: *But wait! We now have* ***two*** *equations:*

$$1 - \frac{V_{0a}^{-}}{V_{0a}^{+}} = \frac{V_{0b}^{+}}{V_{0a}^{+}} - \frac{V_{0b}^{-}}{V_{0a}^{+}} \qquad\qquad 1 + \frac{V_{0a}^{-}}{V_{0a}^{+}} = 3\frac{V_{0b}^{+}}{V_{0a}^{+}} - \frac{V_{0b}^{-}}{V_{0a}^{+}}$$

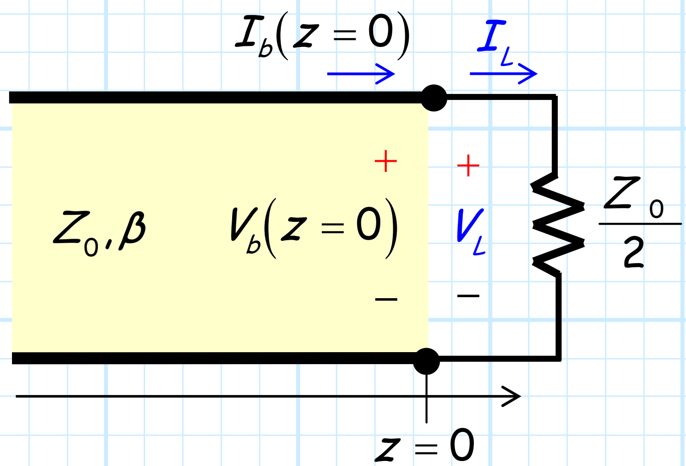
but ***three*** *unknowns:*

$$\frac{V_{0a}^{-}}{V_{0a}^{+}},\ \frac{V_{0b}^{+}}{V_{0a}^{+}},\ \frac{V_{0b}^{-}}{V_{0a}^{+}}$$

*Did we make a **mistake** somewhere?*

A: Nope! We just have more work to do. After all, there is a yet **another boundary** to be analyzed!

Boundary Conditions at $z = 0$



From KVL, we conclude:

$$V_b(z=0) = V_L$$

From KCL:

$$I_b(z=0) = I_L$$

And from Ohm's Law:

$$I_L = \frac{V_L}{Z_0/2} = \frac{2\,V_L}{Z_0}$$

We likewise know from the telegrapher's equation that:

$$\begin{aligned} V_b(z=0) &= V_{0b}^{+}\, e^{-j\beta(0)} + V_{0b}^{-}\, e^{+j\beta(0)} \\ &= V_{0b}^{+}\,(1) + V_{0b}^{-}\,(1) \\ &= V_{0b}^{+} + V_{0b}^{-} \end{aligned}$$

We similarly find that:

$$I_b(z=0) = \frac{V_{0b}^{+} - V_{0b}^{-}}{Z_0}$$

Combing this with the above results:

$$I_L = \frac{2\,V_L}{Z_0}$$

$$I_b(z=0) = \frac{2\,V_b(z=0)}{Z_0}$$

$$\frac{V_{0b}^{+} - V_{0b}^{-}}{Z_0} = \frac{2\left(V_{0b}^{+} + V_{0b}^{-}\right)}{Z_0}$$

From which we conclude:

$$V_{0b}^{+} - V_{0b}^{-} = 2\left(V_{0b}^{+} + V_{0b}^{-}\right) \quad \Rightarrow \quad -3\,V_{0b}^{-} = V_{0b}^{+}$$

And so:

$$V_{0b}^{-} = -\frac{1}{3}\,V_{0b}^{+}$$

Note that we could have also determined this using the load reflection coefficient:

$$\frac{V_b^{-}(z=0)}{V_b^{+}(z=0)} = \Gamma(z=0) = \Gamma_0$$

Where:

$$V_b^{-}(z=0) = V_{0b}^{-}\; e^{+j\beta(0)} = V_{0b}^{-}$$

$$V_b^{+}(z=0) = V_{0b}^{+}\; e^{-j\beta(0)} = V_{0b}^{+}$$

And we use the boundary condition:

$$\Gamma_{0b} = \Gamma_{Lb} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{0.5Z_0 - Z_0}{0.5Z_0 + Z_0} = \frac{-0.5}{1.5} = -\frac{1}{3}$$

Therefore, we arrive at the **same result** as before:

$$\frac{V_b^-(z=0)}{V_b^+(z=0)} = \Gamma_{0b}$$

$$\frac{V_{0b}^-}{V_{0b}^+} = -\frac{1}{3}$$

Either way, we can use this result to simplify our first set of boundary conditions:

$$\begin{aligned} 1 - \frac{V_{0a}^-}{V_{0a}^+} &= \frac{V_{0b}^+}{V_{0a}^+} - \frac{V_{0b}^-}{V_{0a}^+} \\ &= \frac{V_{0b}^+}{V_{0a}^+} - \frac{-V_{0b}^+/3}{V_{0a}^+} \\ &= \frac{V_{0b}^+}{V_{0a}^+} + \frac{1}{3}\frac{V_{0b}^+}{V_{0a}^+} \\ &= \frac{4}{3}\frac{V_{0b}^+}{V_{0a}^+} \end{aligned}$$

And:

$$1+\frac{V_{0a}^-}{V_{0a}^+}=3\frac{V_{0b}^+}{V_{0a}^+}-\frac{V_{0b}^-}{V_{0a}^+}$$
$$=3\frac{V_{0b}^+}{V_{0a}^+}-\frac{-V_{0b}^+/3}{V_{0a}^+}$$
$$=3\frac{V_{0b}^+}{V_{0a}^+}+\frac{1}{3}\frac{V_{0b}^+}{V_{0a}^+}$$
$$=\frac{10}{3}\frac{V_{0b}^+}{V_{0a}^+}$$

NOW we have **two** equations and **two** unknowns:

$$1-\frac{V_{0a}^-}{V_{0a}^+}=\frac{4}{3}\frac{V_{0b}^+}{V_{0a}^+} \qquad\qquad 1+\frac{V_{0a}^-}{V_{0a}^+}=\frac{10}{3}\frac{V_{0b}^+}{V_{0a}^+}$$

Adding the two equations, we find:

$$\left(1-\frac{V_{0a}^-}{V_{0a}^+}\right)+\left(1+\frac{V_{0a}^-}{V_{0a}^+}\right)=\left(\frac{4}{3}\frac{V_{0b}^+}{V_{0a}^+}\right)+\left(\frac{10}{3}\frac{V_{0b}^+}{V_{0a}^+}\right)$$
$$2=\frac{14}{3}\frac{V_{0b}^+}{V_{0a}^+}$$
$$\frac{3}{7}=\frac{V_{0b}^+}{V_{0a}^+}$$

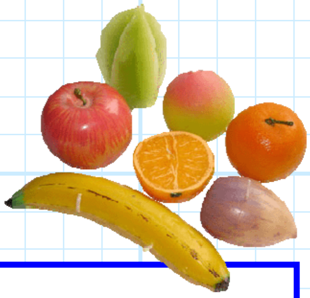
And so using the second equation above:

$$\frac{V_{0a}^{-}}{V_{0a}^{+}} = \frac{10}{3}\frac{V_{0b}^{+}}{V_{0a}^{+}} - 1$$
$$= \frac{10}{3}\frac{3}{7} - 1$$
$$= \frac{3}{7}$$

And finally, from one of our original boundary conditions:

$$\frac{V_{0b}^{-}}{V_{0a}^{+}} = \frac{V_{0b}^{+}}{V_{0a}^{+}} - 1 + \frac{V_{0a}^{-}}{V_{0a}^{+}}$$
$$= \frac{3}{7} - 1 + \frac{3}{7}$$
$$= -\frac{1}{7}$$

And so now we **summarize** the fruit of our labor:



$$\frac{V_{0a}^{-}}{V_{0a}^{+}} = \frac{3}{7} \qquad \frac{V_{0b}^{+}}{V_{0a}^{+}} = \frac{3}{7} \qquad \frac{V_{0b}^{-}}{V_{0a}^{+}} = -\frac{1}{7}$$

Yes it is! It's time for a **sanity check**!!!

The first of our boundary condition equations:

$$1-\frac{V_{0a}^-}{V_{0a}^+}=\frac{V_{0b}^+}{V_{0a}^+}-\frac{V_{0b}^-}{V_{0a}^+}$$

$$1-\frac{3}{7}=\frac{3}{7}-\left(-\frac{1}{7}\right)$$

$$\frac{4}{7}=\frac{4}{7}$$

And from the second:

$$1+\frac{V_a^-}{V_a^+}=3\frac{V_b^+}{V_a^+}-\frac{V_b^-}{V_a^+}$$

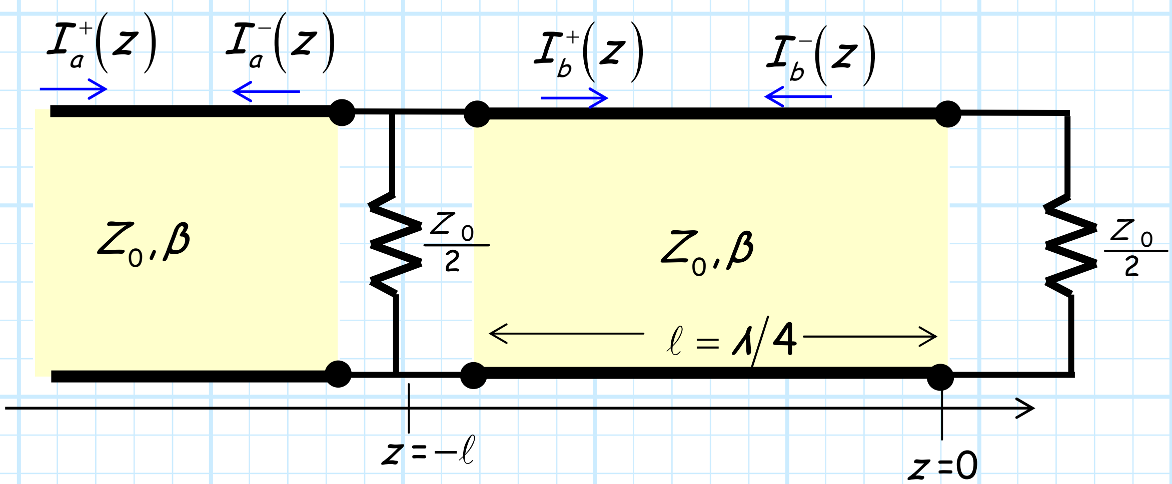
$$1+\frac{3}{7}=3\frac{3}{7}-\left(-\frac{1}{7}\right)$$

$$\frac{10}{7}=\frac{10}{7}$$

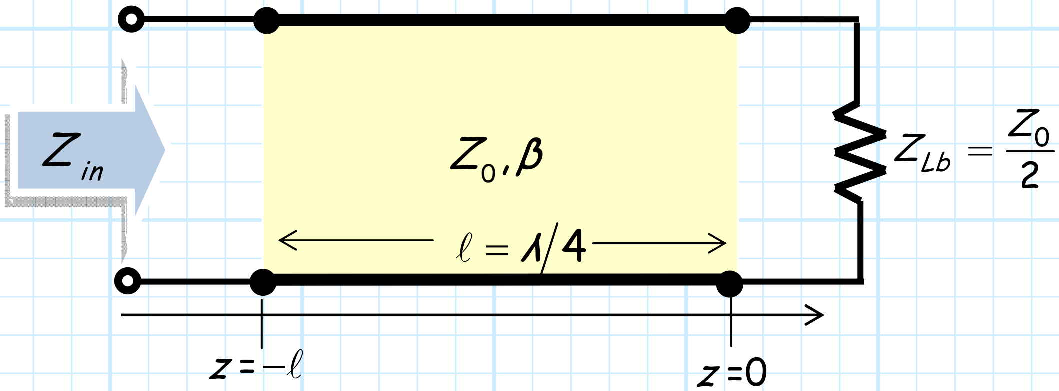
Notice that we can **also verify** the result:

$$\frac{V_{0a}^-}{V_{0a}^+}=\frac{3}{7}$$

By using the **equivalent circuit** of:



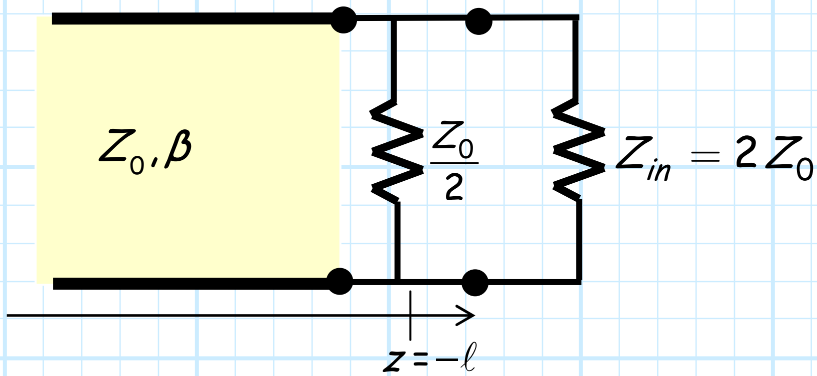
Specifically, we can determine the input impedance of this circuit:



Since the transmission line is the **special case** of one quarter wavelength, we know that:

$$Z_{in} = \frac{Z_0^2}{0.5Z_0} = 2.0\ Z_0$$

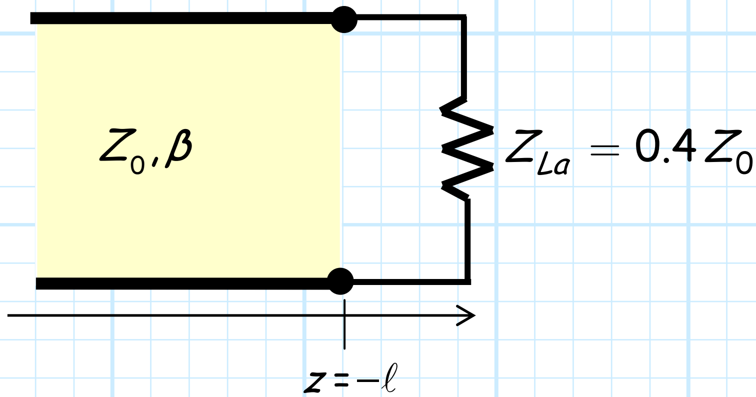
And so the equivalent circuit becomes:



Where the two parallel impedances combine as:

$$0.5Z_0 \| 2Z_0 = \frac{Z_0}{2.5} = 0.4Z_0$$

And so the equivalent load at $z=-\ell$ is $0.4Z_0$:



Now, the reflection coefficient of **this** load is:

$$\Gamma_{La} = \frac{0.4Z_0 - Z_0}{0.4Z_0 + Z_0} = \frac{-0.6}{1.4} = -\frac{3}{7}$$

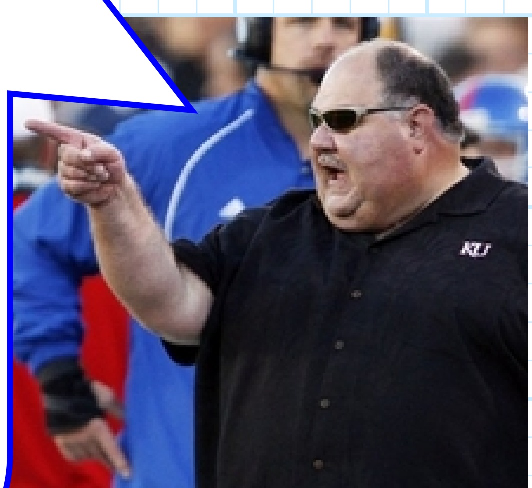
Q: *Wait a second! Using your fancy "boundary conditions" to solve the problem, you **earlier** arrived at the conclusion:*

$$\frac{V_{0a}^-}{V_{0a}^+} = \frac{3}{7}$$

*But **now** we find that instead:*

$$\frac{V_{0a}^-}{V_{0a}^+} = \Gamma_{La} = -\frac{3}{7}$$

*Apparently your annoyingly pretentious boundary condition analysis introduced some sort of **sign error** !*



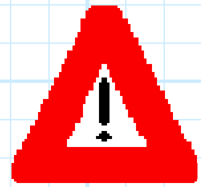
A: Absolutely not! The boundary condition analysis is perfectly correct, and:

$$\frac{V_{0a}^-}{V_{0a}^+} = \frac{3}{7}$$

is the right answer.

The statement:

$$\frac{V_a^-}{V_a^+} = \Gamma_{La} = -\frac{3}{7}$$



is **erroneous**!

Q: *But how could that possibly be? **You** earlier determined that:*

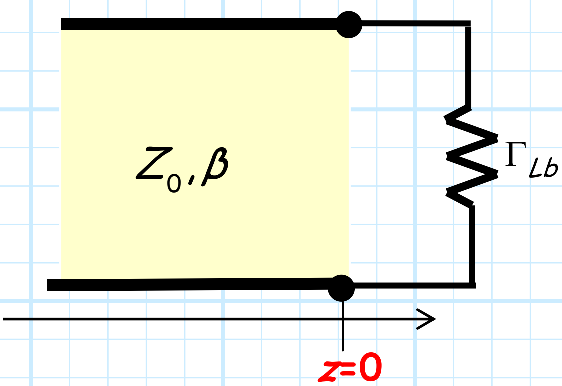
$$\frac{V_{0b}^-}{V_{0b}^+} = \Gamma_{Lb} = -\frac{1}{3}$$

So why then is:

$$\frac{V_{0a}^-}{V_{0a}^+} \neq \Gamma_{La} \quad ????$$

A: In the first case, load Γ_{Lb} is located at position $z = 0$, so that:

$$\Gamma_{Lb} = \frac{V_b^-(z=0)}{V_b^+(z=0)} = \frac{V_b^-}{V_b^+}$$

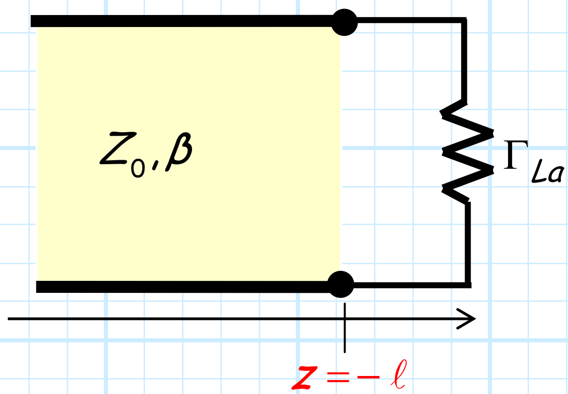


Note this result can be more compactly stated as a boundary condition requirement:

$$\Gamma_{Lb} = \Gamma(z=0) = \frac{V_{0b}^-}{V_{0b}^+} e^{+j\beta(0)} = \frac{V_{0b}^-}{V_{0b}^+} = \Gamma_{0b}$$

For the **second** case, the load Γ_{Lb} is located **instead** at position $z=-\ell$, so that:

$$\Gamma_{La} = \frac{V_a^-(z=-\ell)}{V_a^+(z=-\ell)} = \frac{V_{0a}^- e^{-j\beta\ell}}{V_{0a}^+ e^{+j\beta\ell}} = \frac{V_{0a}^-}{V_{0a}^+} e^{-j2\beta\ell} = \Gamma_{0a} e^{-j2\beta\ell}$$



Note this result can be more compactly stated as a boundary condition requirement:

$$\Gamma_{La} = \Gamma(z=-\ell) = \frac{V_{0a}^-}{V_{0a}^+} e^{-j2\beta\ell}$$

From the equation above we find:

$$\frac{V_{0a}^-}{V_{0a}^+} = \Gamma_{La}\, e^{+j2\beta\ell} = -\frac{3}{7} e^{+j\pi} = +\frac{3}{7}$$

*That's **precisely** the same result as we determined earlier using our **boundary conditions**!*

*Our answers are **good**!*