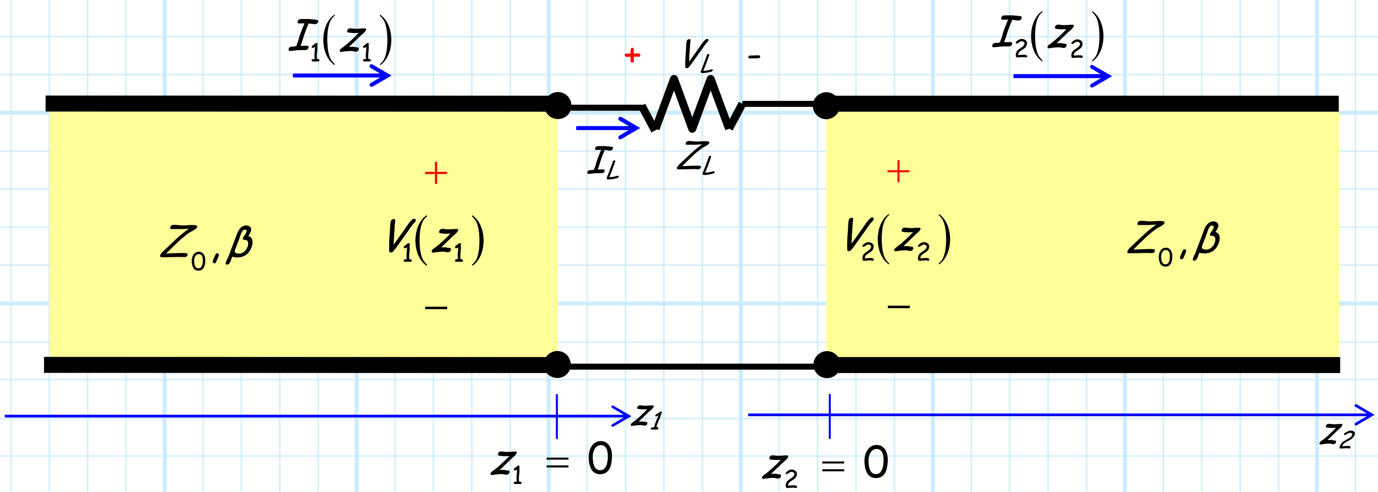
Example: Applying Boundary Conditions

Consider **this** circuit:



I.E., Two transmissions of **identical** characteristic impedance are connect by a **series** impedance Z_L . This second line is eventually **terminated** with a load $Z_L = Z_0$ (i.e., the second line is **matched**).

Q: *What is the voltage and current along each of these two transmission lines? More specifically, what are* V_{01}^{+}, V_{01}^{-}, V_{02}^{+} and V_{02}^{-} *??*

A: Since a source has not been specified, we can only determine V_{01}^{-}, V_{02}^{+} and V_{02}^{-} in terms of complex constant V_{01}^{+}. To accomplish this, we must apply a **boundary conditions** at the end of each line!

$z_1 < 0$

We know that the voltage along the **first** transmission line is:

$$V_1(z_1) = V_{01}^{+} e^{-j\beta z_1} + V_{01}^{-} e^{+j\beta z_1} \qquad [\text{for } z_1 < 0]$$

while the **current** along that same line is described as:

$$I_1(z_1) = \frac{V_{01}^{+}}{Z_0} e^{-j\beta z_1} - \frac{V_{01}^{-}}{Z_0} e^{+j\beta z_1} \qquad [\text{for } z_1 < 0]$$

$z_2 > 0$

We likewise know that the voltage along the **second** transmission line is:

$$V_2(z_2) = V_{02}^{+} e^{-j\beta z_2} + V_{02}^{-} e^{+j\beta z_2} \qquad [\text{for } z_2 > 0]$$

while the **current** along that same line is described as:

$$I_2(z_2) = \frac{V_{02}^{+}}{Z_0} e^{-j\beta z_2} - \frac{V_{02}^{-}}{Z_0} e^{+j\beta z_2} \qquad [\text{for } z_2 > 0]$$

Moreover, since the second line is terminated in a **matched load**, we know that the **reflected** wave from this load must be zero:

$$V_2^{-}(z_2) = V_{02}^{-} e^{-j\beta z_2} = 0$$

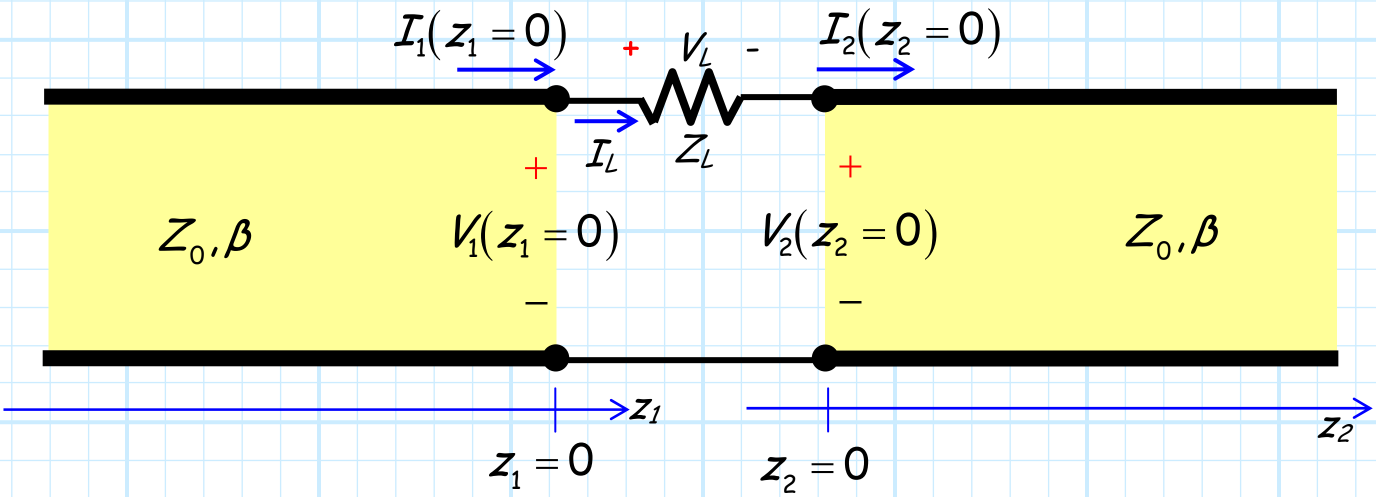
The voltage and current along the **second** transmission line is thus simply:

$$V_2(z_2) = V_2^+(z_2) = V_{02}^+ \, e^{-j\beta z_2} \qquad [\text{for } z_2 > 0]$$

$$I_2(z_2) = I_2^+(z_2) = \frac{V_{02}^+}{Z_2} \, e^{-j\beta z_2} \qquad [\text{for } z_2 > 0]$$

z=0

At the location where these two transmission lines meet, the current and voltage expressions each **must** satisfy some specific **boundary conditions**:



The **first** boundary condition comes from **KVL**, and states that:

$$V_1(z=0) - I_L Z_L = V_2(z=0)$$

$$V_{01}^+ \, e^{-j\beta(0)} + V_{01}^- \, e^{+j\beta(0)} - I_L Z_L = V_{02}^+ \, e^{-j\beta(0)}$$

$$V_{01}^+ + V_{01}^- - I_L Z_L = V_{02}^+$$

the **second** boundary condition comes from **KCL**, and states that:

$$I_1(z=0) = I_L$$

$$\frac{V_{01}^{+}}{Z_0} e^{-j\beta(0)} - \frac{V_{01}^{-}}{Z_0} e^{+j\beta(0)} = I_L$$

$$V_{01}^{+} - V_{01}^{-} = Z_0 I_L$$

while the **third** boundary condition likewise comes from **KCL**, and states that:

$$I_L = I_2(z=0)$$

$$I_L = \frac{V_{02}^{+}}{Z_0} e^{-j\beta(0)}$$

$$Z_0 I_L = V_{02}^{+}$$

Finally, we have Ohm's Law:

$$V_L = Z_L \, I_L$$

Note that we now have **four** equations and **four** unknowns ($V_{01}^{-}, V_{02}^{+}, V_L, I_L$)! We can **solve** for each in terms of V_{01}^{+} (i.e., the **incident** wave).

For **example**, let's determine V_{02}^{+} (in terms of V_{01}^{+}). We combine the **first** and **second** boundary conditions to determine:

$$V_{01}^{+} + V_{01}^{-} - I_L Z_L = V_{02}^{+}$$

$$V_{01}^{+} + \left(V_{01}^{+} - Z_0 I_L\right) - I_L Z_L = V_{02}^{+}$$

$$2V_{01}^{+} - I_L\left(Z_0 + Z_L\right) = V_{02}^{+}$$

And then adding in the **third** boundary condition:

$$2V_{01}^{+} - I_L\left(Z_0 + Z_L\right) = V_{02}^{+}$$

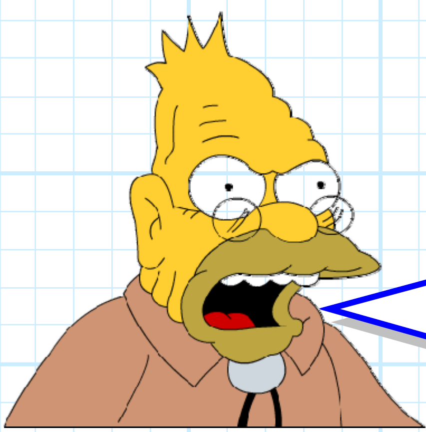
$$2V_{01}^{+} - \frac{V_{02}^{+}}{Z_0}\left(Z_0 + Z_L\right) = V_{02}^{+}$$

$$2V_{01}^{+} = V_{02}^{+}\left(\frac{2Z_0 + Z_L}{Z_0}\right)$$

Thus, we find that $V_{02}^{+} = T_0\, V_{01}^{+}$:

$$T_0 \doteq \frac{V_{02}^{+}}{V_{01}^{+}} = \frac{2Z_0}{2Z_0 + Z_L}$$

Now let's determine V_{01}^{-} (in terms of V_{01}^{+}).



Q: *Why are you wasting our time? Don't we **already** know that $V_{01}^{-} = \Gamma_0\, V_{01}^{+}$, where:*

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0}$$

A: Perhaps. Humor me while I **continue** with our **boundary condition** analysis.

We combine the **first** and **third** boundary conditions to determine:

$$V_{01}^{+} + V_{01}^{-} - I_L Z_L = V_{02}^{+}$$
$$V_{01}^{+} + V_{01}^{-} - I_L Z_L = Z_0 I_L$$
$$V_{01}^{+} + V_{01}^{-} = I_L \left(Z_0 + Z_L \right)$$

And then adding the **second** boundary condition:

$$V_{01}^{+} + V_{01}^{-} = I_L \left(Z_0 + Z_L \right)$$
$$V_{01}^{+} + V_{01}^{-} = \frac{\left(V_{01}^{+} - V_{01}^{-} \right)}{Z_0} \left(Z_0 + Z_L \right)$$
$$V_{01}^{+} \left(\frac{Z_L}{Z_0} \right) = V_{01}^{-} \left(\frac{2Z_0 + Z_L}{Z_0} \right)$$

Thus, we find that $V_{01}^{-} = \Gamma_0 V_{01}^{+}$, where:

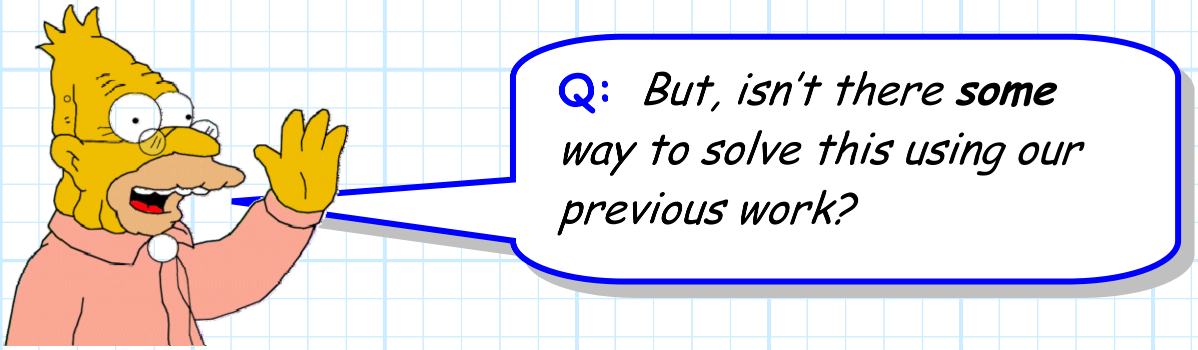
$$\Gamma_0 \doteq \frac{V_{01}^{-}}{V_{01}^{+}} = \frac{Z_L}{Z_L + 2Z_0}$$

Note this is **not** the expression:

$$\Gamma_0 \neq \frac{Z_L - Z_0}{Z_L + Z_0}$$

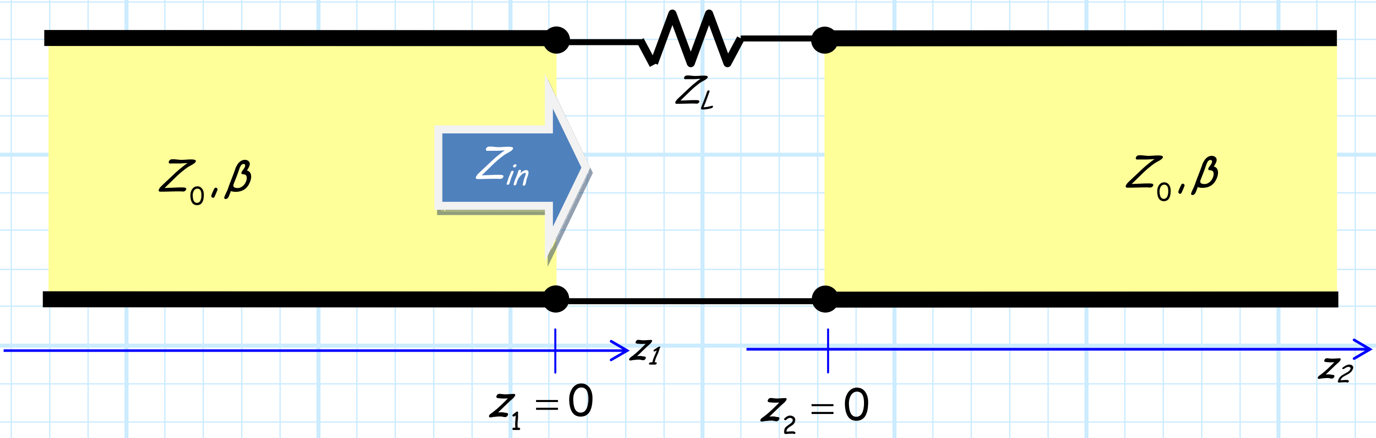
This is a completely **different** problem than the transmission line simply terminated by load Z_L. Thus, the **results** are likewise different. This shows that you must always **carefully** consider the problem you are attempting to solve, and guard against using "shortcuts" with previously derived expressions that may be **inapplicable**.

→ This is why you must know **why** a correct answer is correct!

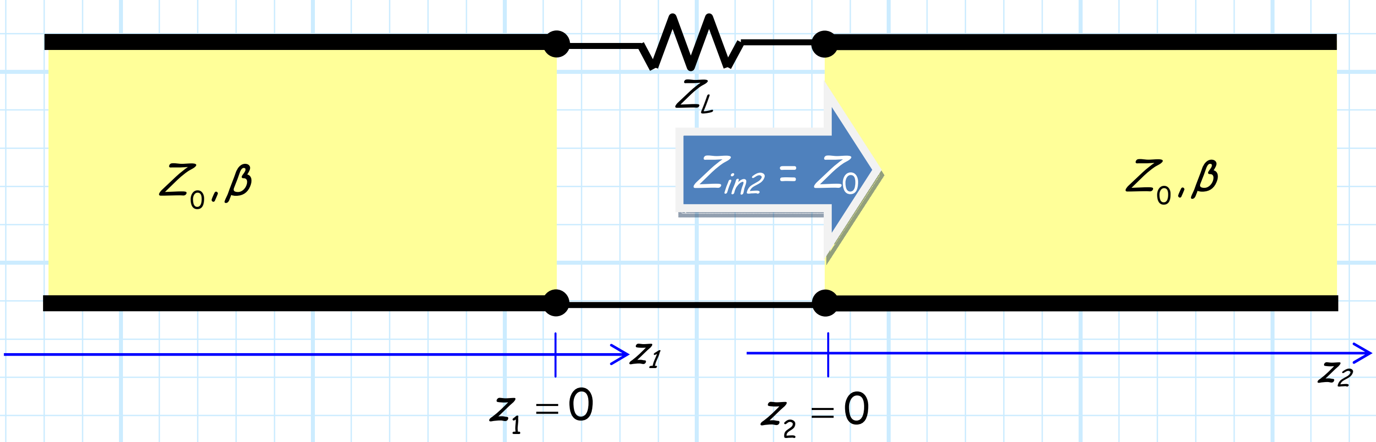


A: Actually, there is!

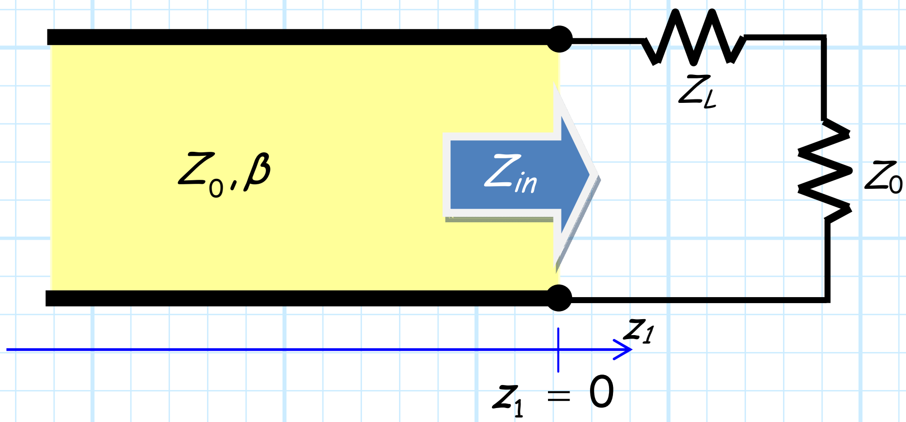
An **alternative** way for finding $\Gamma_0 = V_{01}^- / V_{01}^+$ is to determine the **input** impedance at the **end** of the **first** transmission line:



Note that since the second line is (eventually) terminated in a matched load, the input impedance at the **beginning** of the **second** line is simply equal to Z_0.



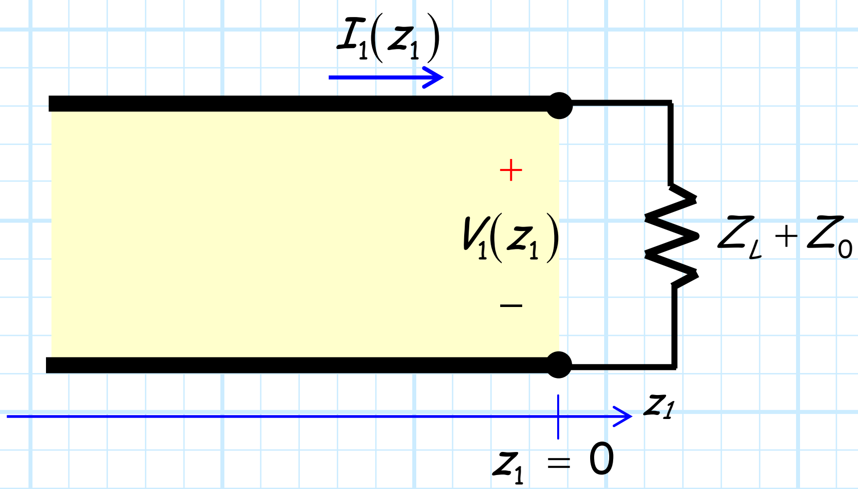
Thus, the **equivalent** circuit becomes:



And it is apparent that:

$$Z_{in} = Z_L + Z_0$$

As far as the first section of transmission line is concerned, it is **terminated** in a load with impedance $Z_L + Z_0$. The current and voltage along this first transmission line is **precisely** the same as if it **actually** were!



Thus, we find that $\Gamma_0 = V_{01}^- / V_{01}^+$, where:

$$\Gamma_0 = \frac{Z(z_1=0) - Z_0}{Z(z_1=0) + Z_0} = \frac{(Z_L + Z_0) - Z_0}{(Z_L + Z_0) + Z_0} = \frac{Z_L}{Z_L + 2Z_0}$$

Precisely the same result as before!

Now, one **more** point. Recall we found in an **earlier** handout that $T_0 = 1 + \Gamma_0$. But for this example we find that this statement is **not valid**:

$$1 + \Gamma_0 = \frac{2(Z_L + Z_0)}{Z_L + 2Z_0} \neq T_0$$

Again, be **careful** when analyzing microwave circuits!

A: An important engineering tool that **you** must master is commonly referred to as the "**sanity check**".

Simply put, a sanity check is simply **thinking** about your result, and determining whether or not it **makes sense**. A great **strategy** is to set one of the variables to a value so that the **physical** problem becomes **trivial**—so trivial that the correct answer is **obvious** to you. Then make sure your results **likewise** provide this obvious answer!

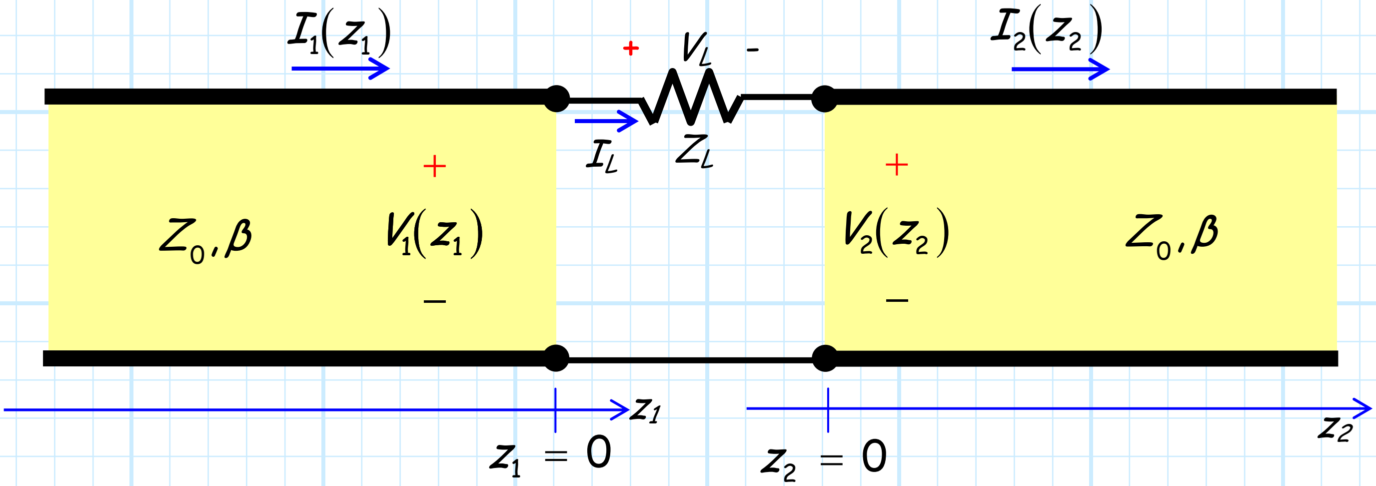
For example, consider the problem we just finished analyzing. Say that the impedance Z_L is actually a **short** circuit (Z_L=0). We find that:

$$\Gamma_0 = \left.\frac{Z_L}{Z_L + 2Z_0}\right|_{Z_L=0} = 0 \qquad\qquad T_0 = \left.\frac{2Z_0}{2Z_0 + Z_L}\right|_{Z_L=0} = 1$$

Likewise, consider the case where Z_L is actually an **open** circuit ($Z_L=\infty$). We find that:

$$\Gamma_0 = \left.\frac{Z_L}{Z_L + 2Z_0}\right|_{Z_L=\infty} = 1 \qquad\qquad T_0 = \left.\frac{2Z_0}{2Z_0 + Z_L}\right|_{Z_L=\infty} = 0$$

Think about what these results mean in terms of the **physical** problem:



Q: *Do these results **make sense**? Have we **passed** the sanity check?*