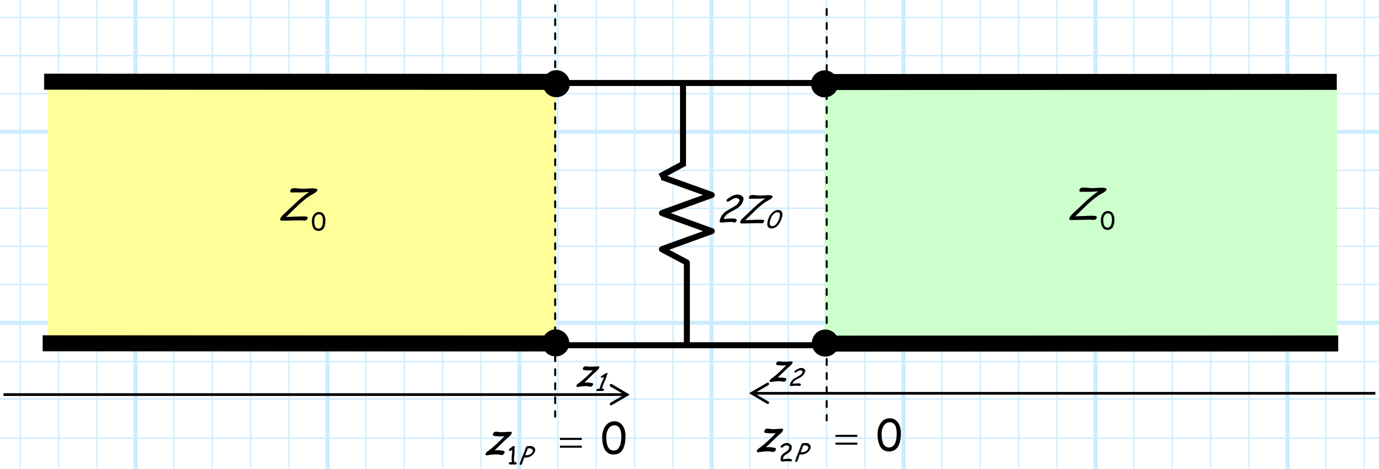
Example: Determining the Scattering Matrix

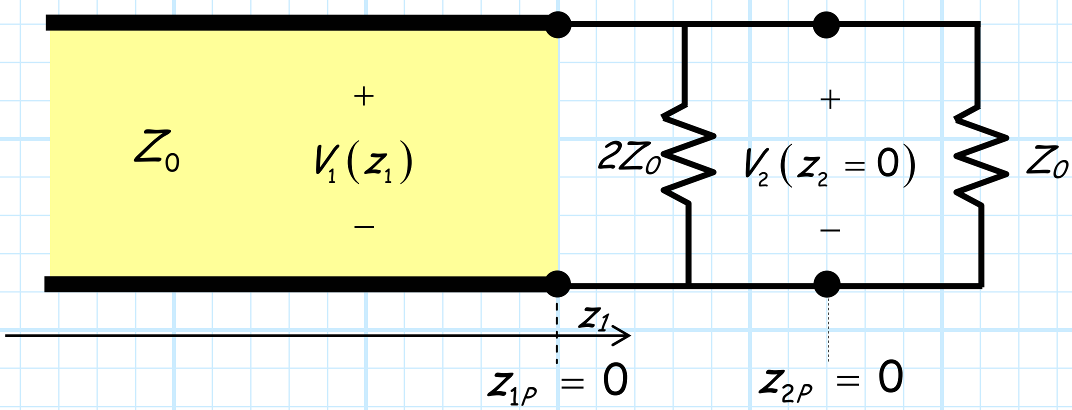
Let's determine the **scattering matrix** of this two-port device:



The **first** step is to terminate port **2** with a **matched** load, and then determine the values:

$$V_1^-\left(z_1 = z_{P1}\right) \qquad \text{and} \qquad V_2^-\left(z_2 = z_{P2}\right)$$

in terms of $V_1^+\left(z_1 = z_{P1}\right)$.



Recall that since port 2 is matched, we know that:

$$V_2^+(z_2 = z_{2P}) = 0$$

And thus:

$$\begin{aligned} V_2(z_2 = 0) &= V_2^+(z_2 = 0) + V_2^-(z_2 = 0) \\ &= 0 + V_2^-(z_2 = 0) \\ &= V_2^-(z_2 = 0) \end{aligned}$$

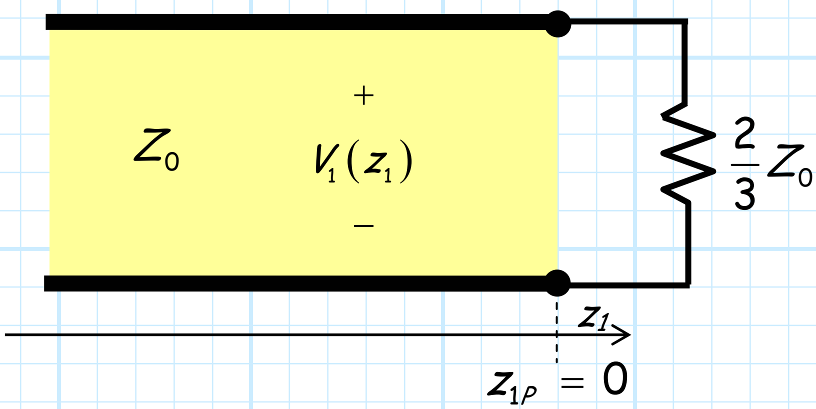
In other words, we **simply** need to determine $V_2(z_2 = 0)$ in order to find $V_2^-(z_2 = 0)$!

However, determining $V_1^-(z_1 = 0)$ is a bit **trickier**. Recall that:

$$V_1(z_1) = V_1^+(z_1) + V_1^-(z_1)$$

Therefore we find $V_1(z_1 = 0) \neq V_1^-(z_1 = 0)$!

Now, we can **simplify** this circuit:



And we know from the **telegraphers equations**:

$$
\begin{aligned}
V_1(z_1) &= V_1^+(z_1) + V_1^-(z_1) \\
&= V_{01}^+ e^{-j\beta z_1} + V_{01}^- e^{+j\beta z_1}
\end{aligned}
$$

Since the load $2Z_0/3$ is located at $z_1 = 0$, we know that the **boundary condition** leads to:

$$
V_1(z_1) = V_{01}^+ \left(e^{-j\beta z_1} + \Gamma_L e^{+j\beta z_1} \right)
$$

where:

$$
\begin{aligned}
\Gamma_L &= \frac{\left(\tfrac{2}{3}\right) Z_0 - Z_0}{\left(\tfrac{2}{3}\right) Z_0 + Z_0} \\
&= \frac{\left(\tfrac{2}{3}\right) - 1}{\left(\tfrac{2}{3}\right) + 1} \\
&= \frac{-\tfrac{1}{3}}{\tfrac{5}{3}} \\
&= -0.2
\end{aligned}
$$

Therefore:

$$
V_1^+(z_1) = V_{01}^+ e^{-j\beta z_1} \quad \text{and} \quad V_1^-(z_1) = V_{01}^+ (-0.2) e^{+j\beta z_1}
$$

and thus:

$$
V_1^+(z_1 = 0) = V_{01}^+ e^{-j\beta(0)} = V_{01}^+
$$

$$
V_1^-(z_1 = 0) = V_{01}^+ (-0.2) e^{+j\beta(0)} = -0.2 V_{01}^+
$$

We can now determine S_{11} !

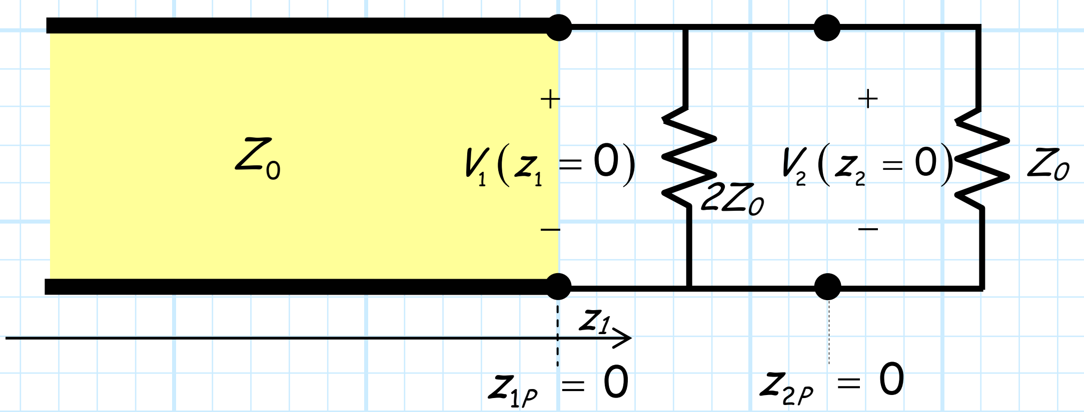
$$
S_{11} = \frac{V_1^-(z_1 = 0)}{V_1^+(z_1 = 0)} = \frac{-0.2\, V_{01}^+}{V_{01}^+} = -0.2
$$

Now its time to find $V_2^-(z_2=0)$!

Again, since port 2 is terminated, the **incident** wave on port 2 must be **zero**, and thus the value of the **exiting** wave at port 2 is equal to the **total** voltage at port 2:

$$V_2^-(z_2=0) = V_2(z_2=0)$$

This **total** voltage is relatively **easy** to determine. Examining the circuit, it is evident that $V_1(z_1=0) = V_2(z_2=0)$.



Therefore:

$$\begin{aligned} V_2(z_2=0) = V_1(z_1=0) &= V_{01}^+\left(e^{-j\beta(0)} - 0.2\,e^{+j\beta(0)}\right) \\ &= V_{01}^+(1-0.2) \\ &= V_{01}^+(0.8) \end{aligned}$$

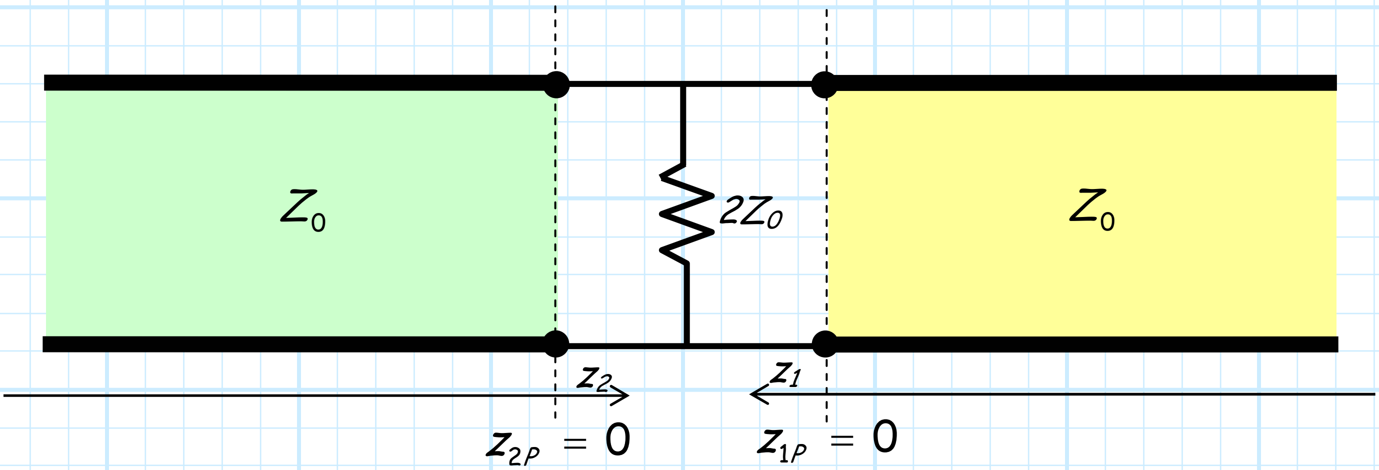
And thus the scattering parameter S_{21} is:

$$S_{21} = \frac{V_2^-(z_2=0)}{V_1^+(z_1=0)} = \frac{0.8\,V_{01}^+}{V_{01}^+} = 0.8$$

Now we **just** need to find S_{12} and S_{22}.

Q: *Yikes! This has been an awful lot of work, and you mean that we are only **half-way** done!?*

A: Actually, we are nearly finished! Note that this circuit is **symmetric**—there is really **no** difference between port 1 and port 2. If we "flip" the circuit, it remains **unchanged**!



Thus, we can conclude due to this **symmetry** that:

$$S_{11} = S_{22} = -0.2$$

and:

$$S_{21} = S_{12} = 0.8$$

Note this last equation is **likewise** a result of **reciprocity**.

Thus, the **scattering matrix** for this two port network is:

$$\mathcal{S} = \begin{bmatrix} -0.2 & 0.8 \\ 0.8 & -0.2 \end{bmatrix}$$