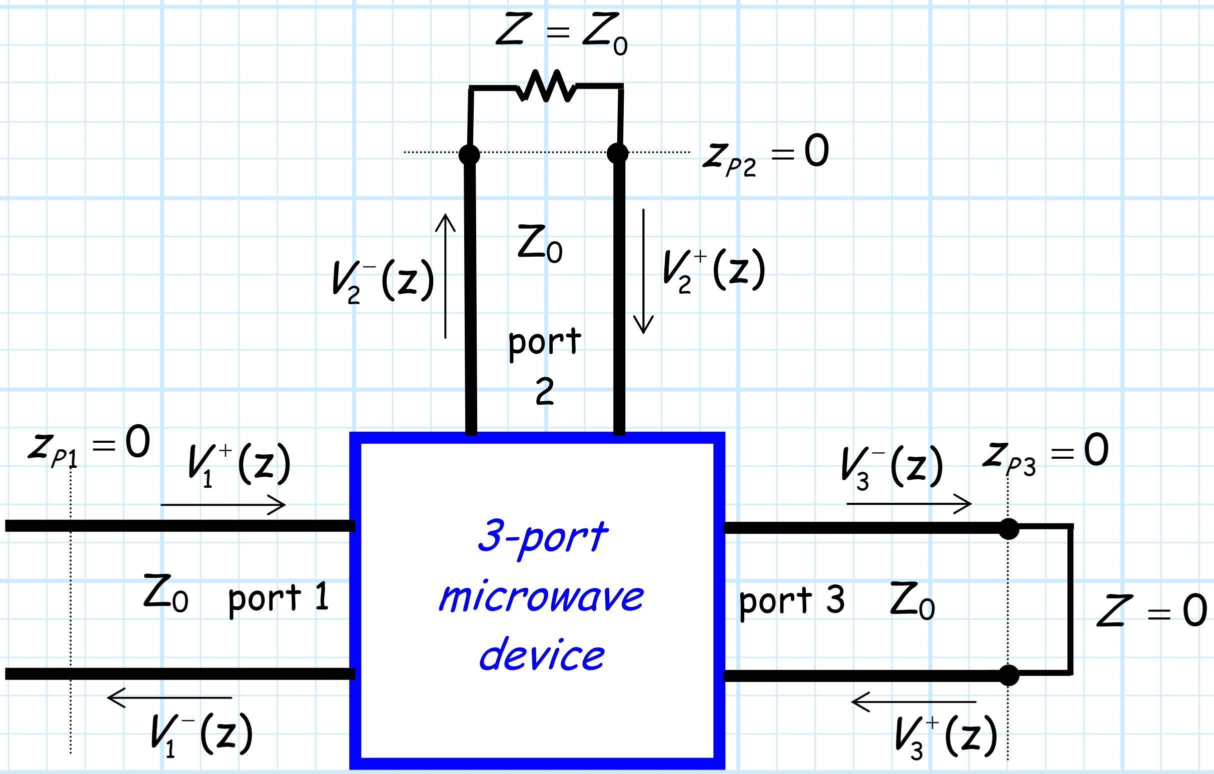
Example: The Scattering Matrix

Say we have a 3-port network that is completely characterized at some frequency ω by the **scattering matrix**:

$$\mathcal{S} = \begin{bmatrix} 0.0 & 0.2 & 0.5 \\ 0.5 & 0.0 & 0.2 \\ 0.5 & 0.5 & 0.0 \end{bmatrix}$$

A **matched load** is attached to port 2, while a **short circuit** has been placed at port 3:



Because of the **matched** load at port 2 (i.e., $\Gamma_L = 0$), we know that:

$$\frac{V_2^+(z_2 = 0)}{V_2^-(z_2 = 0)} = \frac{V_{02}^+}{V_{02}^-} = 0$$

and therefore:

$$V_{02}^+ = 0$$

*You've made a terrible mistake! Fortunately, **I** was here to correct it for you—since $\Gamma_L = 0$, the constant V_{02}^- **(not** V_{02}^+**)** is equal to zero.*

NO!! Remember, the signal $V_2^-(z)$ is **incident** on the matched load, and $V_2^+(z)$ is the **reflected** wave from the load (i.e., $V_2^+(z)$ is incident on **port 2**). Therefore, $V_{02}^+ = 0$ **is correct**!

Likewise, because of the **short** circuit at port 3 ($\Gamma_L = -1$):

$$\frac{V_3^+(z_3 = 0)}{V_3^-(z_3 = 0)} = \frac{V_{03}^+}{V_{03}^-} = -1$$

and therefore:

$$V_{03}^+ = -V_{03}^-$$

Problem:

a) Find the **reflection** coefficient at port 1, i.e.:

$$\Gamma_1 \doteq \frac{V_{01}^-}{V_{01}^+}$$

b) Find the **transmission** coefficient from port 1 to port 2, i.e.,

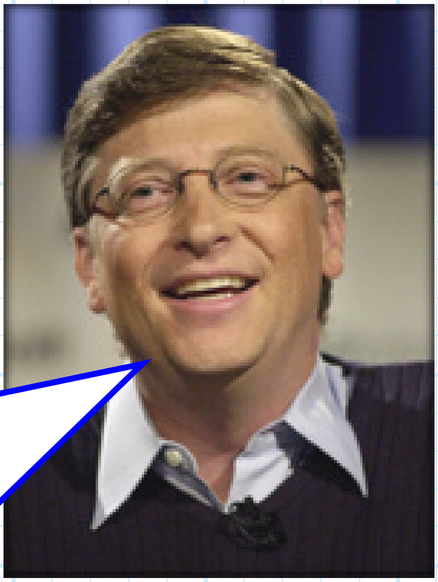
$$T_{21} \doteq \frac{V_{02}^-}{V_{01}^+}$$

*I am amused by the trivial problems that **you** apparently find so difficult. **I** know that:*

$$\Gamma_1 = \frac{V_{01}^-}{V_{01}^+} = S_{11} = 0.0$$

and

$$T_{21} = \frac{V_{02}^-}{V_{01}^+} = S_{21} = 0.5$$



NO!!! The above statement is **not correct**!

Remember, $V_{01}^- / V_{01}^+ = S_{11}$ **only** if ports 2 and 3 are terminated in **matched** loads! In this problem port 3 is terminated with a **short circuit**.

Therefore:

$$\Gamma_1 = \frac{V_{01}^-}{V_{01}^+} \neq S_{11}$$

and similarly:

$$T_{21} = \frac{V_{02}^-}{V_{01}^+} \neq S_{21}$$

To determine the values T_{21} and Γ_1, we must start with the **three** equations provided by the **scattering matrix**:

$$V_{01}^- = \qquad\qquad 0.2\,V_{02}^+ + 0.5\,V_{03}^+$$

$$V_{02}^- = 0.5\,V_{01}^+ \qquad\qquad + 0.2\,V_{03}^+$$

$$V_{03}^- = 0.5\,V_{01}^+ + 0.5\,V_{02}^+$$

and the **two** equations provided by the **attached loads**:

$$V_{02}^+ = 0$$

$$V_{03}^+ = -V_{03}^-$$

We can divide all of these equations by V_{01}^+, resulting in:

$$\Gamma_1 = \frac{V_{01}^-}{V_{01}^+} = \qquad 0.2\,\frac{V_{02}^+}{V_{01}^+} + 0.5\,\frac{V_{03}^+}{V_{01}^+}$$

$$T_{21} = \frac{V_{02}^-}{V_{01}^+} = 0.5 \qquad + 0.2\,\frac{V_{03}^+}{V_{01}^+}$$

$$\frac{V_{03}^-}{V_{01}^+} = 0.5 + 0.5\,\frac{V_{02}^+}{V_{01}^+}$$

$$\frac{V_{02}^+}{V_{01}^+} = 0$$

$$\frac{V_{03}^+}{V_{01}^+} = -\frac{V_{03}^-}{V_{01}^+}$$

Look what we have—**5** equations and **5** unknowns! Inserting equations 4 and 5 into equations 1 through 3, we get:

$$\Gamma_1 = \frac{V_{01}^-}{V_{01}^+} = -0.5\,\frac{V_{03}^+}{V_{01}^+}$$

$$T_{21} = \frac{V_{02}^-}{V_{01}^+} = 0.5 - 0.2\,\frac{V_{03}^+}{V_{01}^+}$$

$$\frac{V_{03}^-}{V_{01}^+} = 0.5$$

Solving, we find:

$$\Gamma_1 = -0.5(0.5) = -0.25$$

$$T_{21} = 0.5 - 0.2(0.5) = 0.4$$