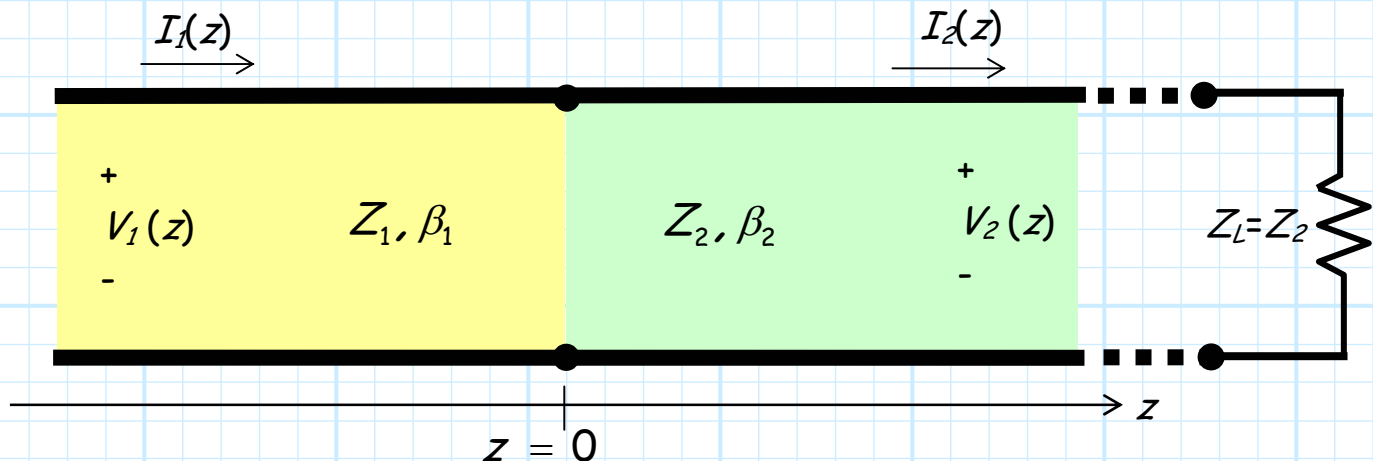


# Example: The Transmission Coefficient T

Consider this circuit:



I.E., a transmission line with characteristic impedance  $Z_1$  **transitions** to a **different** transmission line at location  $z=0$ . This second transmission line has different **characteristic impedance**  $Z_2$  ( $Z_1 \neq Z_2$ ). This second line is **terminated** with a load  $Z_L = Z_2$  (i.e., the second line is **matched**).

**Q:** *What is the voltage and current along each of these two transmission lines? More specifically, what are  $V_{01}^+$ ,  $V_{01}^-$ ,  $V_{02}^+$  and  $V_{02}^-$  ??*

**A:** Since a source has not been specified, we can only determine  $V_{01}^-$ ,  $V_{02}^+$  and  $V_{02}^-$  in terms of complex constant  $V_{01}^+$ . To accomplish this, we must apply a **boundary condition** at  $z=0$ !

$$z < 0$$

We know that the voltage along the **first** transmission line is:

$$V_1(z) = V_{01}^+ e^{-j\beta_1 z} + V_{01}^- e^{+j\beta_1 z} \quad [\text{for } z < 0]$$

while the **current** along that same line is described as:

$$I_1(z) = \frac{V_{01}^+}{Z_1} e^{-j\beta_1 z} - \frac{V_{01}^-}{Z_1} e^{+j\beta_1 z} \quad [\text{for } z < 0]$$

$$z > 0$$

We likewise know that the voltage along the **second** transmission line is:

$$V_2(z) = V_{02}^+ e^{-j\beta_2 z} + V_{02}^- e^{+j\beta_2 z} \quad [\text{for } z > 0]$$

while the **current** along that same line is described as:

$$I_2(z) = \frac{V_{02}^+}{Z_2} e^{-j\beta_2 z} - \frac{V_{02}^-}{Z_2} e^{+j\beta_2 z} \quad [\text{for } z > 0]$$

Moreover, since the second line is terminated in a **matched load**, we know that the **reflected** wave from this load must be zero:

$$V_2^-(z) = V_{02}^- e^{-j\beta_2 z} = 0$$

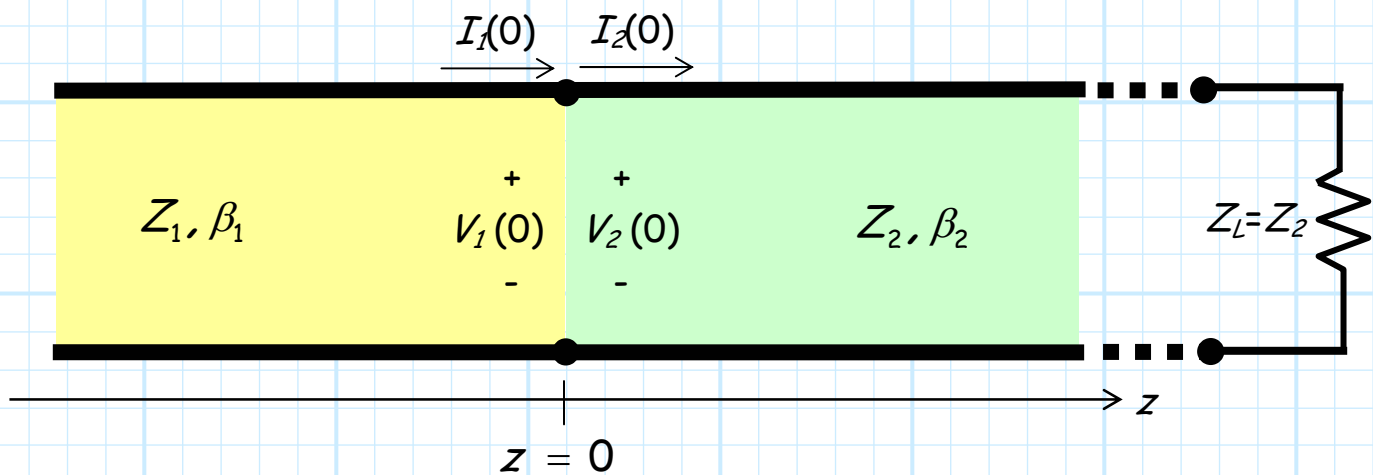
The voltage and current along the **second** transmission line is thus simply:

$$V_2(z) = V_2^+(z) = V_{02}^+ e^{-j\beta_2 z} \quad [\text{for } z > 0]$$

$$I_2(z) = I_2^+(z) = \frac{V_{02}^+}{Z_2} e^{-j\beta_2 z} \quad [\text{for } z > 0]$$

$$z=0$$

At the location where these two transmission lines meet, the current and voltage expressions each **must** satisfy some specific **boundary conditions**:



The **first** boundary condition comes from **KVL**, and states that:

$$V_1(z=0) = V_2(z=0)$$

$$V_{01}^+ e^{-j\beta_1(0)} + V_{01}^- e^{+j\beta_1(0)} = V_{02}^+ e^{-j\beta_2(0)}$$

$$V_{01}^+ + V_{01}^- = V_{02}^+$$

while the **second** boundary condition comes from **KCL**, and states that:

$$I_1(z=0) = I_2(z=0)$$

$$\frac{V_{01}^+}{Z_1} e^{-j\beta_1(0)} - \frac{V_{01}^-}{Z_1} e^{+j\beta_1(0)} = \frac{V_{02}^+}{Z_2} e^{-j\beta_2(0)}$$

$$\frac{V_{01}^+}{Z_1} - \frac{V_{01}^-}{Z_1} = \frac{V_{02}^+}{Z_2}$$

We now have **two** equations and **two** unknowns ( $V_{01}^-$  and  $V_{02}^+$ )! We can **solve** for each in terms of  $V_{01}^+$  (i.e., the **incident** wave).

From the **first** boundary condition we can state:

$$V_{01}^- = V_{02}^+ - V_{01}^+$$

Inserting this into the **second** boundary condition, we find an expression involving **only**  $V_{02}^+$  and  $V_{01}^+$ :

$$\frac{V_{01}^+}{Z_1} - \frac{V_{01}^-}{Z_1} = \frac{V_{02}^+}{Z_2}$$

$$\frac{V_{01}^+}{Z_1} - \frac{V_{02}^+ - V_{01}^+}{Z_1} = \frac{V_{02}^+}{Z_2}$$

$$\frac{2V_{01}^+}{Z_1} = \frac{V_{02}^+}{Z_2} + \frac{V_{02}^+}{Z_1}$$

Solving this expression, we find:

$$V_{02}^+ = \left( \frac{2Z_2}{Z_1 + Z_2} \right) V_{01}^+$$

We can therefore define a **transmission coefficient**, which relates  $V_{02}^+$  to  $V_{01}^+$ :

$$T_0 \doteq \frac{V_{02}^+}{V_{01}^+} = \frac{2Z_2}{Z_1 + Z_2}$$

Meaning that  $V_{02}^+ = T V_{01}^+$ , and thus:

$$V_2(z) = V_2^+(z) = T V_{01}^+ e^{-j\beta_2 z} \quad [\text{for } z > 0]$$

We can **likewise** determine the constant  $V_{01}^-$  in terms of  $V_{01}^+$ . We again start with the **first** boundary condition, from which we concluded:

$$V_{02}^+ = V_{01}^+ + V_{01}^-$$

We can insert this into the **second** boundary condition, and determine an expression involving  $V_{01}^-$  and  $V_{01}^+$  **only**:

$$\begin{aligned} \frac{V_{01}^+}{Z_1} - \frac{V_{01}^-}{Z_1} &= \frac{V_{02}^+}{Z_2} \\ \frac{V_{01}^+}{Z_1} - \frac{V_{01}^-}{Z_1} &= \frac{V_{01}^+ + V_{01}^-}{Z_2} \\ \left( \frac{1}{Z_1} - \frac{1}{Z_2} \right) V_{01}^+ &= \left( \frac{1}{Z_1} + \frac{1}{Z_2} \right) V_{01}^- \end{aligned}$$

Solving this expression, we find:

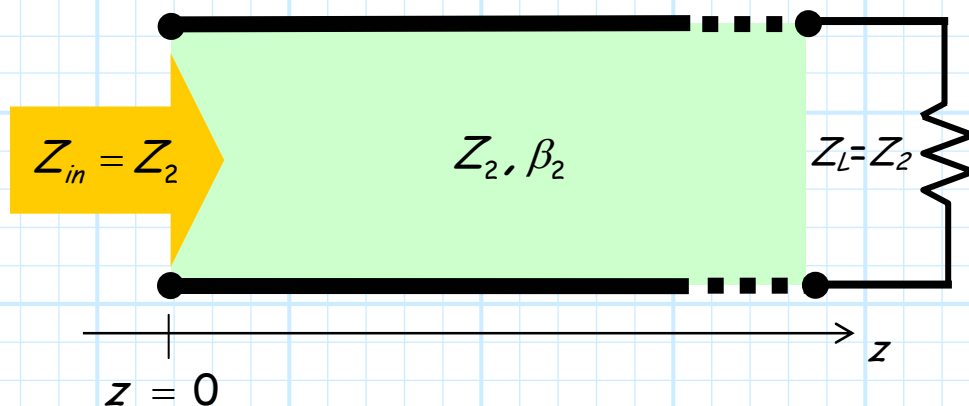
$$V_{01}^- = \left( \frac{Z_2 - Z_1}{Z_2 + Z_1} \right) V_{01}^+$$

We can therefore define a **reflection coefficient**, which relates  $V_{01}^-$  to  $V_{01}^+$ :

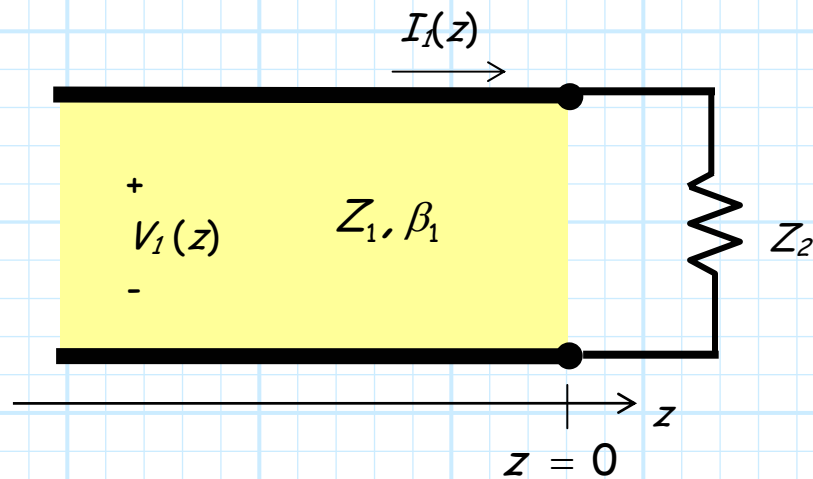
$$\Gamma_0 \doteq \frac{V_{01}^-}{V_{01}^+} = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

This result should **not** surprise us!

Note that because the **second** transmission line is **matched**, its **input impedance** is equal to  $Z_1$ :



and thus we can **equivalently** write the entire circuit as:



We have already analyzed **this** circuit! We know that:

$$\begin{aligned} V_{01}^- &= \Gamma_L V_{01}^+ \\ &= \left( \frac{Z_2 - Z_1}{Z_2 + Z_1} \right) V_{01}^+ \end{aligned}$$

Which is **exactly** the same result as we determined earlier!

The values of the reflection coefficient  $\Gamma_0$  and the transmission coefficient  $T_0$  are **not** independent, but in fact are directly **related**. Recall the **first** boundary expressed was:

$$V_{01}^+ + V_{01}^- = V_{02}^+$$

Dividing this by  $V_{01}^+$ :

$$1 + \frac{V_{01}^-}{V_{01}^+} = \frac{V_{02}^+}{V_{01}^+}$$

Since  $\Gamma_0 = V_{01}^- / V_{01}^+$  and  $\mathcal{T}_0 = V_{02}^+ / V_{01}^+$ :

$$1 + \Gamma_0 = \mathcal{T}_0$$

Note the result  $\mathcal{T}_0 = 1 + \Gamma_0$  is true for **this** particular circuit, and therefore is **not** a universally valid expression for two-port networks!