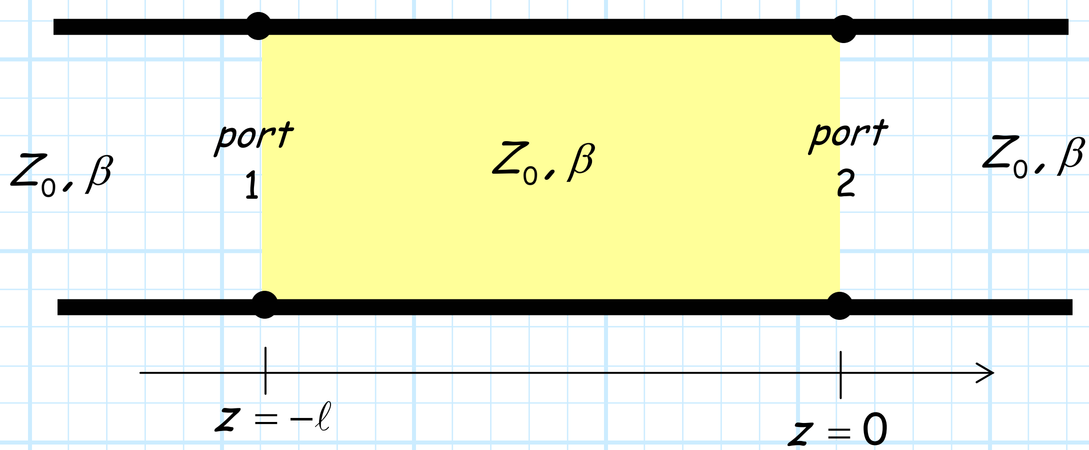
Example: Using Symmetry to Determine a Scattering Matrix

Say we wish to determine the scattering matrix of the simple two-port device shown below:



We note that that attaching transmission lines of characteristic impedance Z_0 to each port of our "circuit" forms a **continuous** transmission line of characteristic impedance Z_0.

Thus, the voltage **all along** this transmission line thus has the form:

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

We begin by defining the location of port 1 as $z_{1P} = -\ell$, and the port location of port 2 as $z_{2P} = 0$:

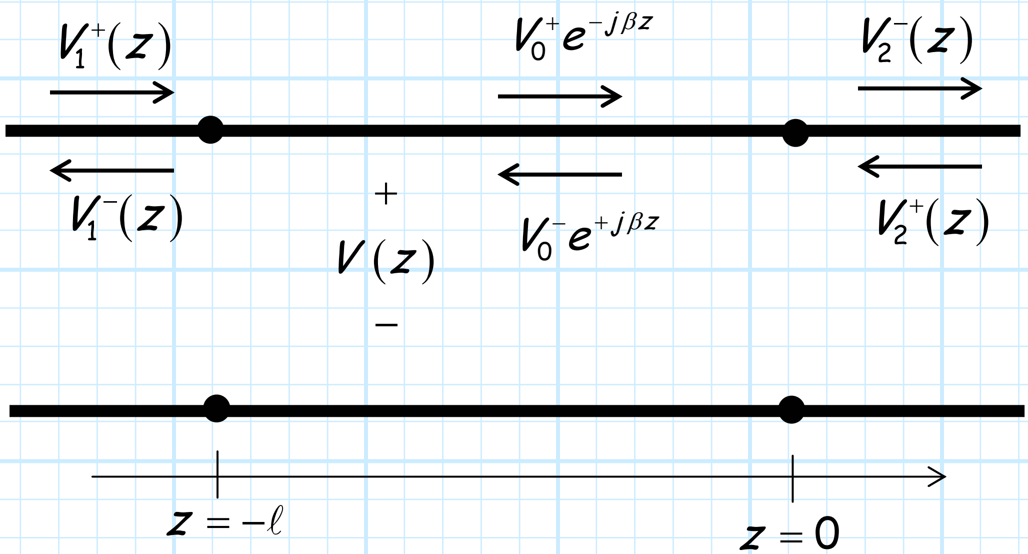
We can thus conclude:

$$V_1^+(z) = V_0^+ e^{-j\beta z} \qquad (z \leq -\ell)$$

$$V_1^-(z) = V_0^- e^{+j\beta z} \qquad (z \leq -\ell)$$

$$V_2^+(z) = V_0^- e^{+j\beta z} \qquad (z \geq 0)$$

$$V_2^-(z) = V_0^+ e^{-j\beta z} \qquad (z \geq 0)$$



Say the transmission line on port 2 is terminated in a **matched load**. We know that the **–z** wave must be **zero** $(V_0^- = 0)$, and so the voltage along the transmission line becomes simply the **+z** wave voltage:

$$V(z) = V_0^+ e^{-j\beta z}$$

and so:

$$V_1^+(z) = V_0^+ e^{-j\beta z} \qquad V_1^-(z) = 0 \qquad (z \leq -\ell)$$

$$V_2^+(z) = 0 \qquad V_2^-(z) = V_0^+ e^{-j\beta z} \qquad (z \geq 0)$$

Now, **because** port 2 is terminated in a matched load, we can determine the scattering parameters S_{11} and S_{21}:

$$S_{11} = \left. \frac{V_1^-(z = z_{1P})}{V_1^+(z = z_{1P})} \right|_{V_2^+=0} = \left. \frac{V^-(z = -\ell)}{V^+(z = -\ell)} \right|_{V_2^+=0} = \frac{0}{V_0^+ \; e^{-j\beta(-\ell)}} = 0$$

$$S_{21} = \left. \frac{V_2^-(z = z_{2P})}{V_1^+(z = z_{1P})} \right|_{V_2^+=0} = \left. \frac{V_2^-(z = 0)}{V_1^+(z = -\ell)} \right|_{V_2^+=0} = \frac{V_0^+ \; e^{-j\beta(0)}}{V_0^+ \; e^{-j\beta(-\ell)}} = \frac{1}{e^{+j\beta\ell}} = e^{-j\beta\ell}$$

From the **symmetry** of the structure, we can conclude:

$$S_{22} = S_{11} = 0$$

And from both reciprocity **and** symmetry:

$$S_{12} = S_{21} = e^{-j\beta\ell}$$

Thus:

$$\mathcal{S} = \begin{bmatrix} 0 & e^{-j\beta\ell} \\ e^{-j\beta\ell} & 0 \end{bmatrix}$$