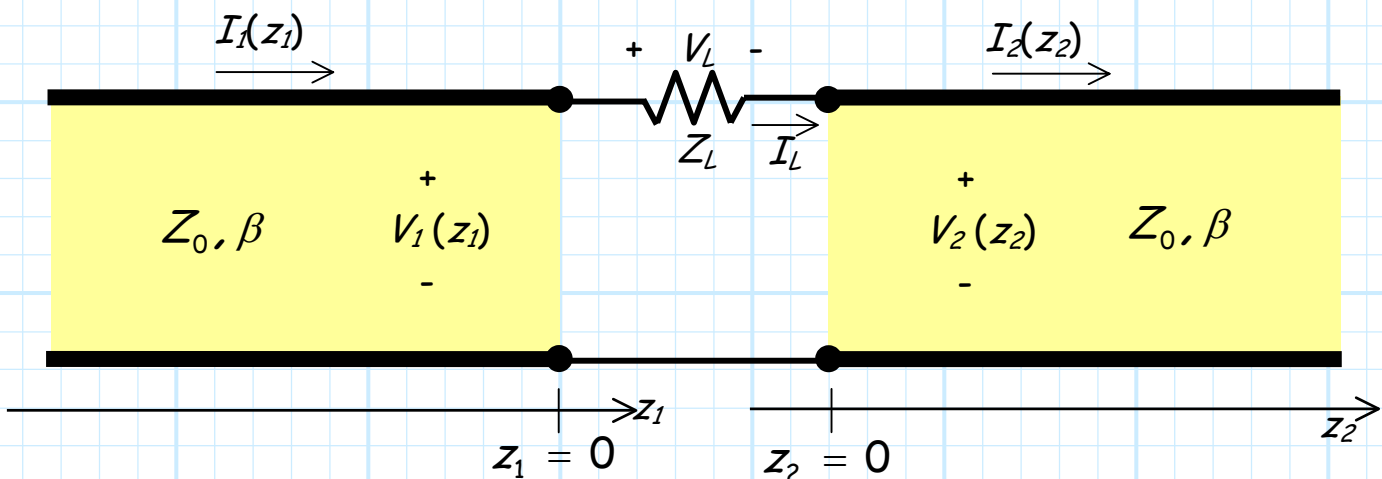


Example: Applying Boundary Conditions

Consider this circuit:



I.E., Two transmissions of **identical** characteristic impedance are connect by a **series** impedance Z_L . This second line is eventually **terminated** with a load $Z_L = Z_0$ (i.e., the second line is **matched**).

Q: *What is the voltage and current along each of these two transmission lines? More specifically, what are V_{01}^+ , V_{01}^- , V_{02}^+ and V_{02}^- ??*

A: Since a source has not been specified, we can only determine V_{01}^- , V_{02}^+ and V_{02}^- in terms of complex constant V_{01}^+ . To accomplish this, we must apply a **boundary conditions** at the end of each line!

$$z_1 < 0$$

We know that the voltage along the **first** transmission line is:

$$V_1(z_1) = V_{01}^+ e^{-j\beta z_1} + V_{01}^- e^{+j\beta z_1} \quad [\text{for } z_1 < 0]$$

while the **current** along that same line is described as:

$$I_1(z_1) = \frac{V_{01}^+}{Z_0} e^{-j\beta z_1} - \frac{V_{01}^-}{Z_0} e^{+j\beta z_1} \quad [\text{for } z_1 < 0]$$

$$z_2 > 0$$

We likewise know that the voltage along the **second** transmission line is:

$$V_2(z_2) = V_{02}^+ e^{-j\beta z_2} + V_{02}^- e^{+j\beta z_2} \quad [\text{for } z_2 > 0]$$

while the **current** along that same line is described as:

$$I_2(z_2) = \frac{V_{02}^+}{Z_0} e^{-j\beta z_2} - \frac{V_{02}^-}{Z_0} e^{+j\beta z_2} \quad [\text{for } z_2 > 0]$$

Moreover, since the second line is terminated in a **matched load**, we know that the **reflected** wave from this load must be zero:

$$V_2^-(z_2) = V_{02}^- e^{-j\beta z_2} = 0$$

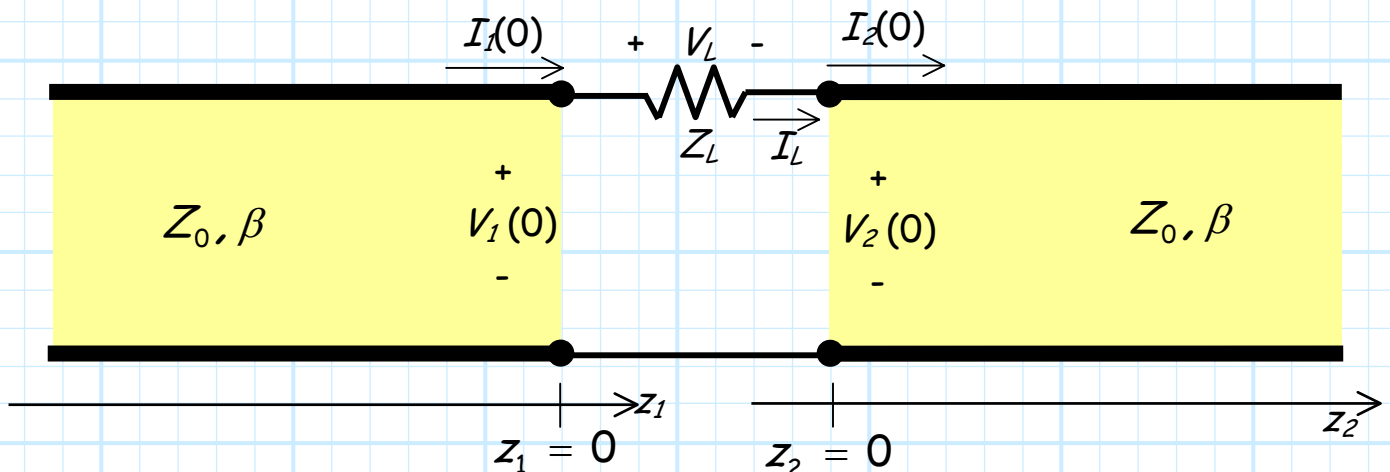
The voltage and current along the **second** transmission line is thus simply:

$$V_2(z_2) = V_2^+(z_2) = V_{02}^+ e^{-j\beta z_2} \quad [\text{for } z_2 > 0]$$

$$I_2(z_2) = I_2^+(z_2) = \frac{V_{02}^+}{Z_2} e^{-j\beta z_2} \quad [\text{for } z_2 > 0]$$

$z=0$

At the location where these two transmission lines meet, the current and voltage expressions each **must** satisfy some specific **boundary conditions**:



The **first** boundary condition comes from **KVL**, and states that:

$$V_1(z=0) - I_L Z_L = V_2(z=0)$$

$$V_{01}^+ e^{-j\beta(0)} + V_{01}^- e^{+j\beta(0)} - I_L Z_L = V_{02}^+ e^{-j\beta(0)}$$

$$V_{01}^+ + V_{01}^- - I_L Z_L = V_{02}^+$$

the **second** boundary condition comes from **KCL**, and states that:

$$I_1(z=0) = I_L$$

$$\frac{V_{01}^+}{Z_0} e^{-j\beta(0)} - \frac{V_{01}^-}{Z_0} e^{+j\beta(0)} = I_L$$

$$V_{01}^+ - V_{01}^- = Z_0 I_L$$

while the **third** boundary condition likewise comes from **KCL**, and states that:

$$I_L = I_2(z=0)$$

$$I_L = \frac{V_{02}^+}{Z_0} e^{-j\beta(0)}$$

$$Z_0 I_L = V_{02}^+$$

Finally, we have Ohm's Law:

$$V_L = Z_L I_L$$

Note that we now have **four** equations and **four** unknowns (V_{01}^- , V_{02}^+ , V_L , I_L)! We can **solve** for each in terms of V_{01}^+ (i.e., the **incident** wave).

For **example**, let's determine V_{02}^+ (in terms of V_{01}^+). We combine the **first** and **second** boundary conditions to determine:

$$V_{01}^+ + V_{01}^- - I_L Z_L = V_{02}^+$$

$$V_{01}^+ + (V_{01}^+ - Z_0 I_L) - I_L Z_L = V_{02}^+$$

$$2V_{01}^+ - I_L (Z_0 + Z_L) = V_{02}^+$$

And then adding in the **third** boundary condition:

$$2V_{01}^+ - I_L (Z_0 + Z_L) = V_{02}^+$$

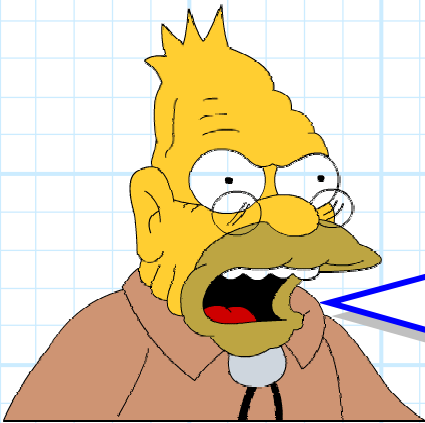
$$2V_{01}^+ - \frac{V_{02}^+}{Z_0} (Z_0 + Z_L) = V_{02}^+$$

$$2V_{01}^+ = V_{02}^+ \left(\frac{2Z_0 + Z_L}{Z_0} \right)$$

Thus, we find that $V_{02}^+ = T_0 V_{01}^+$:

$$T_0 \doteq \frac{V_{02}^+}{V_{01}^+} = \frac{2Z_0}{2Z_0 + Z_L}$$

Now let's determine V_{01}^- (in terms of V_{01}^+).



Q: *Why are you wasting our time? Don't we already know that $V_{01}^- = \Gamma_0 V_{01}^+$, where:*

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0}$$

A: Perhaps. Humor me while I **continue** with our **boundary condition** analysis.

We combine the **first** and **third** boundary conditions to determine:

$$V_{01}^+ + V_{01}^- - I_L Z_L = V_{02}^+$$

$$V_{01}^+ + V_{01}^- - I_L Z_L = Z_0 I_L$$

$$V_{01}^+ + V_{01}^- = I_L (Z_0 + Z_L)$$

And then adding the **second** boundary condition:

$$V_{01}^+ + V_{01}^- = I_L (Z_0 + Z_L)$$

$$V_{01}^+ + V_{01}^- = \frac{(V_{01}^+ - V_{01}^-)}{Z_0} (Z_0 + Z_L)$$

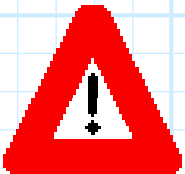
$$V_{01}^+ \left(\frac{Z_L}{Z_0} \right) = V_{01}^- \left(\frac{2Z_0 + Z_L}{Z_0} \right)$$

Thus, we find that $V_{01}^- = \Gamma_0 V_{01}^+$, where:

$$\Gamma_0 \doteq \frac{V_{01}^-}{V_{01}^+} = \frac{Z_L}{Z_L + 2Z_0}$$

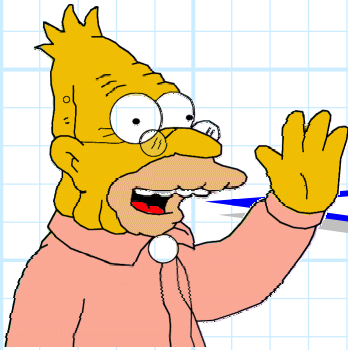
Note this is **not** the expression:

$$\Gamma_0 \neq \frac{Z_L - Z_0}{Z_L + Z_0}$$



This is a completely **different** problem than the transmission line simply terminated by load Z_L . Thus, the **results** are likewise different. This shows that you must always **carefully** consider the problem you are attempting to solve, and guard against using "shortcuts" with previously derived expressions that may be **inapplicable**.

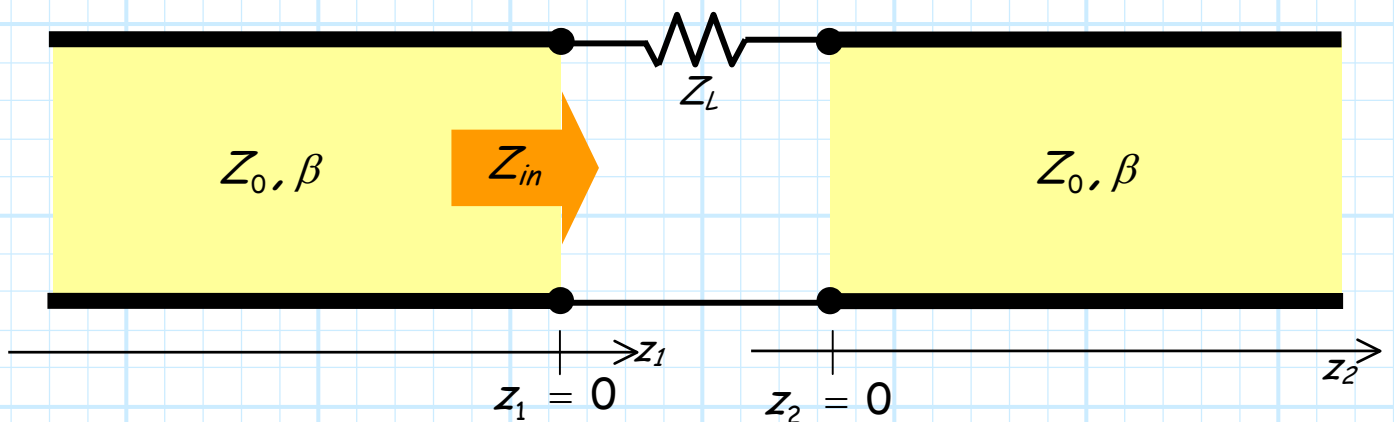
→ This is why you must know **why** a correct answer is correct!



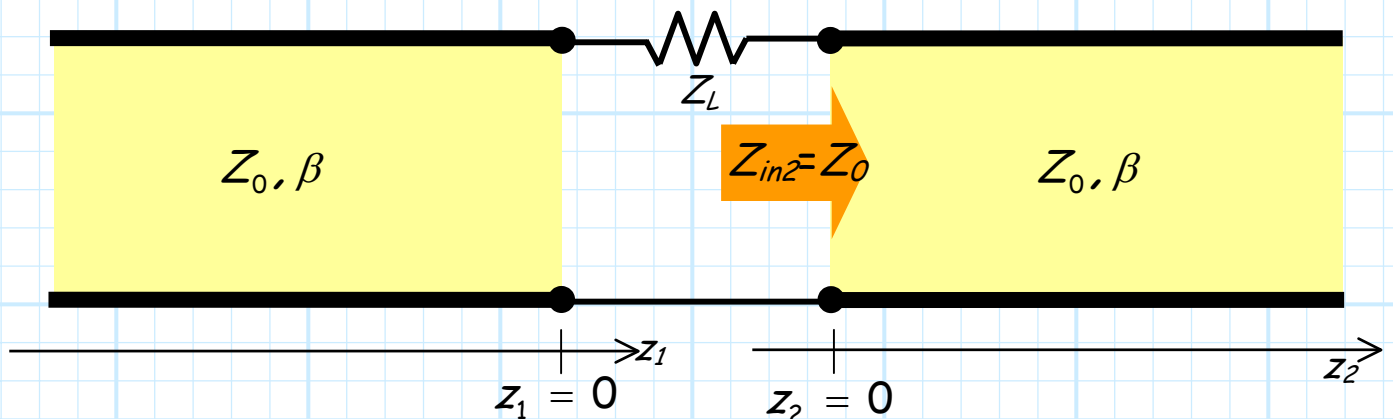
Q: But, isn't there *some* way to solve this using our previous work?

A: Actually, there is!

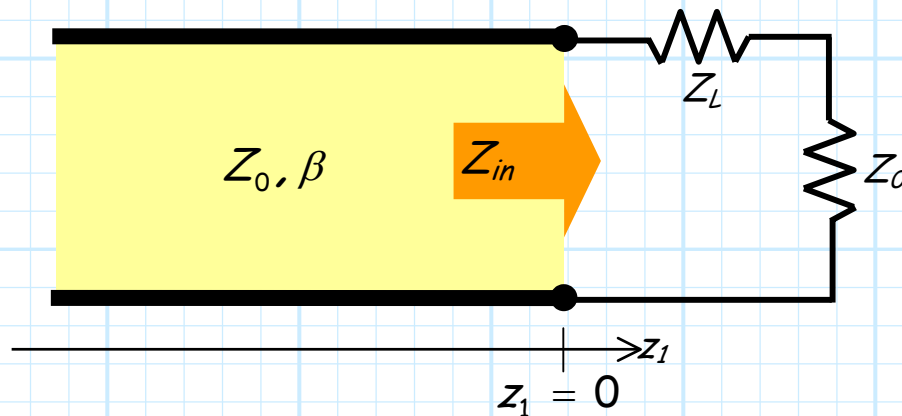
An **alternative** way for finding $\Gamma_0 = V_{01}^- / V_{01}^+$ is to determine the **input impedance at the end of the first transmission line:**



Note that since the second line is (eventually) terminated in a matched load, the input impedance at the **beginning** of the **second** line is simply equal to Z_0 .



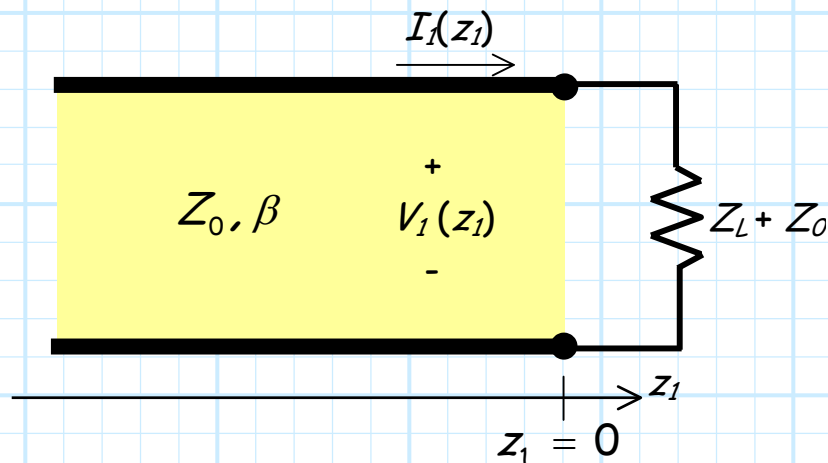
Thus, the **equivalent** circuit becomes:



And it is apparent that:

$$Z_{in} = Z_L + Z_0$$

As far as the first section of transmission line is concerned, it is **terminated** in a load with impedance $Z_L + Z_0$. The current and voltage along this first transmission line is **precisely** the same as if it **actually** were!



Thus, we find that $\Gamma_0 = V_{01}^- / V_{01}^+$, where:

$$\begin{aligned}\Gamma_0 &= \frac{Z(z_1=0) - Z_0}{Z(z_1=0) + Z_0} \\ &= \frac{(Z_L + Z_0) - Z_0}{(Z_L + Z_0) + Z_0} \\ &= \frac{Z_L}{Z_L + 2Z_0}\end{aligned}$$

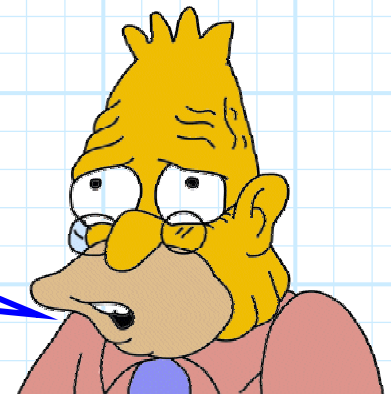
Precisely the same result as before!

Now, one **more** point. Recall we found in an **earlier** handout that $T_0 = 1 + \Gamma_0$. But for this example we find that this statement is **not valid**:

$$1 + \Gamma_0 = \frac{2(Z_L + Z_0)}{Z_L + 2Z_0} \neq T_0$$

Again, be **careful** when analyzing microwave circuits!

Q: *But this seems so difficult. How will I know if I have made a mistake?*



A: An important engineering tool that **you** must master is commonly referred to as the "**sanity check**".

Simply put, a sanity check is simply **thinking** about your result, and determining whether or not it **makes sense**. A great **strategy** is to set one of the variables to a value so that the **physical** problem becomes **trivial**—so trivial that the correct answer is **obvious** to you. Then make sure your results **likewise** provide this obvious answer!

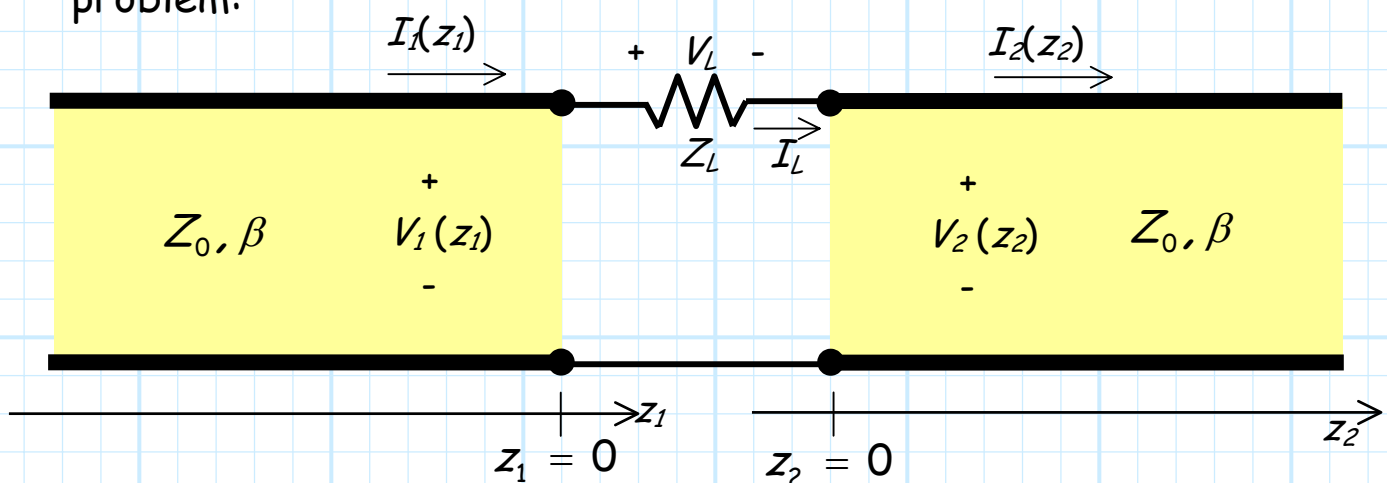
For example, consider the problem we just finished analyzing. Say that the impedance Z_L is actually a **short circuit** ($Z_L=0$). We find that:

$$\Gamma_0 = \frac{Z_L}{Z_L + 2Z_0} \Big|_{Z_L=0} = 0 \qquad T_0 = \frac{2Z_0}{2Z_0 + Z_L} \Big|_{Z_L=0} = 1$$

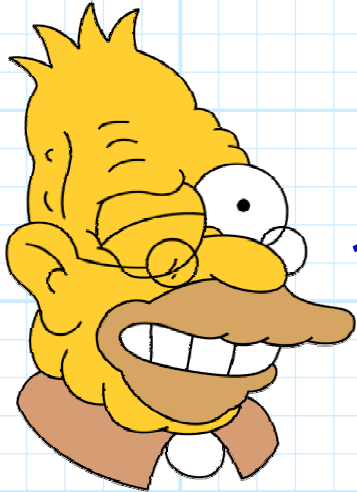
Likewise, consider the case where Z_L is actually an **open circuit** ($Z_L=\infty$). We find that:

$$\Gamma_0 = \frac{Z_L}{Z_L + 2Z_0} \Big|_{Z_L=\infty} = 1 \qquad T_0 = \frac{2Z_0}{2Z_0 + Z_L} \Big|_{Z_L=\infty} = 0$$

Think about what these results mean in terms of the **physical** problem:



Q: Do these results **make sense**? Have we **passed** the sanity check?



A: *I'll let you decide!*
What do you think?