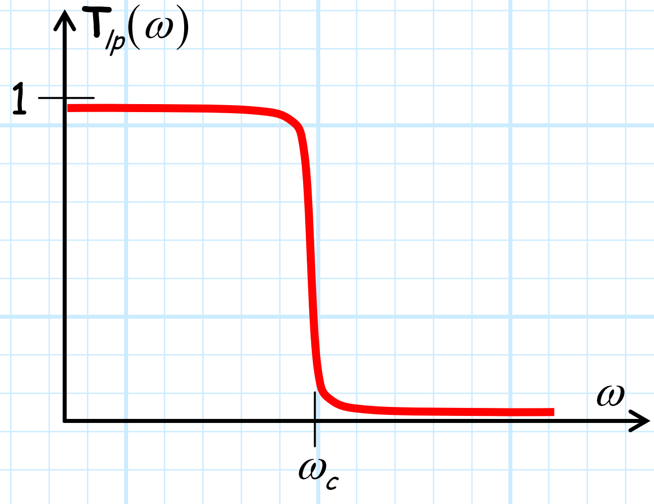
Filter Transformations

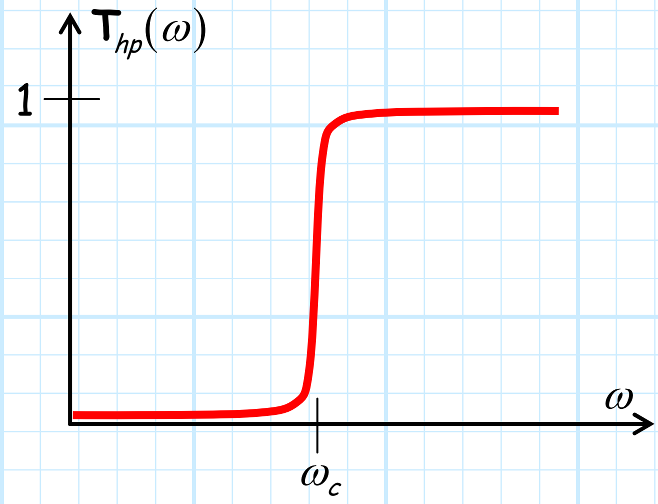
Q: *OK, so we now know how to design a lumped-element **lowpass** filter—how do we design say, a **bandpass** or **highpass** filter??*

A: If we have **already** designed a **lowpass** filter, we are **almost** done!

We can use the concept of **filter transformations** to determine the **new** filter designs from a lowpass design. As a result, we can construct a 3rd-order Butterworth **high-pass** filter or a 5th-order Chebychev **bandpass** filter!

We will find that the mathematics for each filter design will be very **similar**. For example, the difference between a lowpass and highpass filter is essentially an **inverse**—the frequencies below ω_c are mapped into frequencies above ω_c—and vice versa.





For example, we find that:

$$\mathbf{T}_{lp}(\omega = 0) = \mathbf{T}_{hp}(\omega = \infty) = 1$$

likewise:

$$\mathbf{T}_{lp}(\omega = \infty) = \mathbf{T}_{hp}(\omega = 0) = 0$$

but:

$$\mathbf{T}_{lp}(\omega = \omega_c) = \mathbf{T}_{hp}(\omega = \omega_c) = 0.5$$

Thus, in general we find:

$$\mathbf{T}_{lp}(\omega = \alpha\,\omega_c) = \mathbf{T}_{hp}\left(\omega = \frac{1}{\alpha}\,\omega_c\right)$$

where α is some positive, real value (i.e., $0 \le \alpha < \infty$).

For example, if $\alpha = 0.5$, then

$$\mathbf{T}_{lp}(\omega = 0.5\,\omega_c) = \mathbf{T}_{hp}(\omega = 2.0\omega_c)$$

In other words, the transmission through a low-pass filter at one half the cutoff frequency will be equal to the transmission through a (mathematically similar) high-pass filter at twice the cutoff frequency.

Now, recall the loss-ratio functions for Butterworth and Chebychev low-pass filters:

$$P_{LR}^{lp}(\omega) = 1 + \left(\frac{\omega}{\omega_c}\right)^{2N} \qquad\qquad P_{LR}^{lp}(\omega) = 1 + k^2\, T_N^2\left(\frac{\omega}{\omega_c}\right)$$

Note in each case that the argument of the function has the form:

$$\frac{\omega}{\omega_c}$$

In other words, the frequency is **normalized** by the cutoff frequency.

Consider now **this** mapping:

$$\frac{\omega}{\omega_c} \Rightarrow -\frac{\omega_c}{\omega}$$

This mapping **transforms** our lowpass filter response into a corresponding high pass filter response! I.E.:

$$\begin{aligned} P_{LR}^{hp}(\omega) &= 1 + \left(-\frac{\omega_c}{\omega}\right)^{2N} \\ &= 1 + \left(\frac{\omega_c}{\omega}\right)^{2N} \end{aligned}$$

$$\begin{aligned} P_{LR}^{hp}(\omega) &= 1 + k^2\, T_N^2\left(-\frac{\omega_c}{\omega}\right) \\ &= 1 + k^2\, T_N^2\left(\frac{\omega_c}{\omega}\right) \end{aligned}$$

Q: *Yikes! Where did this mapping come from? Are sure this works?*

Consider the again the case where $\omega = \alpha\,\omega_c$; the low pass responses are:

$$\begin{aligned} P_{LR}^{lp}(\omega) &= 1 + \left(\frac{\alpha\,\omega_c}{\omega_c}\right)^{2N} \\ &= 1 + (\alpha)^{2N} \end{aligned}$$

$$\begin{aligned} P_{LR}^{lp}(\omega) &= 1 + k^2\, T_N^2\left(\frac{\alpha\,\omega_c}{\omega_c}\right) \\ &= 1 + k^2\, T_N^2(\alpha) \end{aligned}$$

Now consider the high-pass responses where $\omega = \omega_c / \alpha$:

$$P_{LR}^{hp}(\omega) = 1 + \left(\frac{\omega_c}{\omega_c/\alpha} \right)^{2N} = 1 + (\alpha)^{2N}$$

$$P_{LR}^{hp}(\omega) = 1 + k^2 \, T_N^2 \left(\frac{\omega_c}{\omega_c/\alpha} \right) = 1 + k^2 \, T_N^2 (\alpha)$$

Thus, we can conclude from this mapping that:

$$P_{LR}^{lp}(\omega = \alpha \, \omega_c) = P_{LR}^{hp}\left(\omega = \frac{1}{\alpha} \omega_c \right)$$

And since $\mathbf{T} = P_{LR}^{-1}$:

$$\mathbf{T}_{lp}(\omega = \alpha \, \omega_c) = \mathbf{T}_{hp}\left(\omega = \frac{1}{\alpha} \omega_c \right)$$

Exactly the result that we expected! Our mapping provides a method for transforming a low-pass filter into a high-pass filter!

Q: *OK Poindexter, you have succeeded in providing another one of your "fascinating" mathematical insights, but does this "mapping" provide anything useful for us engineers?*

A: Absolutely! We can apply this mapping one component element (capacitor or inductor) at a time to our low-pass schematic design, and the result will be a direct transformation into a high-pass filter schematic.

Recall the reactance of an inductor element in a low-pass filter design is:

$$jX_n^{lp} = j\omega\, L_n^{lp} = j\omega\, g_n \left(\frac{R_s}{\omega_c}\right) = j\ g_n R_s \left(\frac{\omega}{\omega_c}\right)$$

while that of a capacitor is:

$$jX_n^{lp} = \frac{1}{j\omega\, C_n^{lp}} = \frac{1}{j\omega\left(\frac{g_n}{R_s\,\omega_c}\right)^{-1}} = -j\,\frac{R_s}{g_n}\left(\frac{\omega_c}{\omega}\right)$$

Now applying the mapping:

$$\frac{\omega}{\omega_c} \quad \Rightarrow \quad -\frac{\omega_c}{\omega}$$

we find for the inductor:

$$jX_n^{hp} = j\ g_n R_s \left(-\frac{\omega_c}{\omega}\right) = -j\ \frac{g_n R_s \omega_c}{\omega} = \frac{1}{j\left(g_n R_s \omega_c\right)^{-1}\,\omega}$$

and the capacitor:

$$jX_n^{hp} = -j\,\frac{R_s}{g_n}\left(-\frac{\omega}{\omega_c}\right) = j\omega\left(\frac{R_s}{g_n\omega_c}\right)$$

It is clear (do **you** see why?) that the transformation has converted a positive (i.e., inductive) reactance into a negative (i.e., capacitive) reactance—and vice versa.

As a result, to transform a low-pass filter schematic into a high-pass filter schematic, we:

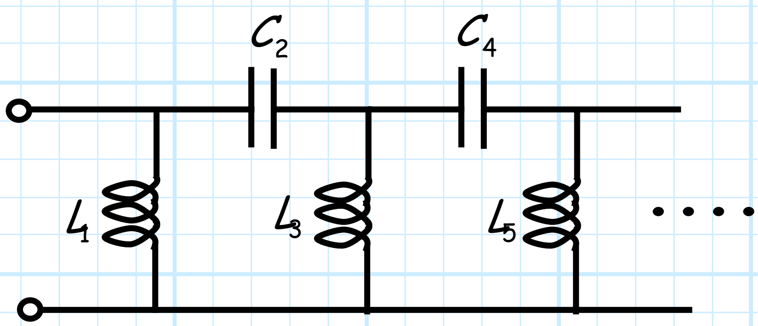
1. Replace each inductor with a capacitor of value:

$$C_n^{hp} = \frac{1}{g_n R_s \omega_c} = \frac{1}{\omega_c^2 \, L_n^{lp}}$$

2. Replace each capacitor with an inductor of value:

$$L_n^{hp} = \frac{R_s}{g_n \omega_c} = \frac{1}{\omega_c^2 \, C_n^{lp}}$$

Thus, a **high-pass ladder circuit** consists of **series capacitors** and **shunt inductors** (compare this to the low-pass) ladder circuit!).



Q: *What about band-pass filters?*

A: The difference between a lowpass and bandpass filter is simply a **shift** in the "center" frequency of the filter, where the center frequency of a lowpass filter is essentially $\omega = 0$.

For this case, we find that the **mapping**:

$$\frac{\omega}{\omega_c} \quad \Rightarrow \quad \frac{1}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

transforms a low-pass function into a **band-pass function**, where Δ is the **normalized bandwidth**:

$$\Delta = \frac{\omega_2 - \omega_1}{\omega_0}$$

and ω_1 and ω_2 define the two **3dB frequencies** of the bandpass filter.

For example, the Butterworth **low-pass** function:

$$P_{LR}(\omega) = 1 + \left(\frac{\omega}{\omega_c}\right)^{2N}$$

becomes a Butterworth **band-pass** function:

$$P_{LR}(\omega) = 1 + \frac{1}{\Delta^{2N}}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^{2N}$$

Applying this transform to the reactance of a low-pass inductive element:

$$jX_n^{bp} = j\, g_n R_s \frac{1}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = j\omega\left(\frac{g_n R_s}{\omega_0 \Delta}\right) + \frac{1}{j\omega\left(\Delta / g_n \omega_0 R_s\right)}$$

Look what happened! The transformation turned the inductive reactance into an inductive reactance in series with a capacitive reactance.

As similar analysis of the transformation of the low-pass capacitive reactance shows that it is transformed into an inductive reactance in parallel with an capacitive reactance.

As a result, to transform a low-pass filter schematic into a band-pass filter schematic, we:

1. Replace each series inductor with a capacitor and inductor in series, with values:

$$L_n^{bp} = g_n \frac{R_s}{\omega_0 \Delta} \qquad\qquad C_n^{bp} = \frac{1}{g_n} \frac{\Delta}{\omega_0 R_s}$$

2. Replace each shunt capacitor with an inductor and capacitor in parallel, with values:

$$L_n^{bp} = \frac{1}{g_n} \frac{\Delta R_s}{\omega_0} \qquad\qquad C_n^{bp} = g_n \frac{1}{\omega_0 \Delta R_s}$$

Thus, the ladder circuit for **band-pass circuit** is simply a ladder network of LC resonators, both series and parallel: