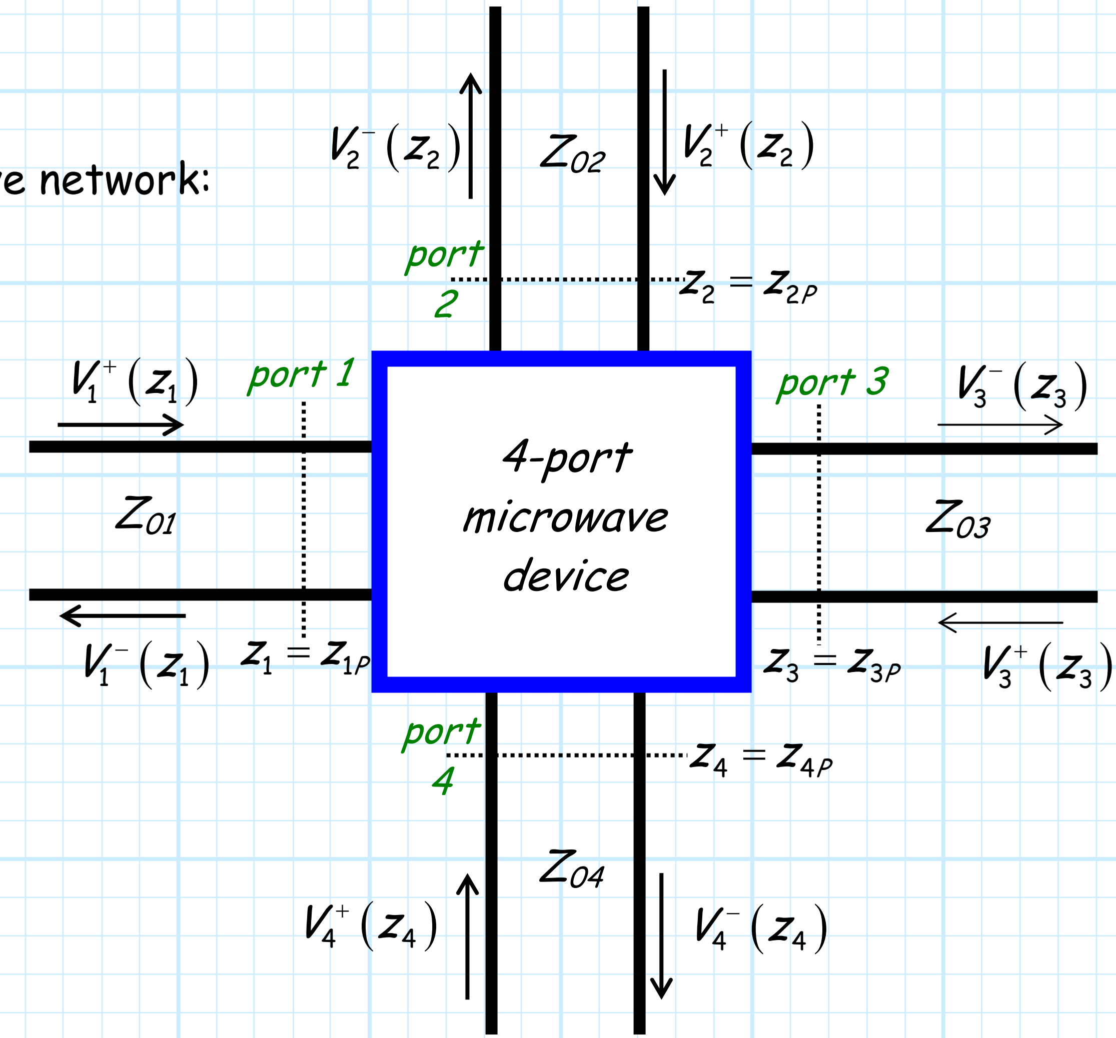
Generalized Scattering Parameters

Consider now **this** microwave network:



Q: ***Boring!*** *We studied this* ***before;*** *this will lead to the definition of scattering parameters, right?*

A: Not exactly. For this network, the **characteristic impedance** of each transmission line is **different** (i.e., $Z_{01} \neq Z_{02} \neq Z_{03} \neq Z_{04}$)!

Q: *Yikes! You said scattering parameters are **dependent** on transmission line characteristic impedance Z_0. If these values are **different** for each port, **which** Z_0 do we use?*

A: For this **general** case, we must use **generalized scattering parameters**! First, we define a slightly new form of complex wave amplitudes:

$$\boxed{a_n = \frac{V_{0n}^{+}}{\sqrt{Z_{0n}}} \qquad\qquad b_n = \frac{V_{0n}^{-}}{\sqrt{Z_{0n}}}}$$

So for example:

$$a_1 = \frac{V_{01}^{+}}{\sqrt{Z_{01}}} \qquad\qquad b_3 = \frac{V_{03}^{-}}{\sqrt{Z_{03}}}$$

The key things to note are:

a A **variable *a*** (e.g., $a_1, a_2, \cdots$) denotes the complex amplitude of an **incident (i.e., plus)** wave.

b A **variable *b*** (e.g., $b_1, b_2, \cdots$) denotes the complex amplitude of an **exiting (i.e., minus)** wave.

We now get to **rewrite** all our transmission line knowledge in terms of these generalized complex amplitudes!

First, our two propagating wave amplitudes (i.e., plus and minus) are **compactly** written as:

$$V_{0n}^{+} = a_n \sqrt{Z_{0n}} \qquad V_{0n}^{-} = b_n \sqrt{Z_{0n}}$$

And so:

$$V_n^{+}(z_n) = a_n \sqrt{Z_{0n}}\; e^{-j\beta z_n} \qquad V_n^{-}(z_n) = b_n \sqrt{Z_{0n}}\; e^{+j\beta z_n} \qquad \Gamma(z_n) = \frac{b_n}{a_n}\, e^{+j2\beta z_n}$$

Likewise, the total voltage, current, and impedance are:

$$V_n(z_n) = \sqrt{Z_{0n}}\left(a_n\, e^{-j\beta z_n} + b_n\, e^{+j\beta z_n}\right)$$

$$I_n(z_n) = \frac{a_n\, e^{-j\beta z_n} - b_n\, e^{+j\beta z_n}}{\sqrt{Z_{0n}}}$$

$$Z(z_n) = \frac{a_n\, e^{-j\beta z_n} + b_n\, e^{+j\beta z_n}}{a_n\, e^{-j\beta z_n} - b_n\, e^{+j\beta z_n}}$$

Assuming that our port planes are defined with $z_{nP} = 0$, we can determine the total voltage, current, and impedance **at port n** as:

$$V_n \doteq V_n(z_n = 0) = \sqrt{Z_{0n}}\,(a_n + b_n) \qquad I_n \doteq I_n(z_n = 0) = \frac{a_n - b_n}{\sqrt{Z_{0n}}}$$

$$Z_n \doteq Z(z_n = 0) = \frac{a_n + b_n}{a_n - b_n}$$

Likewise, the **power** associated with each wave is:

$$P_n^+ = \frac{\left|V_{0n}^+\right|^2}{2Z_{0n}} = \frac{\left|a_n\right|^2}{2} \qquad P_n^- = \frac{\left|V_{0n}^-\right|^2}{2Z_{0n}} = \frac{\left|b_n\right|^2}{2}$$

As such, the power **delivered to** port n (i.e., the power **absorbed by** port n) is:

$$P_n = P_n^+ - P_n^- = \frac{\left|a_n\right|^2 - \left|b_n\right|^2}{2}$$

This result is also **verified**:

$$
\begin{aligned}
P_n &= \frac{1}{2}\mathrm{Re}\left\{V_n I_n^*\right\} \\
&= \frac{1}{2}\mathrm{Re}\left\{\left(a_n + b_n\right)\left(a_n^* - b_n^*\right)\right\} \\
&= \frac{1}{2}\mathrm{Re}\left\{a_n a_n^* + b_n a_n^* - a_n b_n^* - b_n b_n^*\right\} \\
&= \frac{1}{2}\mathrm{Re}\left\{\left|a_n\right|^2 + b_n a_n^* - \left(b_n a_n^*\right)^* - \left|b_n\right|^2\right\} \\
&= \frac{1}{2}\mathrm{Re}\left\{\left|a_n\right|^2 + j\,\mathrm{Im}\left\{b_n a_n^*\right\} - \left|b_n\right|^2\right\} \\
&= \frac{\left|a_n\right|^2 - \left|b_n\right|^2}{2}
\end{aligned}
$$

Q: *So what's the **big deal**? This is yet **another way** to express transmission line activity. Do we **really** need to know this, or is this simply a strategy for making the next exam **even harder**?*

$$Z_1 = \frac{a_1 + b_1}{a_1 - b_1}$$

You may have noticed that this notation (a_n, b_n) provides descriptions that are a bit **"cleaner"** and more symmetric between current and voltage.

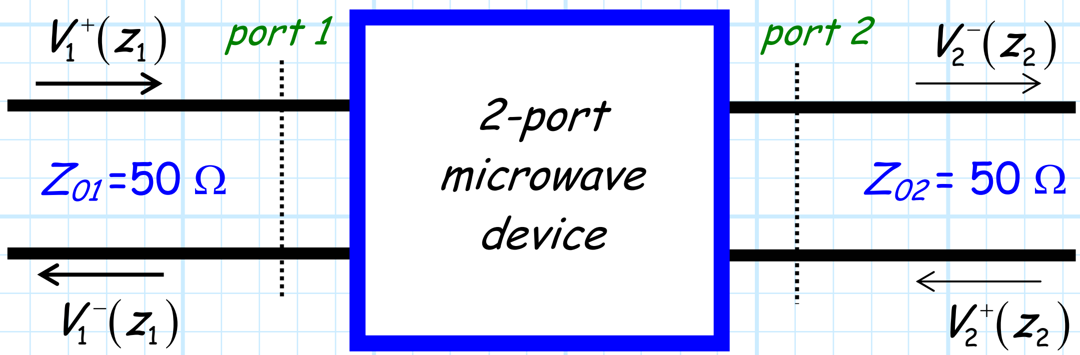
However, the **main reason** for this notation is for evaluating the **scattering parameters** of a device with **dissimilar** transmission line impedance (e.g., $Z_{01} \neq Z_{02} \neq Z_{03} \neq Z_{04}$).

For these cases we must use **generalized scattering parameters**:

$$S_{mn} = \frac{V_{0m}^-}{V_{0n}^+}\frac{\sqrt{Z_{0n}}}{\sqrt{Z_{0m}}} \qquad \text{(when } V_k^+(z_k)=0 \text{ for all } k \neq n\text{)}$$

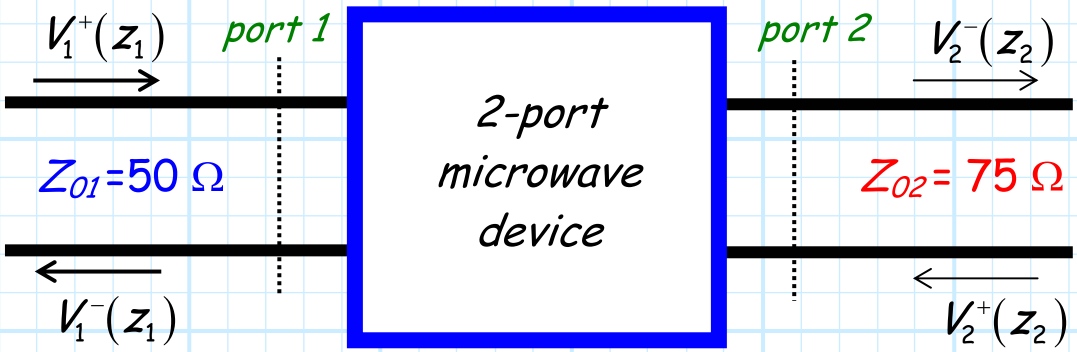
Note that **if** the transmission lines at each port are identical ($Z_{0m} = Z_{0n}$), the scattering parameter definition "reverts back" to the original (i.e., $S_{mn} = V_{0m}^- / V_{0n}^+$ **if** $Z_{0m} = Z_{0n}$). E.G.:

$$S_{21} = \frac{V_{02}^-}{V_{01}^+} \quad \text{when } V_{02}^+ = 0$$



But, **if** the transmission lines at each port are **dissimilar** ($Z_{0m} \neq Z_{0n}$), our original scattering parameter definition is **not correct** (i.e., $S_{mn} \neq V_{0m}^{-}/V_{0n}^{+}$ **if** $Z_{0m} \neq Z_{0n}$)! E.G.:

$$S_{21} \neq \frac{V_{02}^{-}}{V_{01}^{+}} \quad \text{when } V_{02}^{+} = 0$$



$$S_{21} = \frac{V_{02}^{-}}{V_{01}^{+}} \frac{\sqrt{50}}{\sqrt{75}} \quad \text{when } V_{02}^{+} = 0$$

Note that the generalized scattering parameters can be more **compactly** written in terms of our **new** wave amplitude notation:

$$\boxed{S_{mn} = \frac{V_{0m}^{-}}{V_{0n}^{+}} \frac{\sqrt{Z_{0n}}}{\sqrt{Z_{0m}}} = \frac{b_m}{a_n} \qquad (\text{when } a_k = 0 \text{ for all } k \neq n)}$$

Remember, this is the **generalized** form of scattering parameter—it **always** provides the correct answer, **regardless** of the values of Z_{0m} or Z_{0n}!

Q: *But **why can't** we define the scattering parameter as $S_{mn} = V_{0m}^{-} / V_{0n}^{+}$, **regardless** of Z_{0m} or Z_{0n}?? **Who says** we must define it with those **awful** $\sqrt{Z_{0n}}$ values in there?*

A: Good question! Recall that a lossless device is will **always** have a **unitary** scattering matrix. As a result, the scattering parameters of a lossless device will **always** satisfy, for example:

$$1 = \sum_{m=1}^{M} \left| S_{mn} \right|^2$$

This is true **only** if the scattering parameters are **generalized**!

The scattering parameters of a lossless device will form a unitary matrix **only** if defined as $S_{mn} = b_m / a_n$. If we use $S_{mn} = V_{0m}^{-} / V_{0n}^{+}$, the matrix will be unitary **only** if the connecting transmission lines have the **same** characteristic impedance.

Q: *Do we really **care** if the matrix of a lossless device is unitary or not?*

A: **Absolutely** we do! The:

$$\text{lossless device} \Leftrightarrow \text{unitary scattering matrix}$$

relationship is a very powerful one. It allows us to **identify** lossless devices, and it allows us to determine **if** specific lossless devices are **even possible**!