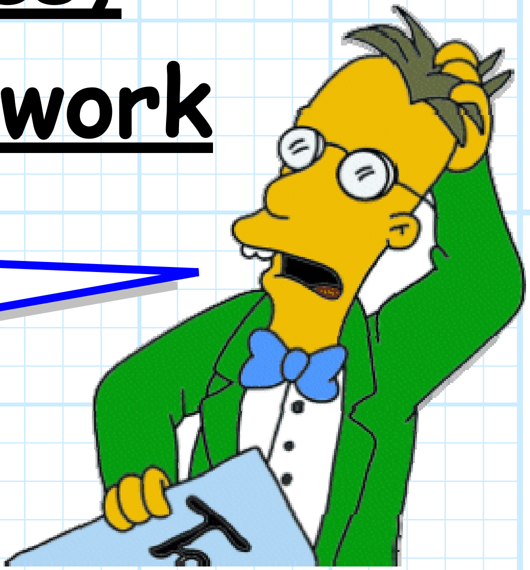
The Matched, Lossless, Reciprocal, 4-Port Network

*Guess what! I have determined that—unlike a **3-port** device—a matched, lossless, reciprocal **4-port** device **is** physically possible! In fact, I've found **two** general solutions!*



The first solution is referred to as the **symmetric** solution:

$$\mathcal{S} = \begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & j\beta \\ j\beta & 0 & 0 & \alpha \\ 0 & j\beta & \alpha & 0 \end{bmatrix}$$

Note for this symmetric solution, every row and every column of the scattering matrix has the **same** four values (i.e., α, $j\beta$, and two zeros)!

The second solution is referred to as the **anti-symmetric** solution:

$$\mathcal{S} = \begin{bmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & -\beta \\ \beta & 0 & 0 & \alpha \\ 0 & -\beta & \alpha & 0 \end{bmatrix}$$

Note that for this anti-symmetric solution, **two** rows and **two** columns have the same four values (i.e., α, β, and two zeros), while the **other** two row and columns have (slightly) **different** values (α, $-\beta$, and two zeros)

It is **quite** evident that each of these solutions are **matched** and **reciprocal**. However, to ensure that the solutions are indeed **lossless**, we must place an **additional** constraint on the values of α, β. Recall that a **necessary** condition for a lossless device is:

$$\sum_{m=1}^{N} |S_{mn}|^2 = 1 \quad \text{for all } n$$

Applying this to the **symmetric** case, we find:

$$|\alpha|^2 + |\beta|^2 = 1$$

Likewise, for the **anti-symmetric** case, we also get

$$|\alpha|^2 + |\beta|^2 = 1$$

It is evident that if the scattering matrix is **unitary** (i.e., lossless), the values α and β **cannot** be independent, but must **related** as:

$$\boxed{|\alpha|^2 + |\beta|^2 = 1}$$

Generally speaking, we will find that $|\alpha| \geq |\beta|$. Given the constraint on these two values, we can thus conclude that:

$$0 \leq |\beta| \leq {}^{1}\!/\!_{\sqrt{2}} \qquad \text{and} \qquad {}^{1}\!/\!_{\sqrt{2}} \leq |\alpha| \leq 1$$