**Mapping Z to \( \Gamma \)**

Recall that line impedance and reflection coefficient are equivalent—either one can be expressed in terms of the other:

\[
\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0} \quad \text{and} \quad Z(z) = Z_0 \left( \frac{1 + \Gamma(z)}{1 - \Gamma(z)} \right)
\]

Note this relationship also depends on the characteristic impedance \( Z_0 \) of the transmission line. To make this relationship more direct, we first define a normalized impedance value \( z' \) (an impedance coefficient!):

\[
z'(z) = \frac{Z(z)}{Z_0} = \frac{R(z)}{Z_0} + j \frac{X(z)}{Z_0} = r(z) + j x(z)
\]

Using this definition, we find:

\[
\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0} = \frac{Z(z)/Z_0 - 1}{Z(z)/Z_0 + 1} = \frac{z'(z) - 1}{z'(z) + 1}
\]
Thus, we can express $\Gamma(z)$ explicitly in terms of normalized impedance $z'$—and vice versa!

$$\Gamma(z) = \frac{z'(z) - 1}{z'(z) + 1} \quad \quad z'(z) = \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$$

The equations above describe a mapping between coefficients $z'$ and $\Gamma$. This means that each and every normalized impedance value likewise corresponds to one specific point on the complex $\Gamma$ plane!

For example, say we wish to mark or somehow indicate the values of normalized impedance $z'$ that correspond to the various points on the complex $\Gamma$ plane.

Some values we already know specifically:

<table>
<thead>
<tr>
<th>case</th>
<th>$Z$</th>
<th>$z'$</th>
<th>$\Gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>$Z_0$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>$jZ_0$</td>
<td>$j$</td>
<td>$j$</td>
</tr>
<tr>
<td>5</td>
<td>$-jZ_0$</td>
<td>$-j$</td>
<td>$-j$</td>
</tr>
</tbody>
</table>
Therefore, we find that these five normalized impedances map onto five specific points on the complex $\Gamma$ plane:

Or, the five complex $\Gamma$ map onto five points on the normalized impedance plane:
Now, the preceding provided examples of the mapping of points between the complex (normalized) impedance plane, and the complex $\Gamma$ plane.

We can likewise map whole contours (i.e., sets of points) between these two complex planes. We shall first look at two familiar cases.

$$Z = R$$

In other words, the case where impedance is purely real, with no reactive component (i.e., $X = 0$).

Meaning that normalized impedance is:

$$Z' = r + j0 \quad (i.e., \ x = 0)$$

where we recall that $r = R/Z_0$.

Remember, this real-valued impedance results in a real-valued reflection coefficient:

$$\Gamma = \frac{r - 1}{r + 1}$$

I.E.,

$$\Gamma_r \triangleq Re\{\Gamma\} = \frac{r - 1}{r + 1} \quad \Gamma_i \triangleq Im\{\Gamma\} = 0$$
Thus, we can determine a mapping between two contours—one contour ($x = 0$) on the normalized impedance plane, the other ($\Gamma_i = 0$) on the complex $\Gamma$ plane:

$$x = 0 \iff \Gamma_i = 0$$
In other words, the case where impedance is purely imaginary, with no resistive component (i.e., \( R = 0 \)).

Meaning that normalized impedance is:

\[
z' = 0 + jx \quad (i.e., \ r = 0)
\]

where we recall that \( x = X/Z_0 \).

Remember, this imaginary impedance results in a reflection coefficient with unity magnitude:

\[
|\Gamma| = 1
\]

Thus, we can determine a mapping between two contours—one contour (\( r = 0 \)) on the normalized impedance plane, the other (\( |\Gamma| = 1 \)) on the complex \( \Gamma \) plane:

\[
r = 0 \quad \Leftrightarrow \quad |\Gamma| = 1
\]
$r = 0$ 
$(|\Gamma| = 1)$ 

Invalid Region

$x = j \infty$ 

Invalid Region

$x = -j \infty$
Q: These two “mappings” may very well be fascinating in an academic sense, but they are not particularly relevant, since actual values of impedance generally have both a real and imaginary component.

Sure, mappings of more general impedance contours (e.g., \( r = 0.5 \) or \( x = -1.5 \)) onto the complex \( \Gamma \) plane would be useful—but it seems clear that those mappings are impossible to achieve!?!?

A: Actually, not only are mappings of more general impedance contours (such as \( r = 0.5 \) and \( x = -1.5 \)) onto the complex \( \Gamma \) plane possible, these mappings have already been achieved—thanks to Dr. Smith and his famous chart!