Mapping Z to Γ

Recall that line impedance and reflection coefficient are **equivalent**—either one can be expressed in terms of the other:

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0} \qquad \text{and} \qquad Z(z) = Z_0\left(\frac{1+\Gamma(z)}{1-\Gamma(z)}\right)$$

Note this relationship also depends on the characteristic impedance Z_0 of the transmission line. To make this relationship **more direct**, we first define a **normalized** impedance value z' (an impedance coefficient!):

$$z'(z) = \frac{Z(z)}{Z_0} = \frac{R(z)}{Z_0} + j\,\frac{X(z)}{Z_0} = r(z) + j\,x(z)$$

Using this definition, we find:

$$\begin{aligned}\Gamma(z) &= \frac{Z(z) - Z_0}{Z(z) + Z_0} \\ &= \frac{Z(z)/Z_0 - 1}{Z(z)/Z_0 + 1} \\ &= \frac{z'(z) - 1}{z'(z) + 1}\end{aligned}$$

Thus, we can express $\Gamma(z)$ explicitly in terms of normalized impedance z'--and vice versa!

$$\Gamma(z) = \frac{z'(z)-1}{z'(z)+1} \qquad\qquad z'(z) = \frac{1+\Gamma(z)}{1-\Gamma(z)}$$

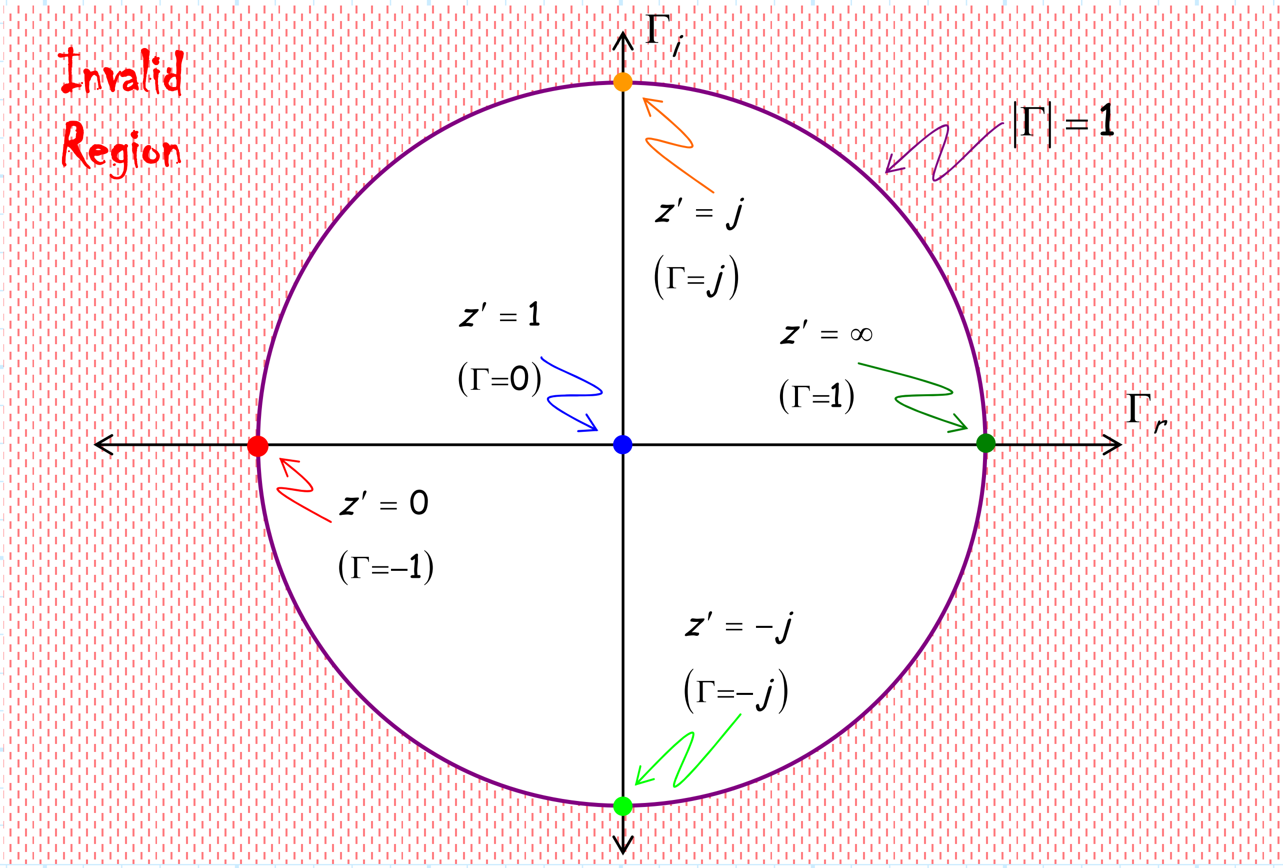
The equations above describe a **mapping** between coefficients z' and Γ. This means that each and every normalized impedance value likewise corresponds to one specific point on the complex Γ plane!

For example, say we wish to mark or somehow indicate the values of normalized impedance z' that correspond to the various points on the complex Γ plane.

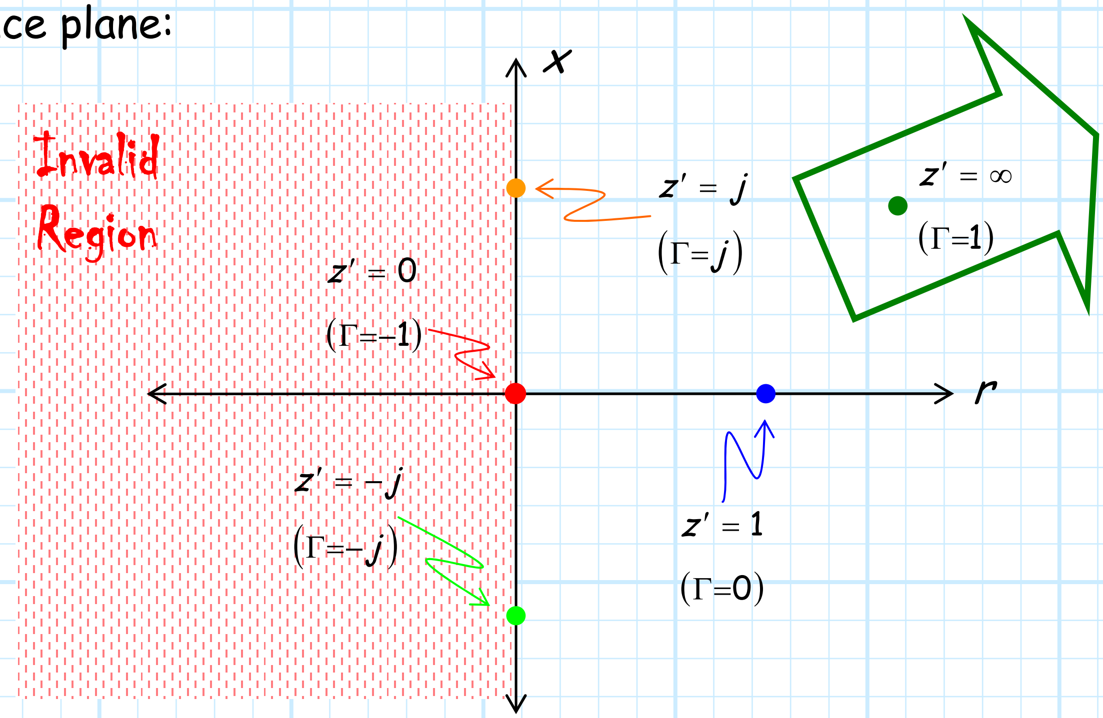
Some values we already know **specifically**:

case	Z	z'	Γ
1	∞	∞	1
2	0	0	-1
3	Z_0	1	0
4	$j\,Z_0$	j	j
5	$-j\,Z_0$	$-j$	$-j$

Therefore, we find that these five normalized impedances map onto five specific points on the complex Γ plane:



Or, the five complex Γ map onto five points on the normalized impedance plane:



Now, the preceding provided examples of the mapping of **points** between the complex (normalized) impedance plane, and the complex Γ plane.

We can likewise **map whole contours** (i.e., sets of points) between these two complex planes. We shall first look at two familiar cases.

$$Z = R$$

In other words, the case where impedance is purely **real**, with **no** reactive component (i.e., $X = 0$).

Meaning that **normalized** impedance is:

$$z' = r + j0 \qquad (i.e., x = 0)$$

where we recall that $r = R/Z_0$.

Remember, this real-valued impedance results in a real-valued reflection coefficient:

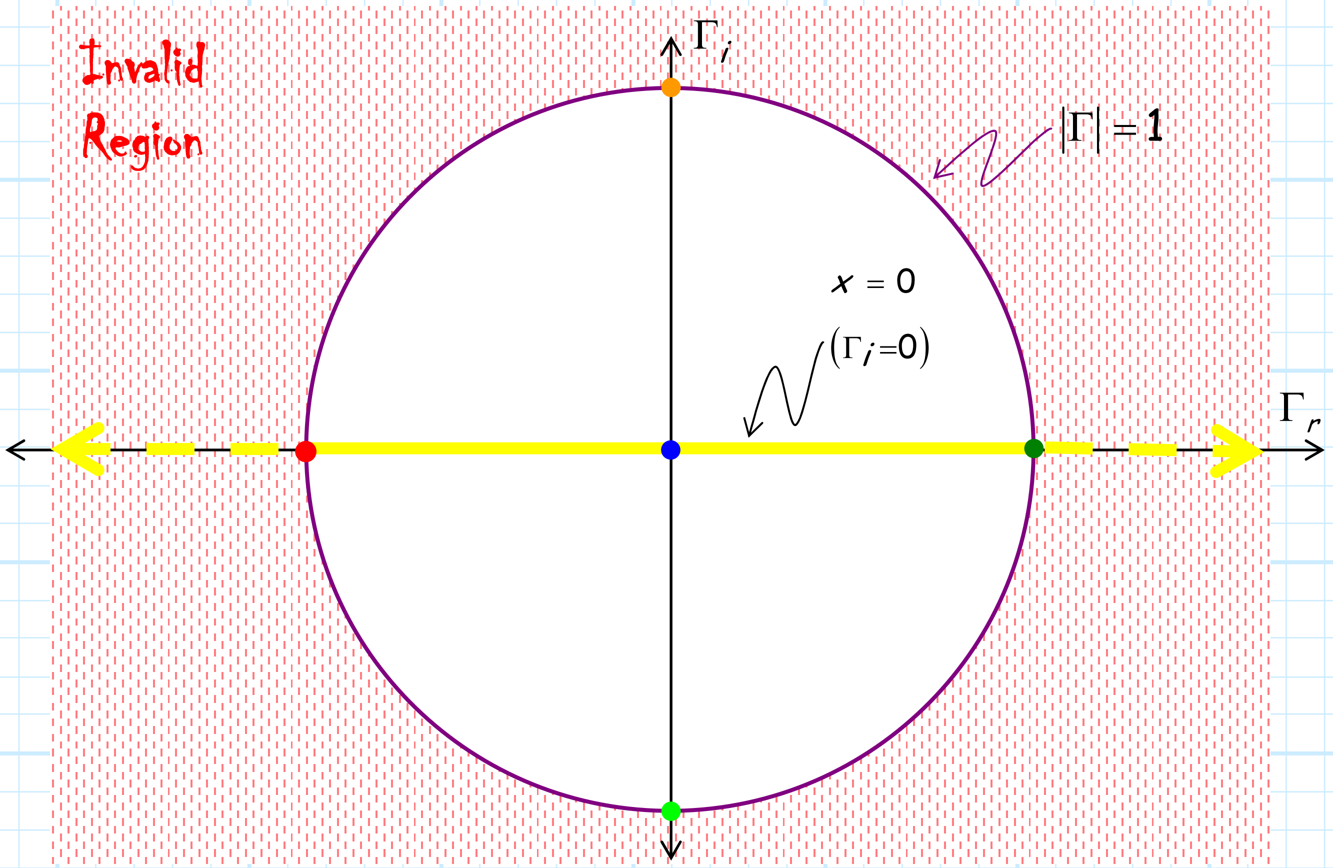
$$\Gamma = \frac{r-1}{r+1}$$

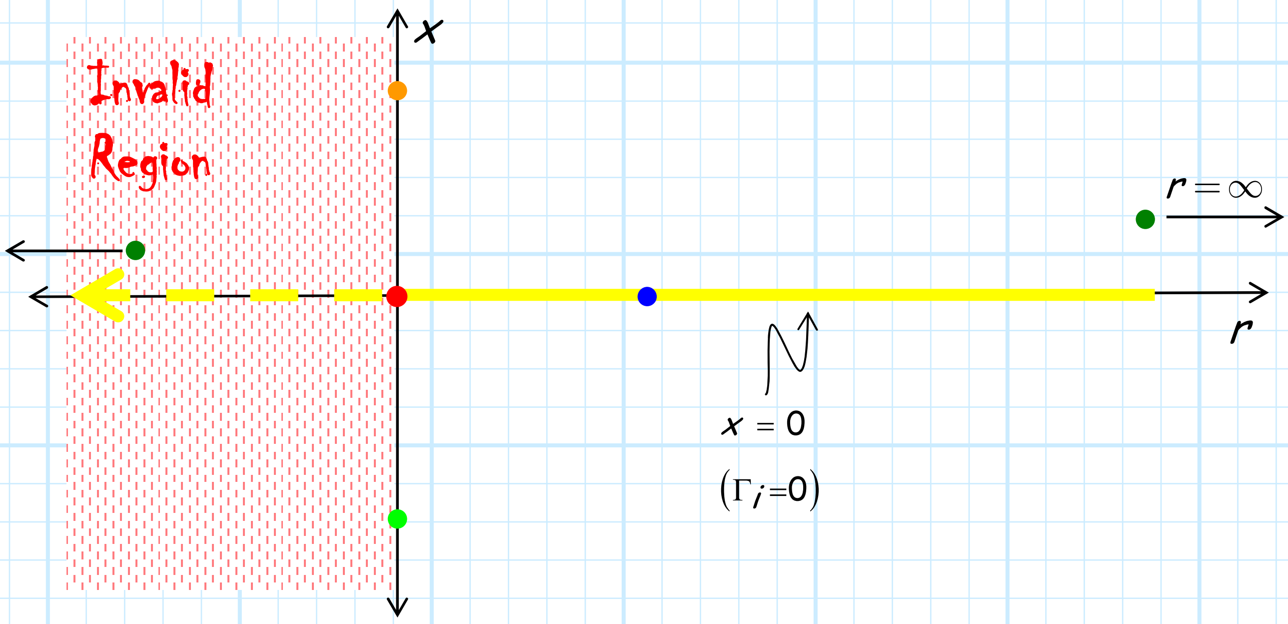
I.E.,:

$$\Gamma_r \doteq Re\{\Gamma\} = \frac{r-1}{r+1} \qquad \Gamma_i \doteq Im\{\Gamma\} = 0$$

Thus, we can determine a mapping between two **contours**—one contour ($x = 0$) on the normalized **impedance** plane, the other ($\Gamma_i = 0$) on the complex Γ **plane**:

$$x = 0 \quad \Leftrightarrow \quad \Gamma_i = 0$$





$$Z = jX$$

In other words, the case where impedance is **purely imaginary**, with **no** resistive component (i.e., $R = 0$).

Meaning that normalized impedance is:

$$z' = 0 + jx \qquad (i.e., r = 0)$$

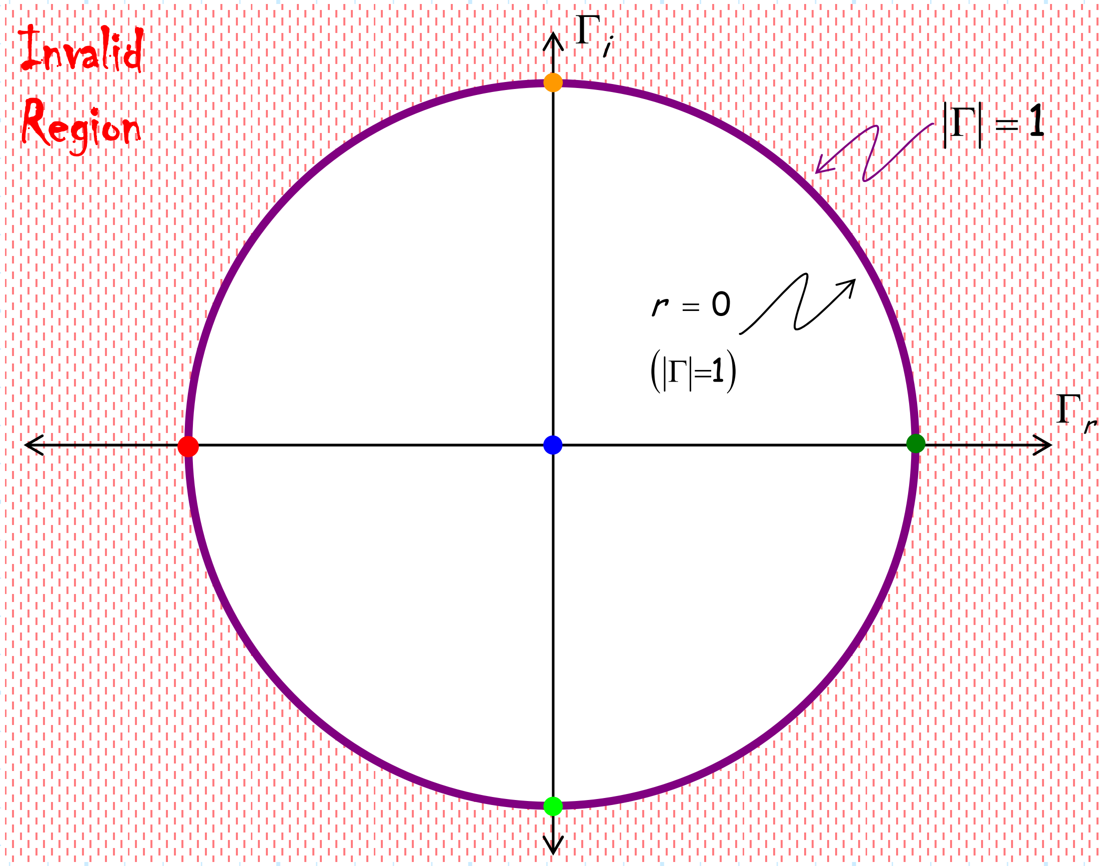
where we recall that $x = X/Z_0$.

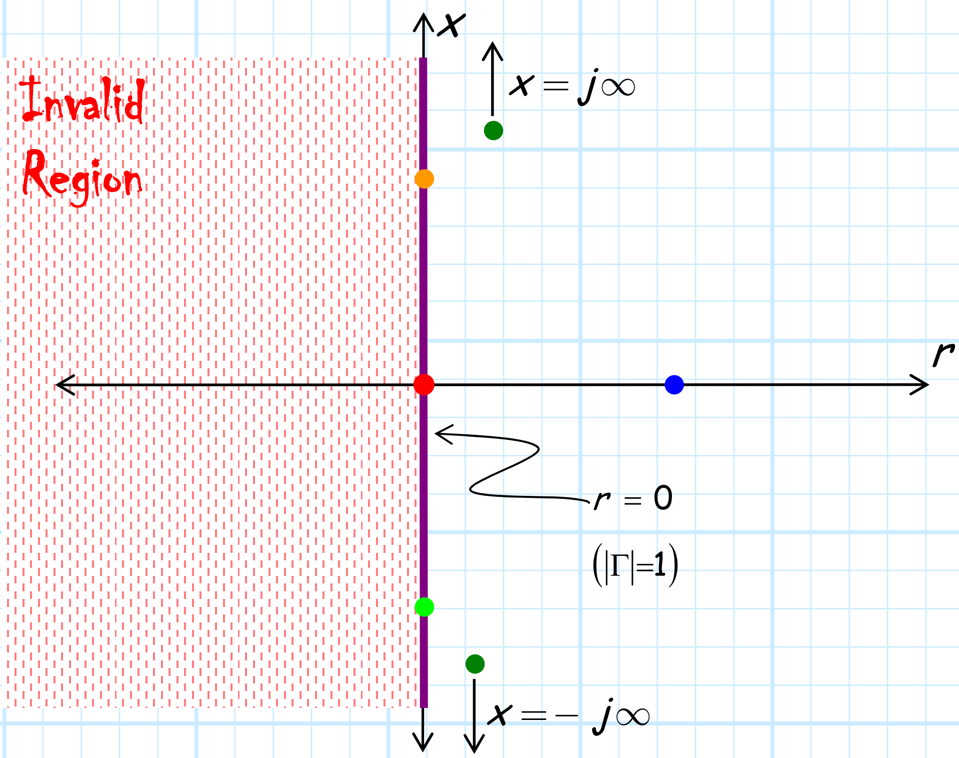
Remember, this **imaginary** impedance results in a reflection coefficient with **unity magnitude**:

$$|\Gamma| = 1$$

Thus, we can determine a mapping between two contours—one contour ($r = 0$) on the normalized impedance plane, the other ($|\Gamma| = 1$) on the complex Γ plane:

$$r = 0 \qquad \Leftrightarrow \qquad |\Gamma| = 1$$


Invalid
Region
Γ_i
$|\Gamma| = 1$
$r = 0$
$(|\Gamma|=1)$
Γ_r


x
$x = j\infty$
Invalid
Region
r
$r = 0$
$(|\Gamma|=1)$
$x = -j\infty$

Q: *These **two** "mappings" may very well be fascinating in an **academic** sense, but they are **not** particularly relevant, since actual values of impedance generally have **both** a real and imaginary component.*

*Sure, mappings of more **general** impedance contours (e.g., $r = 0.5$ or $x = -1.5$) onto the complex Γ **would** be useful—but it seems clear that those mappings are impossible to achieve!?!*

A: Actually, not only are mappings of more general impedance contours (such as $r = 0.5$ and $x = -1.5$) onto the complex Γ plane **possible**, these mappings have **already** been achieved—thanks to **Dr. Smith** and his famous **chart**!