Matched, Lossless, Reciprocal Devices

As we discussed earlier, a device can be **lossless** or **reciprocal**. In addition, we can likewise classify at being **matched**.

Let's examine **each** of these three characteristics, and how they relate to the **scattering matrix**.

Matched

A matched device is another way of saying that the **input impedance** at each port is **equal to Z_0** when **all other** ports are terminated in matched loads. As a result, the **reflection coefficient** of each port is **zero**—no signal will be come out of a port if a signal is incident on that port (and only that port).

In other words, we want:

$$V_m^- = S_{mm} V_m^+ = 0 \quad \text{for all } m$$

a result that occurs when:

$$S_{mm} = 0 \quad \text{for all } m \text{ if matched}$$

We find therefore that a matched device will exhibit a scattering matrix where all **diagonal elements** are **zero**.

Therefore:

$$\bar{\bar{\mathbf{S}}} = \begin{bmatrix} 0 & 0.1 & j0.2 \\ 0.1 & 0 & 0.3 \\ j0.2 & 0.3 & 0 \end{bmatrix}$$

is an example of a scattering matrix for a **matched,** three port device.

Lossless

Recall for a lossless device, all of the power that delivered to each device port must eventually finds its way **out**!

In other words, power is not **absorbed** by the network—no power to be **converted to heat**!

Consider, for example, a **four-port** device. Say a signal is incident on port 1, and that **all** other ports are **terminated**. The power **incident** on port 1 is therefore:

$$P_1^+ = \frac{\left|V_1^+\right|^2}{2Z_0}$$

while the power **leaving** the device at each port is:

$$P_m^- = \frac{\left|V_m^-\right|^2}{2Z_0} = \frac{\left|S_{m1}V_1^-\right|^2}{2Z_0} = |S_{m1}|^2 P_1^+$$

The **total** power leaving the device is therefore:

$$\begin{aligned} P_{out} &= P_1^- + P_2^- + P_3^- + P_4^- \\ &= |S_{11}|^2 P_1^+ + |S_{21}|^2 P_1^+ + |S_{31}|^2 P_1^+ + |S_{41}|^2 P_1^+ \\ &= \left(|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 + |S_{41}|^2\right) P_1^+ \end{aligned}$$

Note therefore that if the device is **lossless**, the output power will be **equal** to the input power, i.e., $P_{out} = P_1^+$. This is true **only** if:

$$|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 + |S_{41}|^2 = 1$$

If the device is lossless, this will likewise be true for each of the **other** ports:

$$\begin{aligned} |S_{12}|^2 + |S_{22}|^2 + |S_{32}|^2 + |S_{42}|^2 &= 1 \\ |S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 + |S_{43}|^2 &= 1 \\ |S_{14}|^2 + |S_{24}|^2 + |S_{34}|^2 + |S_{44}|^2 &= 1 \end{aligned}$$

We can state in general then:

$$\sum_{m=1}^{N} |S_{mn}|^2 = 1 \quad \text{for all } n$$

In fact, it can be shown that a lossless device will have a **unitary** scattering matrix, i.e.:

$$\bar{\bar{\mathbf{S}}}^H \bar{\bar{\mathbf{S}}} = \bar{\bar{\mathbf{I}}} \qquad \text{if lossless}$$

where H indicates **conjugate transpose** and $\bar{\bar{\mathbf{I}}}$ is the **identity** matrix.

The columns of a unitary matrix form an **orthonormal set**—that is, the **magnitude** of each column is 1 (as shown above) and dissimilar column vector are mutually **orthogonal**. In other words, the inner product (i.e., dot product) of dissimilar vectors is zero:

$$\sum_{n=1}^{N} S_{1i} S_{1j}^* = S_{1i} S_{1j}^* + S_{2i} S_{2j}^* + \cdots + S_{Ni} S_{Nj}^* = 0 \quad \text{for all } i \neq j$$

An **example** of a (unitary) scattering matrix for a **lossless** device is:

$$\bar{\bar{\mathbf{S}}} = \begin{bmatrix} 0 & \frac{1}{2} & j\frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 0 & 0 & j\frac{\sqrt{3}}{2} \\ j\frac{\sqrt{3}}{2} & 0 & 0 & \frac{1}{2} \\ 0 & j\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Reciprocal

Recall **reciprocity** results when we build a **passive** (i.e., unpowered) device with **simple** materials.

For a reciprocal network, we find that the elements of the scattering matrix are **related** as:

$$S_{mn} = S_{nm}$$

For example, a **reciprocal** device will have $S_{21} = S_{12}$ or $S_{32} = S_{23}$. We can write reciprocity in matrix form as:

$$\bar{\bar{\mathbf{S}}}^T = \bar{\bar{\mathbf{S}}} \qquad \text{if reciprocal}$$

where T indicates (non-conjugate) transpose.

An **example** of a scattering matrix describing a **reciprocal**, but **lossy** and **non-matched** device is:

$$\bar{\bar{\mathbf{S}}} = \begin{bmatrix} 0.10 & -0.40 & -j0.20 & 0.05 \\ -0.40 & j0.20 & 0 & j0.10 \\ -j0.20 & 0 & 0.10-j0.30 & -0.12 \\ 0.05 & j0.10 & -0.12 & 0 \end{bmatrix}$$