**Matched, Lossless, Reciprocal Devices**

As we discussed earlier, a device can be **lossless** or **reciprocal**. In addition, we can likewise classify it as being **matched**.

Let’s examine each of these three characteristics, and how they relate to the **scattering matrix**.

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**Matched**

A matched device is another way of saying that the input **impedance** at each port is equal to $Z_0$ when all other ports are terminated in matched loads. As a result, the **reflection coefficient** of each port is zero—no signal will be come out of a port if a signal is incident on that port (and only that port).

In other words, we want:

$$ V_m^- = S_{mm} V_m^+ = 0 \quad \text{for all } m $$

a result that occurs when:

$$ S_{mm} = 0 \quad \text{for all } m \text{ if matched} $$
We find therefore that a matched device will exhibit a scattering matrix where all diagonal elements are zero.

Therefore:

\[
\mathbf{S} = \begin{bmatrix}
0 & 0.1 & j0.2 \\
0.1 & 0 & 0.3 \\
j0.2 & 0.3 & 0 \\
\end{bmatrix}
\]

is an example of a scattering matrix for a matched, three port device.

**Lossless**

Recall for a lossless device, all of the power that delivered to each device port must eventually finds its way out!

In other words, power is not absorbed by the network—no power to be converted to heat!

Consider, for example, a four-port device. Say a signal is incident on port 1, and that all other ports are terminated. The power incident on port 1 is therefore:

\[
\rho_1^+ = \frac{|V_1^+|^2}{2Z_0}
\]

while the power leaving the device at each port is:
\[ P_m^- = \frac{|V_m^-|^2}{2Z_0} = \frac{|S_{m1}V_1^-|^2}{2Z_0} = |S_{m1}|^2 P_1^+ \]

The total power leaving the device is therefore:

\[ P_{out} = P_1^- + P_2^- + P_3^- + P_4^- \]

\[ = |S_{11}|^2 P_1^+ + |S_{21}|^2 P_1^+ + |S_{31}|^2 P_1^+ + |S_{41}|^2 P_1^+ \]

\[ = \left( |S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 + |S_{41}|^2 \right) P_1^+ \]

Note therefore that if the device is lossless, the output power will be equal to the input power, i.e., \( P_{out} = P_1^+ \). This is true only if:

\[ |S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 + |S_{41}|^2 = 1 \]

If the device is lossless, this will likewise be true for each of the other ports:

\[ |S_{12}|^2 + |S_{22}|^2 + |S_{32}|^2 + |S_{42}|^2 = 1 \]

\[ |S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 + |S_{43}|^2 = 1 \]

\[ |S_{14}|^2 + |S_{24}|^2 + |S_{34}|^2 + |S_{44}|^2 = 1 \]

We can state in general then:

\[ \sum_{m=1}^{N} |S_{mn}|^2 = 1 \quad \text{for all } n \]

In fact, it can be shown that a lossless device will have a unitary scattering matrix, i.e.:
\[ \overline{S}^H \overline{S} = \overline{I} \quad \text{if lossless} \]

where \( H \) indicates conjugate transpose and \( \overline{I} \) is the identity matrix.

The columns of a unitary matrix form an orthonormal set—that is, the magnitude of each column is 1 (as shown above) and dissimilar column vector are mutually orthogonal. In other words, the inner product (i.e., dot product) of dissimilar vectors is zero:

\[
\sum_{n=1}^{N} S_{1i}^* S_{1j} + S_{2i}^* S_{2j} + \cdots + S_{Ni}^* S_{Nj} = 0 \quad \text{for all } i \neq j
\]

An example of a (unitary) scattering matrix for a lossless device is:

\[
S = \begin{bmatrix}
0 & \frac{1}{2} & j^{\frac{3}{2}} & 0 \\
\frac{1}{2} & 0 & 0 & j^{\frac{3}{2}} \\
j^{\frac{3}{2}} & 0 & 0 & \frac{1}{2} \\
0 & j^{\frac{3}{2}} & \frac{1}{2} & 0
\end{bmatrix}
\]

Recall reciprocity results when we build a passive (i.e., unpowered) device with simple materials.
For a reciprocal network, we find that the elements of the scattering matrix are related as:

\[ S_{mn} = S_{nm} \]

For example, a reciprocal device will have \( S_{21} = S_{12} \) or \( S_{32} = S_{23} \). We can write reciprocity in matrix form as:

\[ \bar{S}^T = \bar{S} \quad \text{if reciprocal} \]

where \( T \) indicates (non-conjugate) transpose.

An example of a scattering matrix describing a reciprocal, but lossy and non-matched device is:

\[
\begin{bmatrix}
0.10 & -0.40 & -j0.20 & 0.05 \\
-0.40 & j0.20 & 0 & j0.10 \\
-j0.20 & 0 & 0.10 - j0.30 & -0.12 \\
0.05 & j0.10 & -0.12 & 0
\end{bmatrix}
\]