Matched, Lossless, Reciprocal Devices

As we discussed earlier, a device can be **lossless** or **reciprocal**. In addition, we can likewise classify it as being **matched**. Let's examine **each** of these three characteristics, and how they relate to the **scattering matrix**.

Matched

A matched device is another way of saying that the **input impedance** at each port is **equal to Z_0** when **all other** ports are terminated in matched loads. As a result, the **reflection coefficient** of each port is **zero**—no signal will be come out of a port if a signal is incident on that port (but **only** that port!).

In other words, we want:

$$V_m^- = S_{mm} V_m^+ = 0 \quad \text{for all } m$$

a result that occurs when:

$$\boxed{S_{mm} = 0 \quad \text{for all } m \text{ if matched}}$$

We find therefore that a matched device will exhibit a scattering matrix where all **diagonal elements** are **zero**.

Therefore:

$$\mathcal{S} = \begin{bmatrix} 0 & 0.1 & j0.2 \\ 0.1 & 0 & 0.3 \\ j0.2 & 0.3 & 0 \end{bmatrix}$$

is an example of a scattering matrix for a **matched,** three port device.

Lossless

For a lossless device, all of the power that delivered to each device port must eventually find its way **out**!

In other words, power is not **absorbed** by the network—no power to be **converted to heat**!

Recall the **power incident** on some port m is related to the amplitude of the **incident wave** (V_{0m}^{+}) as:

$$P_m^{+} = \frac{\left|V_{0m}^{+}\right|^2}{2Z_0}$$

While power of the **wave exiting** the port is:

$$P_m^- = \frac{\left|V_{0m}^-\right|^2}{2Z_0}$$

Thus, the power **delivered** to (absorbed by) that port is the **difference** of the two:

$$\Delta P_m = P_m^+ - P_m^- = \frac{\left|V_{0m}^+\right|^2}{2Z_0} - \frac{\left|V_{0m}^-\right|^2}{2Z_0}$$

Thus, the **total power incident** on an N-port device is:

$$P^+ = \sum_{m=1}^{N} P_m^+ = \frac{1}{2Z_0} \sum_{m=1}^{N} \left|V_{0m}^+\right|^2$$

Note that:

$$\sum_{m=1}^{N} \left|V_{0m}^+\right|^2 = \left(\mathbf{V}^+\right)^H \mathbf{V}^+$$

where operator H indicates the **conjugate transpose** (i.e., Hermetian transpose) operation, so that $\left(\mathbf{V}^+\right)^H \mathbf{V}^+$ is the **inner product** (i.e., dot product, or scalar product) of complex vector $\mathbf{V}^+$ with itself.

Thus, we can write the **total power incident** on the device as:

$$P^{+} = \frac{1}{2Z_0}\sum_{m=1}^{N}\left|V_{0m}^{+}\right|^2 = \frac{\left(\mathbf{V}^{+}\right)^H \mathbf{V}^{+}}{2Z_0}$$

Similarly, we can express the **total power** of the **waves exiting** our M-port network to be:

$$P^{-} = \frac{1}{2Z_0}\sum_{m=1}^{N}\left|V_{0m}^{-}\right|^2 = \frac{\left(\mathbf{V}^{-}\right)^H \mathbf{V}^{-}}{2Z_0}$$

Now, recalling that the incident and exiting wave amplitudes are **related** by the **scattering matrix** of the device:

$$\mathbf{V}^{-} = \mathcal{S}\ \mathbf{V}^{+}$$

Thus we find:

$$P^{-} = \frac{\left(\mathbf{V}^{-}\right)^H \mathbf{V}^{-}}{2Z_0} = \frac{\left(\mathbf{V}^{+}\right)^H \mathcal{S}^H \mathcal{S}\ \mathbf{V}^{+}}{2Z_0}$$

Now, the **total power delivered** to the network is:

$$\Delta P = \sum_{m=1}^{M}\Delta P = P^{+} - P^{-}$$

Or explicitly:

$$\begin{aligned}\Delta P &= P^+ - P^- \\ &= \frac{\left(\mathbf{V}^+\right)^H \mathbf{V}^+}{2Z_0} - \frac{\left(\mathbf{V}^+\right)^H \mathcal{S}^H \mathcal{S}\ \mathbf{V}^+}{2Z_0} \\ &= \frac{1}{2Z_0}\left(\mathbf{V}^+\right)^H \left(\mathcal{I} - \mathcal{S}^H \mathcal{S}\right)\mathbf{V}^+\end{aligned}$$

where $\mathcal{I}$ is the **identity matrix**.

Q: *Is there actually some* ***point*** *to this long, rambling, complex presentation?*

A: Absolutely! If our M-port device is lossless then the total power exiting the device must **always** be equal to the total power incident on it.

If network is **lossless,** then $P^+ = P^-$.

Or stated another way, the total **power delivered** to the device (i.e., the power absorbed by the device) must always be **zero** if the device is lossless!

If network is **lossless,** then $\Delta P = 0$

Thus, we can conclude from our math that for a **lossless device**:

$$\Delta P = \frac{1}{2Z_0}\left(\mathbf{V}^+\right)^H\left(\mathcal{I} - \mathcal{S}^H\mathcal{S}\right)\mathbf{V}^+ = 0 \qquad \text{for all } \mathbf{V}^+$$

This is true **only** if:

$$\mathcal{I} - \mathcal{S}^H\mathcal{S} = 0 \qquad \Rightarrow \qquad \mathcal{S}^H\mathcal{S} = \mathcal{I}$$

Thus, we can conclude that the **scattering matrix** of a **lossless** device has the **characteristic**:

If a network is **lossless**, then $\mathcal{S}^H\mathcal{S} = \mathcal{I}$

Q: *Huh? What exactly is this supposed to **tell** us?*

A: A matrix that satisfies $\mathcal{S}^H\mathcal{S} = \mathcal{I}$ is a special kind of matrix known as a **unitary matrix**.

If a network is **lossless**, then its scattering matrix $\mathcal{S}$ is **unitary**.

Q: *How do I **recognize** a unitary matrix if I **see** one?*

A: The **columns** of a unitary matrix form an **orthonormal set**!

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{22} \\ S_{31} & S_{33} & S_{33} & S_{33} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix}$$

matrix columns

In other words, each **column** of the scattering matrix will have a **magnitude equal to one**:

$$\sum_{m=1}^{N} \left| S_{mn} \right|^2 = 1 \quad \text{for all } n$$

while the inner product (i.e., dot product) of **dissimilar columns** must be **zero**.

$$\sum_{n=1}^{N} S_{ni} S_{nj}^* = S_{1i} S_{1j}^* + S_{2i} S_{2j}^* + \cdots + S_{Ni} S_{Nj}^* = 0 \quad \text{for all } i \neq j$$

In other words, dissimilar columns are **orthogonal**.

Consider, for example, a lossless **three-port** device. Say a signal is incident on port 1, and that **all** other ports are **terminated**. The power **incident** on port 1 is therefore:

$$P_1^+ = \frac{\left|V_{01}^+\right|^2}{2Z_0}$$

while the power **exiting** the device at each port is:

$$P_m^- = \frac{\left|V_{0m}^-\right|^2}{2Z_0} = \frac{\left|S_{m1}V_{01}^-\right|^2}{2Z_0} = |S_{m1}|^2 P_1^+$$

The **total** power exiting the device is therefore:

$$\begin{aligned} P^- &= P_1^- + P_2^- + P_3^- \\ &= |S_{11}|^2 P_1^+ + |S_{21}|^2 P_1^+ + |S_{31}|^2 P_1^+ \\ &= \left(|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2\right) P_1^+ \end{aligned}$$

Since this device is **lossless**, then the incident power (**only** on port 1) is **equal** to exiting power (i.e, $P^- = P_1^+$). This is true **only** if:

$$|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 = 1$$

Of course, this will likewise be true if the incident wave is placed on **any** of the **other** ports of this lossless device:

$$|S_{12}|^2 + |S_{22}|^2 + |S_{32}|^2 = 1$$
$$|S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 = 1$$

We can state in general then that:

$$\sum_{m=1}^{3} |S_{mn}|^2 = 1 \quad \text{for all } n$$

In other words, the columns of the scattering matrix must have **unit magnitude** (a requirement of all **unitary** matrices). It is apparent that this must be true for energy to be conserved.

An **example** of a (unitary) scattering matrix for a **lossless** device is:

$$\mathcal{S} = \begin{bmatrix} 0 & \tfrac{1}{2} & j\,\tfrac{\sqrt{3}}{2} & 0 \\ \tfrac{1}{2} & 0 & 0 & j\,\tfrac{\sqrt{3}}{2} \\ j\,\tfrac{\sqrt{3}}{2} & 0 & 0 & \tfrac{1}{2} \\ 0 & j\,\tfrac{\sqrt{3}}{2} & \tfrac{1}{2} & 0 \end{bmatrix}$$

Reciprocal

Recall **reciprocity** results when we build a **passive** (i.e., unpowered) device with **simple** materials.

For a reciprocal network, we find that the elements of the scattering matrix are **related** as:

$$S_{mn} = S_{nm}$$

For example, a **reciprocal** device will have $S_{21} = S_{12}$ or $S_{32} = S_{23}$. We can write reciprocity in matrix form as:

$$\boxed{\mathcal{S}^T = \mathcal{S} \qquad \text{if reciprocal}}$$

where T indicates (non-conjugate) transpose.

An **example** of a scattering matrix describing a **reciprocal**, but **lossy** and **non-matched** device is:

$$\bar{\bar{\mathbf{S}}} = \begin{bmatrix} 0.10 & -0.40 & -j0.20 & 0.05 \\ -0.40 & j0.20 & 0 & j0.10 \\ -j0.20 & 0 & 0.10-j0.30 & -0.12 \\ 0.05 & j0.10 & -0.12 & 0 \end{bmatrix}$$